The Dogma is Wrong

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. Preliminary writeup [BV1], fuller writeup [BV2]. More at ωεβ/talks.

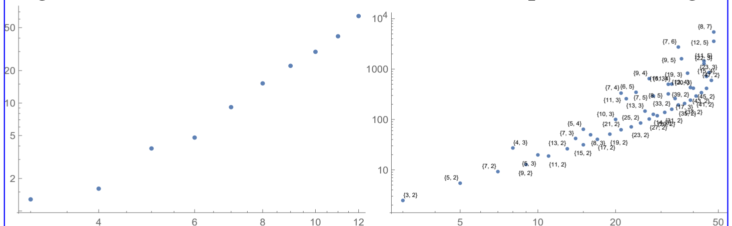
Happy Birthday Anton!

ωεβ:=http://drorbn.net/ld17/

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

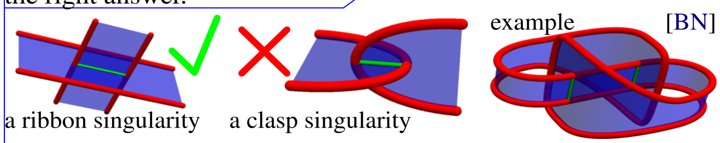
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



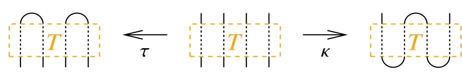
Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

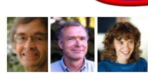
Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \leq 2g-1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Ribbon Knots.



a ribbon singularity a clasp singularity example [BN]

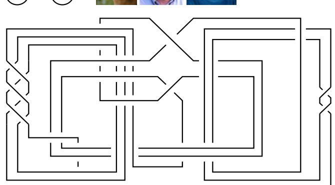




Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccc} & U \in \mathcal{T}_n & & 1 \in \mathcal{A}_n \\ & \nearrow\tau & & \nearrow\tau \\ \mathcal{T}_{2n} & & \xrightarrow{z} & \mathcal{A}_{2n} \\ & \searrow\kappa & & \searrow\kappa \\ & \text{ribbon } K \in \mathcal{T}_1 & & z(K) \in \mathcal{R} \subseteq \mathcal{A}_1 \end{array}$$

with $\mathcal{R} := \kappa(\tau^{-1}(1))$



[Vo]: Works for Alexander!

$A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$

$\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$

Faster is better, leaner is meaner!

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of $poly$ in $S(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,

$$\mathbb{O}\left(a_1^3 y_1 a_2 e^{y_3} x_3^9 \mid x_3 a_1 \otimes y_1 y_3 a_2\right) = x^9 a^3 \otimes y e^y a \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

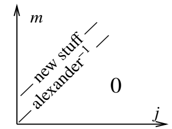
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

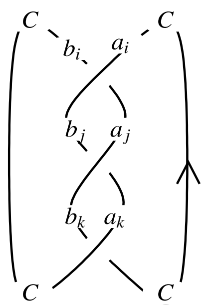
$$\left.\frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot \omega(K)(e^\hbar) = 1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})\omega(K)(q^d)} \left(1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

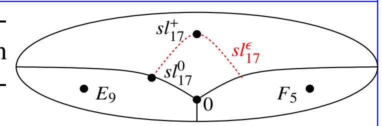
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

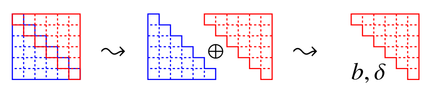
The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

$$m_k^{ij} \circlearrowright \{\mathcal{F}_S\} \xrightarrow{\mathbb{E}} \{U^{\otimes S}\} \circlearrowleft m_k^{ij}$$

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon/(\epsilon^{k+1} = 0)$.



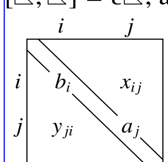
Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\triangledown, b, \delta)$:



$b(\triangledown) = b : \triangledown \otimes \triangledown \to \triangledown$

$b(\triangle) \rightsquigarrow \delta : \triangledown \to \triangledown \otimes \triangledown$

Now define $gl_n^\epsilon := \mathcal{D}(\triangledown, b, \epsilon\delta)$. Schematically, this is $[\triangledown, \triangledown] = \triangledown$, $[\triangle, \triangle] = \epsilon\triangle$, and $[\triangledown, \triangle] = \triangle + \epsilon\triangledown$. In detail, it is



$$[x_{ij}, x_{kl}] = \delta_{jk} x_{il} - \delta_{li} x_{kj} \qquad [y_{ij}, y_{kl}] = \epsilon\delta_{jk} y_{il} - \epsilon\delta_{li} y_{kj}$$
$$[x_{ij}, y_{kl}] = \delta_{jk}(\epsilon\delta_{j<k} x_{il} + \delta_{il}(b_i + \epsilon a_i)/2 + \delta_{i>l} y_{il}) - \delta_{li}(\epsilon\delta_{k<j} x_{kj} + \delta_{kj}(b_j + \epsilon a_j)/2 + \delta_{k>j} y_{kj})$$
$$[a_i, x_{jk}] = (\delta_{ij} - \delta_{ik}) x_{jk} \qquad [b_i, x_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) x_{jk}$$
$$[a_i, y_{jk}] = (\delta_{ij} - \delta_{ik}) y_{jk} \qquad [b_i, y_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) y_{jk}$$

The Main sl_2 Theorem. Let $\mathfrak{g}^\epsilon = \langle t, y, a, x \rangle / ([t, \cdot] = 0, [a, x] = x, [a, y] = -y, [x, y] = t - 2\epsilon a)$ and let $\mathfrak{g}_k = \mathfrak{g}^\epsilon/(\epsilon^{k+1} = 0)$. The $\mathfrak{g}_k$-invariant of any S-component tangle K can be written in the form $Z(K) = \mathbb{O}\left(\omega \mathbb{e}^{L+Q+P} : \bigotimes_{i\in S} y_i a_i x_i\right)$, where ω is a scalar (a rational function in the variables t_i and their exponentials $T_i := \mathbb{e}^{t_i}$), where $L = \sum l_{ij} t_i a_j$ is a quadratic in t_i and a_j with integer coefficients l_{ij}, where $Q = \sum q_{ij} y_i x_j$ is a quadratic in the variables y_i and x_j with scalar coefficients q_{ij}, and where P is a polynomial in $\{\epsilon, y_i, a_i, x_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{y_i, \sqrt{a_i}, x_i\}$. Furthermore, after setting $t_i = t$ and $T_i = T$ for all i, the invariant $Z(K)$ is poly-time computable.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1708/