Dror Bar-Natan: Talks: Ohio-1901:

Computation without Representation

Thanks for inviting me to Ohio! ωεβ:=http://drorbn.net/o19/

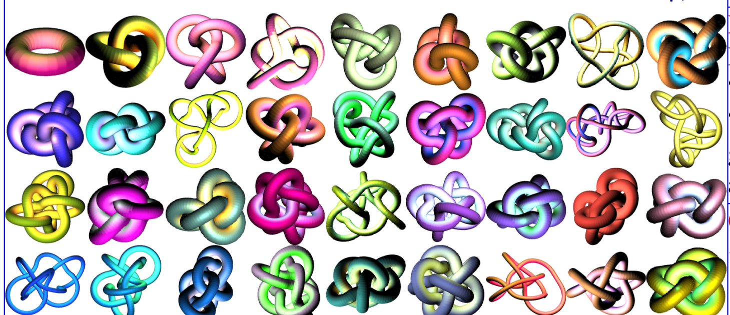
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at [BV] and at ωεβ/talks.

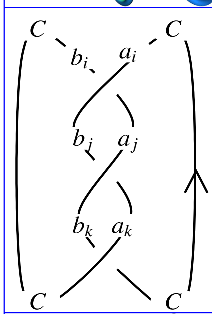
Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast.

In my talk, I will describe a direct method for carrying out such computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know. (experimental analysis @ωεβ/kiw)

Knotted Candies ωεβ/kc





The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad\text{and}\quad C \in U,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

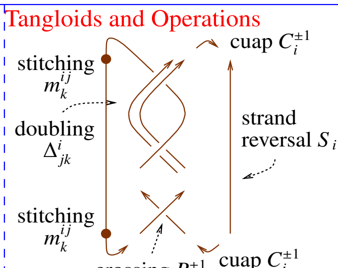
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

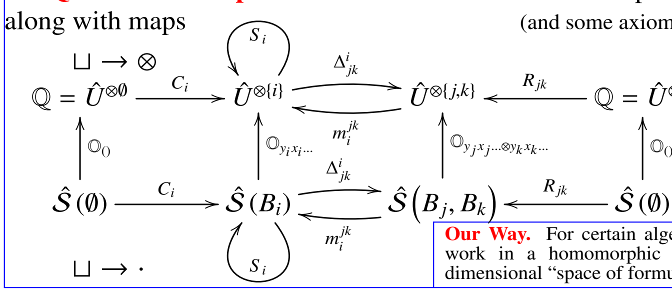
A Knot Theory Portfolio.

- Has operations $\sqcup$, m^{ij}_k, Δ^i_{jk}, S_i.
- All tangloids are generated by $R^{\pm1}$ and $C^{\pm1}$ (so "easy" to produce invariants).
- Makes some knot properties ("genus", "ribbon") become "definable".

Tangloids and Operations

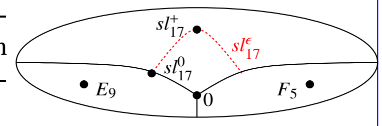


A "Quantum Group" Portfolio consists of a vector space U along with maps (and some axioms...)

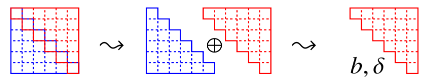


Our Way. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl^k_{17} := sl^\epsilon_{17}/(\epsilon^{k+1} = 0)$.



Solvable Approximation. In gl_n, half is enough! Indeed $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\ulcorner, b, \delta)$:



$b(\ulcorner) = b\colon \ulcorner \otimes \ulcorner \to \ulcorner$

$b(\llcorner) \rightsquigarrow \delta\colon \ulcorner \to \ulcorner \otimes \ulcorner$

Now define $gl^\epsilon_n := \mathcal{D}(\ulcorner, b, \epsilon\delta)$. Schematically, this is $[\ulcorner, \ulcorner] = \ulcorner$, $[\llcorner, \llcorner] = \epsilon\llcorner$, and $[\ulcorner, \llcorner] = \llcorner + \epsilon\ulcorner$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1} = 0$ always yields a solvable Lie algebra.

