The Taylor Remainder Formulas. Let f be a smooth function, let $P_{n,a}(x)$ be the nth order Taylor polynomial of f around a and evaluated at x, so with $a_k = f^{(k)}(a)/k!$,

Brook Taylor

$$P_{n,a}(x) := \sum_{k=0}^{n} a_k(x-a)^k,$$

and let $R_{n,a}(x) := f(x) - P_{n,a}(x)$ be the "mistake" or "remainder term". Then

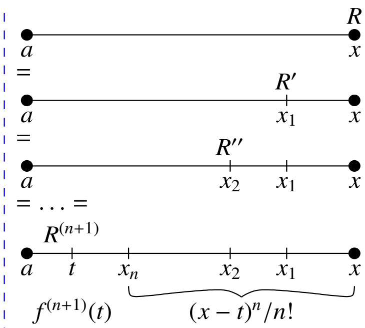
$$R_{n,a}(x) = \int_a^x dt\, \frac{f^{(n+1)}(t)}{n!}(x-t)^n, \tag{1}$$

or alternatively, for some t between a and x,

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}. \tag{2}$$

(In particular, the Taylor expansions of sin, cos, exp, and of several other lovely functions converges to these functions *everywhere*, no matter the odds.)

Proof of (1) (for adults; I learned it from my son Itai). The fundamental theorem of calculus says that if $g(a) = 0$ then $g(x) = \int_a^x dx_1 g(x_1)$. By design, $R^{(k)}_{n,a}(a) = 0$ for $0 \le k \le n$. Therefore



$$R_{n,a}(x) = \int_a^x dx_1 R'_{n,a}(x_1)$$

$$= \int_a^x dx_1 \int_a^{x_1} dx_2 R''_{n,a}(x_2)$$

$$= \ldots = \int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_n} dx_n \int_a^t dt\, R^{(n+1)}_{n,a}(t)$$

$$= \int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_n} dx_n \int_a^t dt\, f^{(n+1)}(t),$$

when $x > a$, and with similar logic when $x < a$,

$$= \int_{a\le t\le x_n \le \ldots \le x_1 \le x} f^{(n+1)}(t) = \int_a^t dt\, f^{(n+1)}(t) \int_{t \le x_n \le \ldots \le x_1 \le x} 1$$

$$= \int_a^t dt \frac{f^{(n+1)}(t)}{n!} \int_{(x_1,\ldots,x_n)\in[t,x]^n} 1 = \int_a^x dt \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

□

de-Fubini (obfuscation in the name of simplicity). Prematurely aborting the above chain of equalities, we find that for any $1 \le k \le n+1$,

Guido Fubini

$$R(x) = \int_a^x dt\, R^{(k)}(t)\frac{(x-t)^{k-1}}{(k-1)!}.$$

But these are easy to prove by induction using integration by parts, and there's no need to invoke Fubini.

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Partial Derivatives Commute. Make Fubini Smile Again!

If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is C^2 near $a \in \mathbb{R}^2$, then $f_{12}(a) = f_{21}(a)$.

Proof. Let $x \in \mathbb{R}^2$ be small, and let $R := [a_1, a_1+x_1]\times[a_2, a_2+x_2]$.

$$f_{12}(a) \sim \boxed{\int f_{12}} = \boxed{\sum f} = \boxed{\int f_{21}} \sim f_{21}(a)$$

$$f_{12}(a) \sim \frac{1}{|R|}\int_R f_{12} = \frac{1}{|R|}\int_{a_1}^{a_1+x_1} dt_1\,(f_1(t_1, a_2+x_2) - f_1(t_1, a_2))$$

$$= \frac{1}{|R|}\begin{pmatrix} f(a_1+x_1, a_2+x_2) - f(a_1+x_1, a_2) \\ -f(a_1, a_2+x_2) + f(a_1, a_2)\end{pmatrix}.$$

