

$$\begin{aligned}
dm_{i,j \rightarrow k} &\rightarrow a_k (\alpha_i + \alpha_j) + b_k (\beta_i + \beta_j) + y_k \eta_i + \frac{y_k \eta_j}{\alpha_i} + \frac{x_k \xi_i}{\alpha_j} + \eta_j \xi_i - \\
& B_k \eta_j \xi_i + \frac{1}{4 \alpha_i \alpha_j} \in \left(2 y_k \eta_j (2 x_k \xi_i + \alpha_j (-2 \beta_i + (1 - 3 B_k) \eta_j \xi_i)) + \right. \\
& \quad \alpha_i \xi_i (x_k (-4 \beta_j + 2 (1 - 3 B_k) \eta_j \xi_i) + \\
& \quad \left. \alpha_j \eta_j (4 a_k B_k + (1 - 4 B_k + 3 B_k^2) \eta_j \xi_i) \right) + x_k \xi_j \\
d\Delta_{i \rightarrow j, k} &\rightarrow a_j \alpha_i + a_k \alpha_i + b_j \beta_i + b_k \beta_i + y_j \eta_i + B_j y_k \eta_i + \\
& x_j \xi_i + x_k \xi_i + \frac{1}{2} \in (B_j y_j y_k \eta_i^2 + x_k \xi_i (-2 a_j + x_j \xi_i)) \\
dS_i &\rightarrow -a_i \alpha_i - b_i \beta_i - \frac{\alpha_i (y_i \eta_i + (-\eta_i + B_i (x_i + \eta_i)) \xi_i)}{B_i} - \\
& \frac{1}{4 B_i^2} \in \alpha_i (\alpha_i \eta_i^2 (2 y_i^2 - 6 y_i \xi_i + 3 \xi_i^2) + B_i^2 \xi_i (4 a_i x_i + 2 x_i^2 \alpha_i \xi_i + \\
& \quad 2 x_i (2 \beta_i + \alpha_i \eta_i \xi_i) + \eta_i (-4 + 4 \beta_i + \alpha_i \eta_i \xi_i)) + \\
& \quad 2 B_i \eta_i (y_i (-2 + 2 \beta_i + 2 x_i \alpha_i \xi_i + \alpha_i \eta_i \xi_i) - \\
& \quad \xi_i (-2 + 2 a_i + 2 \beta_i + 3 x_i \alpha_i \xi_i + 2 \alpha_i \eta_i \xi_i)) \\
R_{i,j} &\rightarrow a_j b_i + x_j y_i - \frac{1}{4} \in x_j^2 y_i^2 \\
P_{i,j} &\rightarrow \alpha_j \beta_i + \eta_i \xi_j + \frac{1}{4} \in \eta_i^2 \xi_j^2
\end{aligned}$$

$$\begin{aligned}
E_{() \rightarrow (1)} &\left[\mathbf{0}, \mathbf{0}, \frac{B}{1 - B + B^2} + \right. \\
& \left. \frac{B (-B + 2 B^2 + 2 B^4 + a (-1 + B - B^3 + B^4) - 2 x y - B^3 (3 + 2 x y))}{(1 - B + B^2)^3} \in + \right. \\
& \left. \frac{1}{2 (1 - B + B^2)^5} \right. \\
& \left. B (4 B^8 + a^2 (1 - B + B^2)^2 (1 + B - 6 B^2 + B^3 + B^4) + 6 B^5 x^2 y^2 + \right. \\
& \quad 2 x y (-2 + 3 x y) - B^7 (11 + 4 x y) - 2 B^2 (1 + 6 x^2 y^2) - \\
& \quad 2 B^4 (1 - 2 x y + 6 x^2 y^2) + B (1 + 8 x y + 6 x^2 y^2) + \\
& \quad B^6 (6 + 8 x y + 6 x^2 y^2) + B^3 (4 + 4 x y + 30 x^2 y^2) + \\
& \quad 2 a (1 - B + B^2) (2 B^6 + 2 x y + 8 B^3 (1 + x y) - 5 B^2 (1 + 2 x y) - \\
& \quad \left. \left. 2 B^5 (1 + 2 x y) - B^4 (7 + 2 x y) + B (2 + 4 x y) \right) \right) \in^2 + 0[\in]^3 \left. \right]
\end{aligned}$$

A Quantum Algebra Example.

Proto-Proposition^{†0} (with Jesse Frohlich and Roland van der Veen, near [Ma, Proposition 1.7.3]). Let H be a finite dimensional Hopf algebra and let $U = H^{*cop} \otimes H$ be its Drinfel'd double, with R -matrix $R \in H^* \otimes H \subset U \otimes U$. Write $R^{\dagger 1} = \sum \rho_a \otimes r_a$, and let $\langle \cdot | \cdot \rangle: H^* \otimes H \rightarrow \mathbb{F}$ be the duality pairing. Then the functional $\int \in U^*$ defined by

$$\int \phi \otimes x := \sum \langle \phi \rho_a^{\dagger 2} | x r_a^{\dagger 3} \rangle$$

is a right^{†4} integral in U^* . (Meaning $\Delta_{jk}^i // \int_j = \int_i // \epsilon_k$ in $\text{Hom}(U^{\otimes \{i\}} \rightarrow U^{\otimes \{k\}})$).

†0 A “proto-proposition” is something that will become a proposition once you figure out the correct statement. †1 Or did we want it to be $R // S_1^2$? Or $R // S_2^2$? †2 Or is it $\rho_a \phi$? †3 Or is it $r_a x$? †4 Or maybe “left”?

`inp = E_{() \rightarrow (1)} [3 a_1 b_1, 5 x_1 y_1, 1] // dm_{i,1 \rightarrow i};`

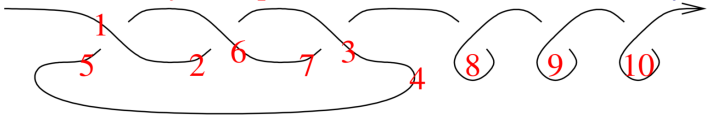
`Table[`

```

HL@TrueQ[
  (inp // (S_{Y_{i \rightarrow 1,1,2,2}} RR) // BM // AM // P_{1,2}) de_j =
  (inp // \Delta \Delta // (S_{Y_{i \rightarrow 1,1,2,2}} RR) // BM // AM // P_{1,2}) ],
{ \Delta \Delta, { d\Delta_{i \rightarrow j, i}, d\Delta_{i \rightarrow j, i} }, { AM, { dm_{2,4 \rightarrow 2}, dm_{4,2 \rightarrow 2} },
{ BM, { dm_{1,3 \rightarrow 1}, dm_{3,1 \rightarrow 1} },
{ RR, { R_{3,4}, R_{3,4} // dS_3 // dS_3, R_{3,4} // dS_4 // dS_4 }
] // MatrixForm
( ( False False False ) ( False False True )
  ( False False False ) ( False False False )
  ( False False False ) ( False False False )
  ( False False True ) ( False False False ) )

```

A Knot Theory Example.



`$k = 2;`

`Simplify[`

```

R_{1,5} R_{6,2} R_{3,7} \bar{C}_4 \bar{Kink}_8 \bar{Kink}_9 \bar{Kink}_{10} // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1} //
dm_{1,4 \rightarrow 1} // dm_{1,5 \rightarrow 1} // dm_{1,6 \rightarrow 1} // dm_{1,7 \rightarrow 1} // dm_{1,8 \rightarrow 1} //
dm_{1,9 \rightarrow 1} // dm_{1,10 \rightarrow 1} ] \cdot v_{-1} \rightarrow v

```

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