CU **and** ***QU***. Starting from sl_2, get $CU_\epsilon = \langle y, a, x, t\rangle/([t, -] = 0,\ [a, y] = -y,\ [a, x] = x,\ [x, y] = 2\epsilon a - t)$. Quantize using standard tools (I'm sorry) and get $QU_\epsilon = \langle y, a, x, t\rangle/([t, -] = 0,\ [a, y] = -y,\ [a, x] = x,\ xy - \mathbb{e}^{\hbar\epsilon} yx = (1 - T\mathbb{e}^{-2\hbar\epsilon a})/\hbar)$.

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set $B = \{y, x, \ldots\}$ of "generators" and isomorphisms $\mathbb{O}_{y,x,\ldots}\colon \hat{\mathcal{S}}(B) \to U$ defined by "ordering monomials" to some fixed $y, x, \ldots$ order. The quantum group portfolio now becomes a "symmetric algebra" portfolio, or a "power series" portfolio.

Operations are Objects.

★ $B^* := \{z^*_i = \zeta_i\colon z_i \in B\}$,

$\langle z^m_i, \zeta^n_i\rangle = \delta_{mn} n!$,

$$\left\langle \prod z^{m_i}_i, \prod \zeta^{n_i}_i \right\rangle = \prod \delta_{m_i n_i} n_i!,$$

in general, for $f \in \mathcal{S}(z_i)$ and $g \in \mathcal{S}(\zeta_i)$,

$$\langle f, g\rangle = f(\partial_{\zeta_i}) g\big|_{\zeta_i=0} = g(\partial_{z_i}) f\big|_{z_i=0}.$$

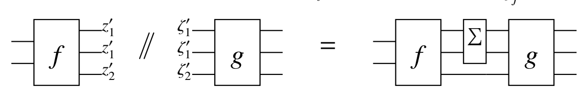
$f \in \mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B) \to \mathcal{S}(B'))$ $\,\|\,$ $\mathcal{S}(B)^* \otimes \mathcal{S}(B')$ $\,\|★\,$ $\mathcal{S}(B^*) \otimes \mathcal{S}(B')$ $\,\|\,$ $\mathcal{S}(B^* \sqcup B')$ $\,\|\,$ $\tilde{f} \in \mathbb{Q}[\zeta_i, z'_i]$

The Composition Law. If

$$\mathcal{S}(B) \xrightarrow[\tilde{f} \in \mathbb{Q}[\![\zeta_i, z'_j]\!]]{f} \mathcal{S}(B') \xrightarrow[\tilde{g} \in \mathbb{Q}[\![\zeta'_j, z''_k]\!]]{g} \mathcal{S}(B'')$$

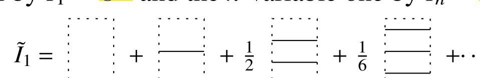
then

$$\widetilde{(f /\!\!/ g)} = \widetilde{(g \circ f)} = \left(\tilde{g}\big|_{\zeta'_j \to \partial_{z'_j}} \tilde{f}\right)_{z'_j=0} = \left(\tilde{f}\big|_{z'_j \to \partial_{\zeta'_j}} \tilde{g}\right)_{\zeta'_j=0}:$$

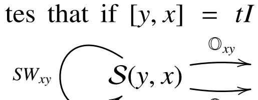


Examples

1. The 1-variable identity map $I\colon \mathcal{S}(z) \to \mathcal{S}(z)$ is given by $\tilde{I}_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by $\tilde{I}_n = \mathbb{e}^{z_1\zeta_1 + \cdots + z_n\zeta_n}$:



2. The "archetypal multiplication map $m^{ij}_k\colon \mathcal{S}(z_i, z_j) \to \mathcal{S}(z_k)$" has $\tilde{m} = \mathbb{e}^{z_k(\zeta_i + \zeta_j)}$.
3. The "archetypal coproduct $\Delta^i_{jk}\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j, z_k)$", given by $z_i \to z_j + z_k$ or $\Delta z = z \otimes 1 + 1 \otimes z$, has $\tilde{\Delta} = \mathbb{e}^{(z_j + z_k)\zeta_i}$.
4. R-matrices tend to have terms of the form $\mathbb{e}^{\hbar y_1 x_2}_q \in \mathcal{U}_q \otimes \mathcal{U}_q$. The "baby R-matrix" is $\tilde{R} = \mathbb{e}^{\hbar yx} \in \mathcal{S}(y, x)$.
5. The "Weyl form of the canonical commutation relations" states that if $[y, x] = tI$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. So with



we have $\widetilde{SW}_{xy} = \mathbb{e}^{\eta y + \xi x - \eta\xi t}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901