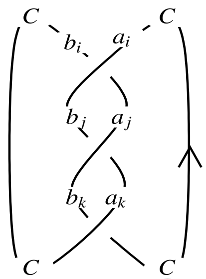


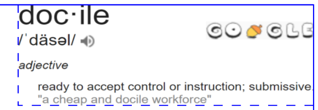
The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R=\sum a_i\otimes b_i\in U\otimes U \quad\text{and}\quad C\in U,$$

form $$Z=\sum_{i,j,k} Ca_ib_ja_kC^2b_ia_jb_kC.$$

Problem. Extract information from Z.
The Dogma. Use representation theory. In principle finite, but *slow*.

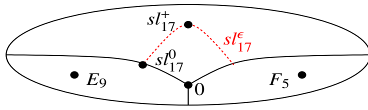
Definition. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form



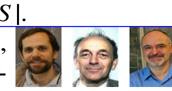
$$\mathbb{e}^{q^{ij}z_iz_j}P=\mathbb{e}^{q^{ij}z_iz_j}\left(\sum_{k\geq 0}\epsilon^kP_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k\leq 4k$.
Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$.

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl^k_{17}:=sl^\epsilon_{17}/(\epsilon^{k+1}=0)$.

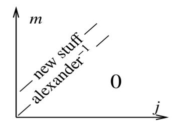


Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing


Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar}=\sum_{j,m\geq 0}a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



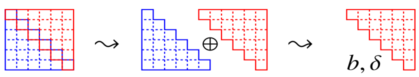
"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q)=\frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

Solvable Approximation. A quantized universal enveloping algebra (aka "quantum group") is an ∞-dimensional inverse limit.

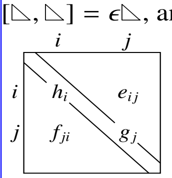
untrimmed ⟶ halfway-trimmed ⟶ almost fully-trimmed
↕ ↕ ↕
too hard — "solvable approximation" (a parameter is hidden) — "finite-type"

Recomposing gl_n. Half is enough! $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(◹,b,\delta)$:



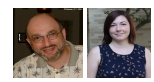
$b(◹)=b\colon ◹\otimes◹\to◹$
$b(◺)\rightsquigarrow\delta\colon ◹\to◹\otimes◹$

Now define $gl_n^\epsilon:=\mathcal{D}(◹,b,\epsilon\delta)$. Schematically, this is $[◹,◹]=◹$, $[◺,◺]=\epsilon◺$, and $[◹,◺]=◺+\epsilon◹$. In detail, it is

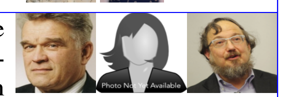


$$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{li}e_{kj}\qquad [f_{ij},f_{kl}]=\epsilon\delta_{jk}f_{il}-\epsilon\delta_{li}f_{kj}$$
$$[e_{ij},f_{kl}]=\delta_{jk}(\epsilon\delta_{j<k}e_{il}+\delta_{il}(h_i+\epsilon g_i)/2+\delta_{i>l}f_{il})-\delta_{li}(\epsilon\delta_{k<j}e_{kj}+\delta_{kj}(h_j+\epsilon g_j)/2+\delta_{k>j}f_{kj})$$
$$[g_i,e_{jk}]=(\delta_{ij}-\delta_{ik})e_{jk}\qquad [h_i,e_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})e_{jk}$$
$$[g_i,f_{jk}]=(\delta_{ij}-\delta_{ik})f_{jk}\qquad [h_i,f_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})f_{jk}$$

Prior art. Some amazing computations by Rozansky and Overbay in [Ro2, Ro3] and in [Ov].


