The Real Thing. In the algebra QU_ϵ, over $\mathbb{Q}[\![\hbar]\!]$ using the *yaxt* order, $T = \mathbb{e}^{\hbar t}$, $\bar{T} = T^{-1}$, $\mathcal{A} = \mathbb{e}^{\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

$$\tilde{R}_{ij} = \mathbb{e}^{\hbar(y_ix_j - t_ia_j)}\left(1 + \epsilon\hbar\left(a_ia_j - \hbar^2y_i^2x_j^2/4\right) + O(\epsilon^2)\right)$$

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$$\tilde{m} = \mathbb{e}^{(\alpha_1+\alpha_2)a+\eta_2\xi_1(1-T)/\hbar+(\xi_1\bar{\mathcal{A}}_2+\xi_2)x+(\eta_1+\eta_2\bar{\mathcal{A}}_1)y}\left(1 + \epsilon\lambda + O(\epsilon^2)\right),$$

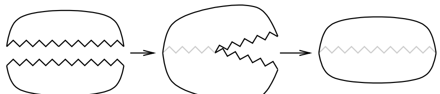
where $\lambda = 2a\eta_2\xi_1T + \eta_2^2\xi_1^2\left(3T^2 - 4T + 1\right)/4\hbar - \eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2/2$
$-\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1/2 + \eta_2\xi_1xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$.

Finally,

$$\tilde{\Delta} = \mathbb{e}^{\tau(t_1+t_1)+\eta(y_1+T_1y_2)+\alpha(a_1+a_2)+\xi(x_1+x_2)}\left(1 + O(\epsilon)\right) \in \mathcal{S}(B^*, B_1, B_2),$$

and $\tilde{S} = \mathbb{e}^{-\tau t-\alpha a-\eta\xi(1-\bar{T})\mathcal{A}/\hbar-\bar{T}\eta y\mathcal{A}-\xi x\mathcal{A}}\left(1 + O(\epsilon)\right) \in \mathcal{S}(B^*, B)$.

The Zipping Issue. (between unbound and bound lies half-zipped).



Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set $\left\langle P(\zeta^j, z_i)\right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i=0}$. (E.g., if $P = \sum a_{nm}\zeta^nz^m$ then $\langle P\rangle_\zeta = \sum a_{nm}\,\partial_z^nz^m\Big|_{z=0} = \sum n!a_{nn}$).

The Zipping / Contraction Theorem. If $P = P(\zeta^j, z_i)$ has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P\mathbb{e}^{c+\eta^iz_i+y_j\zeta^j+q_j^i z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde{q})\mathbb{e}^{c+\eta^i\tilde{q}_i^ky_k}\left\langle P\,\middle|\,{}^{\zeta^j\to\zeta^j+\eta^i\tilde{q}_i^j}_{z_i\to\tilde{q}_i^k(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

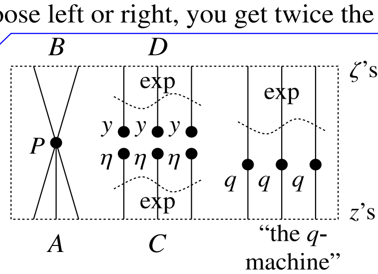
where $\tilde{q}$ is the inverse matrix of $1-q$: $(\delta_j^i - q_j^i)\tilde{q}_k^j = \delta_k^i$.

Exponential Reservoirs. The true Hilbert hotel is exp! Remove one x from an "exponential reservoir" of x's and you are left with the same exponential reservoir:

$$\mathbb{e}^x = \left[\ldots + \frac{xxxxx}{120} + \ldots\right] \xrightarrow{\partial_x} \left[\ldots + \frac{x\cancel{x}xxx}{120} + \ldots\right] = (\mathbb{e}^x)' = \mathbb{e}^x,$$

and if you let each element choose left or right, you get twice the same reservoir:

$$\mathbb{e}^x \xrightarrow{x\to x_l+x_r} \mathbb{e}^{x_l+x_r} = \mathbb{e}^{x_l}\mathbb{e}^{x_r}.$$



A Graphical Proof. Glue top to bottom on the right, in all possible ways. Several scenarios occur:

1. Start at A, go through the q-machine $k \geq 0$ times, stop at B. Get $\left\langle P\left(\zeta, \sum_{k\geq 0} q^kz\right)\right\rangle = \langle P(\zeta, \tilde{q}z)\rangle$.
2. Loop through the q-machine and swallow your own tail. Get $\exp\left(\sum q^k/k\right) = \exp(-\log(1-q)) = \tilde{q}$. 3. …

By the reservoir splitting principle, these scenarios contribute multiplicatively. □

Implementation. ($\mathbb{E}[Q, P]$ means $\mathbb{e}^QP$) ωεβ/Zip

```
Zip_{ζs_List}@𝔼[Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   ζrule = Thread[ζs → ζs + ηs.qt];
   Simplify /@
    𝔼[c + ηs.qt.ys, Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]] ];
```

Real Zipping is a minor mess, and is done in two phases:

	τa-phase		ξy-phase	
ζ-like variables	τ	a	ξ	y
z-like variables	t	α	x	η

Already at $\epsilon = 0$ we get the best known formulas for the Alexander polynomial!

Generic Docility. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form

$$\mathbb{e}^{q^{ij}z_iz_j}P = \mathbb{e}^{q^{ij}z_iz_j}\left(\sum_{k\geq 0}\epsilon^kP_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 4k$.

Our Docility. In the case of QU_ϵ, all invariants and operations are of the form $\mathbb{e}^{L+Q}P$, where

- L is a quadratic of the form $\sum l_{z\zeta}z\zeta$, where z runs over $\{t_i, \alpha_i\}_{i\in S}$ and ζ over $\{\tau_i, a_i\}_{i\in S}$, with integer coefficients $l_{z\zeta}$.
- Q is a quadratic of the form $\sum q_{z\zeta}z\zeta$, where z runs over $\{x_i, \eta_i\}_{i\in S}$ and ζ over $\{\xi_i, y_i\}_{i\in S}$, with coefficients $q_{z\zeta}$ in the ring R_S of rational functions in $\{T_i, \mathcal{A}_i\}_{i\in S}$.
- P is a docile power series in $\{y_i, a_i, x_i, \eta_i, \xi_i\}_{i\in S}$ with coefficients in R_S, and where $\deg(y_i, a_i, x_i, \eta_i, \xi_i) = (1, 2, 1, 1, 1)$.

Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$. !!!!!

- At $\epsilon^2 = 0$ we get the Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] invariant, which is stronger than HOMFLY-PT polynomial and Khovanov homology taken together!
- In general, get "higher diagonals in the Melvin-Morton-Rozansky expansion of the coloured Jones polynomial" [MM, BNG], but why spoil something good?

References.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BV] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* arXiv:1708.04853.

[Fa] L. Faddeev, *Modular Double of a Quantum Group,* arXiv:math/9912078.

[GR] S. Garoufalidis and L. Rozansky, *The Loop Exapnsion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

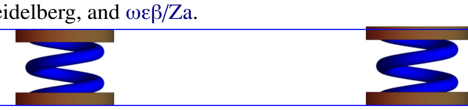
[Qu] C. Quesne, *Jackson's q-Exponential as the Exponential of a Series,* arXiv:math-ph/0305003.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Za] D. Zagier, *The Dilogarithm Function,* in Cartier, Moussa, Julia, and Vanhove (eds) *Frontiers in Number Theory, Physics, and Geometry II.* Springer, Berlin, Heidelberg, and ωεβ/Za.



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901