Solvable Approximations of the Quantum sl_2 Portfolio

See also [BV]  With Roland van der Veen

ωεβ:=http://drorbn.net/mm18/

W.I.P. Warning  Brute Warning

Our Main Theorem (loosely stated). Everything that matters in the quantum sl_2 portfolio can be continuously expressed in terms of docile perturbed Gaussians using solvable approximations. ○

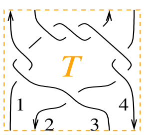
Our Main Points.

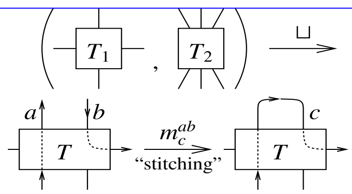
- What's the "quantum sl_2 portfolio"?
- What in it "matters" and why? (the most important question)
- What's "solvable approximation"? What's "continuously"?
- What are "docile perturbed Gaussians"?
- Why do they matter? (2^{nd} most important)
- How proven? (docile)
- How implemented? (sacred; the work of unsung heroes)
- Some context and background.
- What's next?

The quantum sl_2 Portfolio includes a classical universal enveloping algebra CU, its quantization QU, their tensor powers $CU^{\otimes S}$ and $QU^{\otimes S}$ with the "tensor operations" $\otimes$, their products m_k^{ij}, coproducts Δ^i_{jk} and antipodes S_i, their Cartan automophisms $C\theta\colon CU \to CU$ and $Q\theta\colon QU \to QU$, the "dequantizators" $A\mathbb{D}\colon QU \to CU$ and $S\mathbb{D}\colon QU \to CU$, and most importantly, the R-matrix R and the Drinfel'd element s. All this in any PBW basis, and change of basis maps are included.

$$R, s \in \{QU^{\otimes S}\} \circlearrowleft^{\otimes, m_k^{ij}, \Delta^i_{jk}, S_i, \theta} \xrightarrow{A\mathbb{D},\, S\mathbb{D}} \{CU^{\otimes S}\} \circlearrowleft^{\otimes, m_k^{ij}, \Delta^i_{jk}, S_i, \theta}$$

(v-)Tangles.



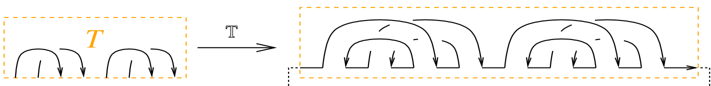


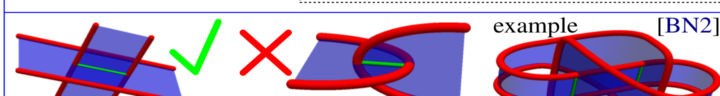
(meta-associativity: $m^{ab}_x /\!\!/ m^{xc}_y = m^{bc}_x /\!\!/ m^{ax}_y$) (tangles are generated by ⤯ and ⤰)

Strand doubling: $a \xrightarrow{\Delta^a_{bc}} \begin{smallmatrix} b \\ c \end{smallmatrix}$ Strand reversal: $a \xrightarrow{S_a} a$

Robert Engman's "The Loop"

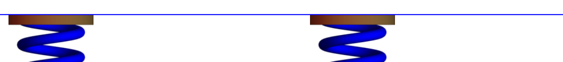
Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.
Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via

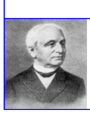




a ribbon singularity a clasp singularity

A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,
Conjecture. Some slice knots are not ribbon.
Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)





"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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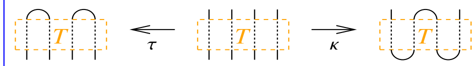
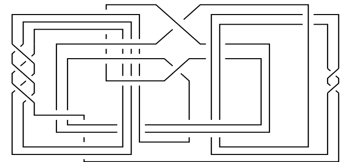
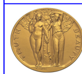


Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccccc} & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n & \\ \mathcal{T}_{2n} & \begin{smallmatrix}\nearrow^{\tau} \\ \searrow_{\kappa}\end{smallmatrix} & \xrightarrow{z} \mathcal{A}_{2n} & \begin{smallmatrix}\nearrow^{\tau} \\ \searrow_{\kappa}\end{smallmatrix} & \text{with } \mathcal{R} := \kappa(\tau^{-1}(1)) \\ & \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1 & \end{array}$$



Faster is better, leaner is meaner!



The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\}$ with $R_S := \mathbb{Z}(\{t_a\colon a \in S\})$:

$$\left({}_a\nearrow_b, {}_b\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array} \right)$$

(Roland: "add to A the product of column b and row a, divide by $(1 - A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu:=T_a-1 \\ \nu:=\alpha-\sigma_a \\ T_a\mapsto T_bT_c}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu \\ c & (T_c-1)\nu/\mu & (\alpha - \sigma_a T_a - \nu T_c)/\mu & (T_c-1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array} \right)$$

$$dS^a \Big\downarrow T_a \to T_a^{-1}$$

$$\left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array} \right)$$

Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left({}_a\nearrow_b, {}_b\nwarrow_a\right) = (a \to 1, b \to t_a^{\pm 1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma /\!\!/ m^{ab}_c = (\sigma \setminus \{a, b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a,t_b\to t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Implementation key idea: ωεβ/AlexDemo

$$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \textstyle\sum \alpha_{ab} t_a h_b)$$

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Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_ b_ → c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{t_a,h_a} λ  ∂_{t_a,h_b} λ  ∂_{t_a} λ; ∂_{t_b,h_a} λ  ∂_{t_b,h_b} λ  ∂_{t_b} λ; ∂_{h_a} λ  ∂_{h_b} λ  λ) /. (t | h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ / μ   ϵ + δ θ / μ; ϕ + α ψ / μ   Ξ + ψ θ / μ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_ b_} := Γ[1, {t_a, t_b}.(1  1 - T_a; 0  T_a).{h_a, h_b}];
Rm_{a_ b_} := Rp_{ab} /. T_a → 1 / T_a;

ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_{a_} :→ a, ∞];
   M = Outer[Factor[∂_{h_{#1} t_{#2}} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
```

Meta-Associativity **Runs.**

```
ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
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True
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R3 . . . divide and conquer!

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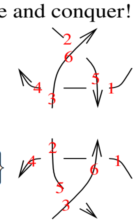
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{ \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix} \right\}$$

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z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```
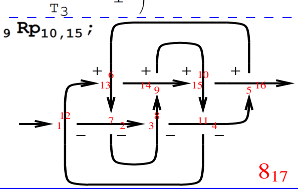

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$



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Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/