Do Not Turn Over Until Instructed

Dror Bar-Natan: Talks: MAASeaway-1810:

Thanks for inviting me to the fall 2018 MAA Seaway Section meeting!

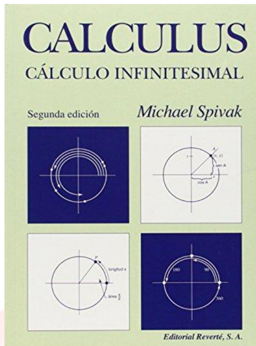
My Favourite First-Year Analysis Theorem

Handout, video, links at ωεβ:=http://drorbn.net/maa18/

Abstract. Whatever it may be, it should say something useful and exciting and it should not be *about* rigour, yet it should *demand* rigour. You can't guess. You probably think it the dreariest. You are wrong.

Contents Ⓢ

Several excerpts here are from Spivak's "Calculus" Ⓢ. I believe they fall under "fair use".



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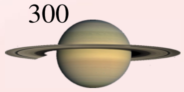


6 Continuous Functions.

for every $\varepsilon > 0$ there is $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Ⓢ

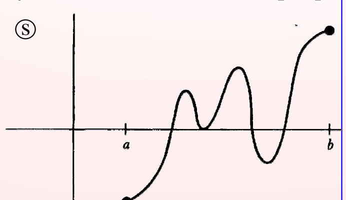
If f and g are continuous at a, then

(1) $f + g$ is continuous at a,

(2) $f \cdot g$ is continuous at a.

7 Three Hard Theorems.

If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$. Ⓢ



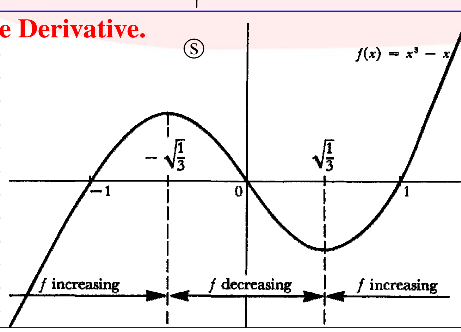
11 Significance of the Derivative. Ⓢ

$y = x^3 - x$

$y' = 3x^2 - 1$

$= (\sqrt{3}x+1)(\sqrt{3}x-1)$

$= \begin{cases} >0 & x > \sqrt{1/3} \\ <0 & -\sqrt{1/3} < x < \sqrt{1/3} \\ >0 & x < -\sqrt{1/3} \end{cases}$

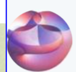


14 The Fundamental Theorem of Calculus.

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

Ⓢ $$\int_a^b f = g(b) - g(a).$$

Tweets Tweets & replies $*$16 π is Irrational.



Dror Bar-Natan @drorbarnatan · 2 Apr 2013
π=a/b, f(x)=x^n(a-bx)^n/n!, n large => 0<V=∫(0,π)f(x)sin(x)dx<1. Repeated integration by parts & f(x)=f(π-x) => V∈Z. So π is irrational.

1

20 Approximation by Polynomial Functions.

Suppose that f is a function for which

$$f'(a), \ldots, f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \le k \le n,$$

and define

$$P_{n,a}(x) = a_0 + a_1(x - a) + \cdots + a_n(x - a)^n.$$

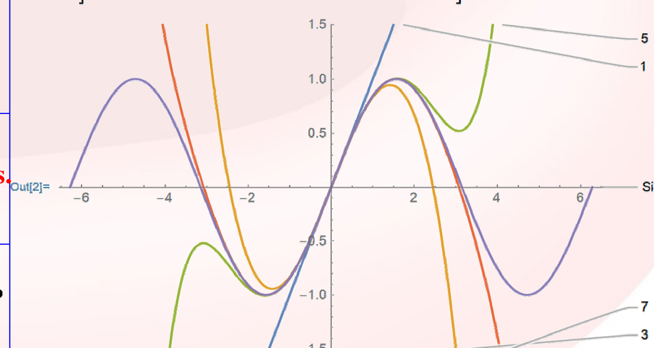
Then

$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x - a)^n} = 0.$$ Ⓢ

For example for $f(x) = \sin(x)$ at $a = 0$, $f^{(k)} = \sin, \cos, -\sin, -\cos, \sin, \ldots$, so

$$a_k = \begin{cases} \frac{(-1)^{(k-1)/2}}{k!} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

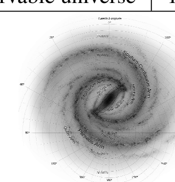
```
In[1]:= a_k_ := { (-1)^((k-1)/2) / k!   OddQ[k]
                  0                     EvenQ[k] ;
        Plot[Evaluate@Append[
           Table[Labeled[Sum_{k=0}^{n} a_k x^k, n], {n, {1, 3, 5, 7}}],
           Labeled[Sin[x], Sin]
        ], {x, -2π, 2π}, PlotRange → {-1.5, 1.5}]
```




```
In[3]:= Column@Table[k → N[a_k 157^k], {k, {0, 3, 9, 13, 29, 35, 157, 223, 457}}]
Out[3]= 0 → 0.
        3 → -644982.
        9 → 1.59711 × 10^14
        13 → 5.65477 × 10^18
        29 → 5.42689 × 10^32
        35 → -6.95433 × 10^36
        157 → 4.86366 × 10^66
        223 → -1.94045 × 10^61
        457 → 4.87404 × 10^-16
In[4]:= {N[Sum_{k=0}^{457} a_k 157^k], Sum_{k=0}^{457} N[a_k 157^k]}
Out[4]= {-0.0795485, 5.10624 × 10^50}
In[5]:= N@Sin[157]
Out[5]= -0.0795485
```

Some sizes (in multiples of the diameter of a Hydrogen atom:

A red blood cell	1.56×10^5
The CN Tower	1.11×10^{13}
The rings of Saturn	5.6×10^{18}
The Milky Way galaxy	1.89×10^{31}
The observable universe	1.76×10^{37}



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Video and more at http://www.math.toronto.edu/~drorbn/Talks/MAASeaway-1810/