Rozansky, Overbay

Faddeev's Formula (In as much as we can tell, first appeared w/o proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q:=\frac{q^n-1}{q-1}$, with $[n]_q!:=[1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x:=\sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have


Faddeev Quesne Zagier

$$\log\mathbb{e}_q^x=\sum_{k\geq 1}\frac{(1-q)^kx^k}{k(1-q^k)}=x+\frac{(1-q)^2x^2}{2(1-q^2)}+\dots.$$

Proof. We have that $\mathbb{e}_q^x=\frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx}=(1+(1-q)x)\mathbb{e}_q^x$, and

$$\log\mathbb{e}_q^{qx}=\log(1+(1-q)x)+\log\mathbb{e}_q^x.$$

Writing $\log\mathbb{e}_q^x=\sum_{k\geq 1}a_kx^k$ and comparing powers of x, we get $q^ka_k=-(1-q)^k/k+a_k$, or $a_k=\frac{(1-q)^k}{k(1-q^k)}$. □

Solvable Approximation (2). At $\epsilon=1$ and modulo $h=g$, the above is just gl_n. By rescaling at $\epsilon\neq 0$, gl_n^ϵ is independent of ϵ. We let gl_n^k be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1}=0$. It is the "$k$-smidgen solvable approximation" of gl_n!
Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2:=[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}_3:=[\mathfrak{g}_2,\mathfrak{g}_2]$, …, $\mathfrak{g}_d=0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.
Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

***GDO*-Categories.** Given $\mathfrak{g}$ with basis $B=\{x,y,\dots\}$, consider the following diagram:

$$\begin{tikzcd}
\mathbb{Q}=\hat{\mathcal{U}}_{(q)}\left(\bigoplus_0\mathfrak{g}\right) \arrow[r, "Z"] & \hat{\mathcal{U}}_{(q)}(\mathfrak{g}) \arrow[r, shift left, "\Delta"] & \hat{\mathcal{U}}_{(q)}\left(\bigoplus_2\mathfrak{g}\right) \arrow[l, shift left, "m"] \\
\hat{\mathcal{S}}(\emptyset) \arrow[u] \arrow[r, "Z"] & \hat{\mathcal{S}}(B) \arrow[u, shift left, "\mathbb{O}(xy\dots:\,\cdot)"] \arrow[u, shift right, "\mathbb{O}(yx\dots:\,\cdot)"'] \arrow[r, shift left, "\Delta"] \arrow[loop below, "SW_{xy}"] & \hat{\mathcal{S}}(B_1,B_2) \arrow[l, shift left, "m"] \arrow[u, "\mathbb{O}(y_1x_1\dots\otimes y_2x_2\dots:\,\cdot)"']
\end{tikzcd}$$

Hence Z, SW_{xy}, m, Δ, (and likewise S and θ) are morphisms in the *completion* of the monoidal category $\mathcal{F}$ whose objects are finite sets B and whose morphisms are $\mathrm{mor}_{\mathcal{F}}(B,B'):=\mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B)\to\mathcal{S}(B'))=\mathcal{S}(B^*,B')$ (by convention, $x^*=\xi$, $y^*=\eta$, etc.). Ergo we need to *consolidate* (at least parts of) said completion.

Aside. "Consolidate" means "give a finite name to an infinite object, and figure out how to sufficiently manipulate such finite names". E.g., solving $f''=-f$ we encounter and set $\sum\frac{(-1)^kx^{2k}}{(2k)!}\rightsquigarrow\cos x$, $\sum\frac{(-1)^kx^{2k+1}}{(2k+1)!}\rightsquigarrow\sin x$, and then $\cos^2x+\sin^2x=1$ and $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

The Composition Law. If

$$\mathcal{S}(B_0)\xrightarrow[{}^tf\in\mathbb{Q}[\![\zeta_{0i},z_{1j}]\!]]{f}\mathcal{S}(B_1)\xrightarrow[{}^tg\in\mathbb{Q}[\![\zeta_{1j},z_{2k}]\!]]{g}\mathcal{S}(B_2)$$

then ${}^t(f/\!/g)={}^t(g\circ f)=\left(g|_{\zeta_{1j}\to\partial_{z_{1j}}}f\right)_{z_{1j}=0}$.

Examples.

1. The 1-variable identity map $I\colon\mathcal{S}(z)\to\mathcal{S}(z)$ is given by ${}^tI_1=\mathbb{e}^{z\zeta}$ and the n-variable one by ${}^tI_n=\mathbb{e}^{z_1\zeta_1+\cdots+z_n\zeta_n}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/