But the answer here is the same as in

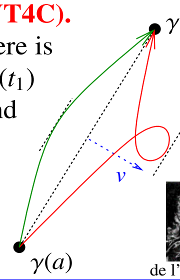
$$f_{21}(a) \sim \frac{1}{|R|}\int_R f_{21} = \frac{1}{|R|}\int_{a_2}^{a_2+x_2} dt_2\,(f_2(a_1+x_1, t_2) - f_2(a_1, t_2))$$

$$= \frac{1}{|R|}\begin{pmatrix} f(a_1+x_1, a_2+x_2) - f(a_1, a_2+x_2) \\ -f(a_1+x_1, a_2) + f(a_1, a_2)\end{pmatrix},$$

and both of these approximations get better and better as $x \to 0$. □

The Mean Value Theorem for Curves (MVT4C). If $\gamma\colon [a,b] \to \mathbb{R}^2$ is a smooth curve, then there is some $t_1 \in (a,b)$ for which $\gamma(b) - \gamma(a)$ and $\dot\gamma(t_1)$ are linearly dependent. If also $\gamma(a) = 0$, and $\gamma = \binom{\xi}{\eta}$ and $\eta \neq 0 \neq \dot\eta$ on (a,b), then

$$\frac{\xi(b)}{\eta(b)} = \frac{\dot\xi(t_1)}{\dot\eta(t_1)} \quad \left(\text{when lucky, } = \frac{\ddot\xi(t_2)}{\ddot\eta(t_2)} \ldots\right).$$



de l'Hôpital

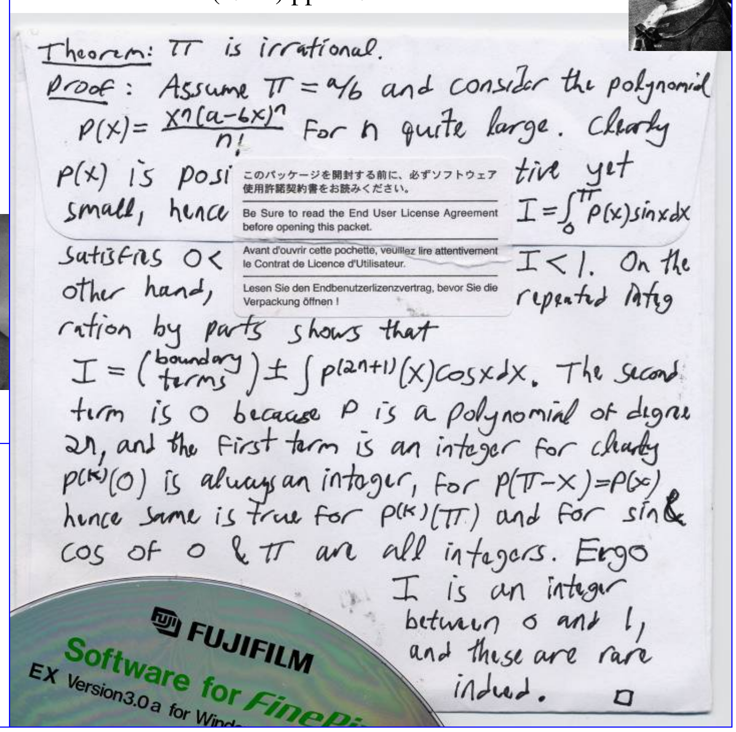
Proof of (2). Iterate the lucky MVT4C as follows:

$$\frac{R_{n,a}(x)}{(x-a)^{n+1}} = \frac{R'_{n,a}(t_1)}{(n+1)(t_1-a)^n} = \ldots = \frac{R^{(n+1)}_{n,a}(t_{n+1})}{(n+1)!} = \frac{f^{(n+1)}(t)}{(n+1)!}.$$

□

J.H. Lambert

π is Irrational following Ivan Niven, Bull. Amer. Math. Soc. (1947) pp. 509:



Theorem: π is irrational.
Proof: Assume π = a/b and consider the polynomial P(x) = x^n(a−bx)^n/n! for n quite large. Clearly P(x) is positive yet small, hence I = ∫₀^π P(x) sin x dx satisfies 0 < I < 1. On the other hand, repeated integration by parts shows that I = (boundary terms) ± ∫ P^(2n+1)(x) cos x dx. The second term is 0 because P is a polynomial of degree 2n, and the first term is an integer for clearly P^(k)(0) is always an integer, for P(π−x) = P(x) & hence same is true for P^(k)(π) and for sin & cos of 0 & π are all integers. Ergo I is an integer between 0 and 1, and these are rare indeed. □

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MAASeaway-1810/