Everything around sl^ϵ_{2+} is DoPeGDO. So what?

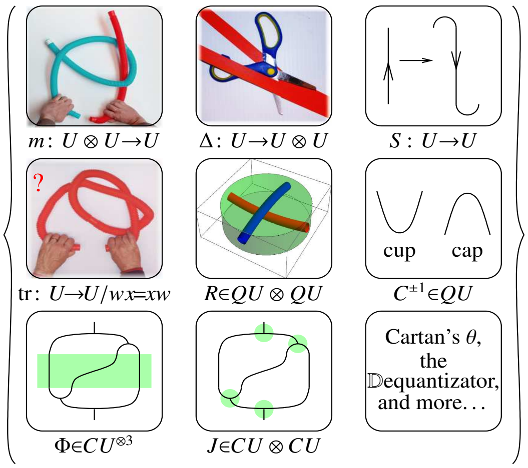
Thanks for inviting me to UCLA! ωεβ:=http://drorbn.net/la19/ More at ωεβ/talks

Continues Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen [BV].

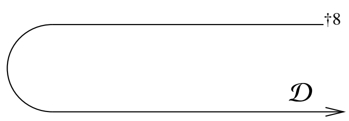
Abstract. I'll explain what "everything around" means: classical and quantum m, Δ, S, tr, R, C, and θ, as well as P, Φ, J, $\mathbb{D}$, and more, and all of their compositions. What **DoPeGDO** means: the category of Docile Perturbed Gaussian Differential Operators. And what sl^ϵ_{2+} means: a solvable approximation of the semi-simple Lie algebra sl_2.

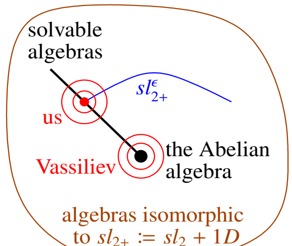
Knot theorists should rejoice because all this leads to very powerful and well-behaved poly-time-computable knot invariants. Quantum algebraists should rejoice because it's a realistic playground for testing complicated equations and theories.

Conventions. 1. For a set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$.†1 2. Everything converges!



Less Abstract





DoPeGDO := The category with objects finite sets†2 and mor($A\to B$):

$$\{\mathcal{F} = \omega\exp(Q+P)\} \subset \mathbb{Q}[\![\zeta_A, z_B, \epsilon]\!]$$

Where: • ω is a scalar.†3 • Q is a "small" ϵ-free quadratic in $\zeta_A \cup z_B$.†4 • P is a "docile perturbation": $P = \sum_{k\geq 1}\epsilon^k P^{(k)}$, where $\deg P^{(k)} \leq 2k+2$.†5

• Compositions:†6

$$\mathcal{F}/\!\!/\mathcal{G} = \mathcal{G}\circ\mathcal{F} := \left(\mathcal{G}|_{\zeta_i\to\partial_{z_i}}\mathcal{F}\right)_{z_i=0} = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

Cool! $(V^*)^{\otimes\Sigma}\otimes V^{\otimes S}$ explodes; the ranks of quadratics and bounded-degree polynomials grow slowly!†7 Representation theory is over-rated!

Cool! How often do you see a computational toolbox so successful?

Our Algebras. Let $sl^\epsilon_{2+} := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t:=\epsilon a-b$ is central and if $\exists\epsilon^{-1}$, $sl^\epsilon_{2+}/\langle t\rangle \cong sl_2$. ωεβ/oa

U is either $CU = \mathcal{U}(sl^\epsilon_{2+})[\![\hbar]\!]$ or $QU = \mathcal{U}_\hbar(sl^\epsilon_{2+}) = A\langle y,b,a,x\rangle[\![\hbar]\!]$ with $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $xy-qyx=(1-AB)/\hbar$, where $q=e^{\hbar\epsilon}$, $A=e^{-\hbar\epsilon a}$, and $B=e^{-\hbar b}$. Set also $T=A^{-1}B=e^{\hbar t}$.

The Quantum Leap. Also decree that in QU,

$$\Delta(y,b,a,x) = (y_1+B_1y_2, b_1+b_2, a_1+a_2, x_1+A_1x_2),$$
$$S(y,b,a,x) = (-B^{-1}y, -b, -a, -A^{-1}x),$$

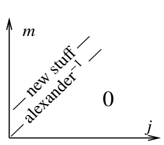
and $R = \sum \hbar^{j+k}y^kb^j\otimes a^jx^k/j![k]_q!$.

Mid-Talk Debts. • What is this good for in quantum algebra?
• In knot theory?
• How does the "inclusion" $\mathcal{D}$: Hom($U^{\otimes\Sigma}\to U^{\otimes S}$) $\rightsquigarrow$ **DoPeGDO** work?
• Proofs that everything around sl^ϵ_{2+} really is **DoPeGDO**.
• Relations with prior art.
• The rest of the "compositions" story.

Theorem ([BG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

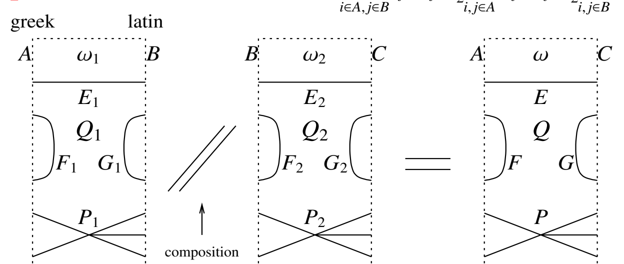
"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

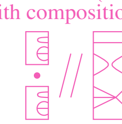
Compositions (1). In mor($A\to B$), $Q=\sum_{i\in A, j\in B}E_{ij}\zeta_iz_j + \frac12\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j + \frac12\sum_{i,j\in B}G_{ij}z_iz_j$



Where • $E = E_1(I-F_2G_1)^{-1}E_2$.
• $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$.
• $G = G_2 + E_2^TG_1(I-F_2G_1)^{-1}E_2$.
• $\omega = \omega_1\omega_2\det(I-F_2G_1)^{-1}$.
• P is computed using "connected Feynman diagrams" or as the solution of a messy PDE (yet we're still in algebra!).

Alexander lives here

One abstraction level up from tangles! {tangles} → {…} with compositions:



DoPeGDO Footnotes. †1. Each variable has a "weight" $\in\{0,1,2\}$, and always $\mathrm{wt}\,z_i + \mathrm{wt}\,\zeta_i = 2$.

†2. Really, "weight-graded finite sets" $A = A_0\sqcup A_1\sqcup A_2$.

†3. Really, a power series in the weight-0 variables†9.

†4. The weight of Q must be 2, so it decomposes as $Q = Q_{20}+Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series†9.

†5. Setting $\mathrm{wt}\,\epsilon=-2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained†9).

†6. There's also an obvious product
$$\mathrm{mor}(A_1\to B_1)\times\mathrm{mor}(A_2\to B_2)\to\mathrm{mor}(A_1\sqcup A_2\to B_1\sqcup B_2).$$

†7. That is, if the weight-0 variables are ignored. Otherwise more care is needed yet the conclusion remains.

†8. Hom($U^{\otimes\Sigma}\to U^{\otimes S}$) $\rightsquigarrow$ mor($\{\eta_i,\beta_i,\tau_i,\alpha_i,\xi_i\}_{i\in\Sigma}\to\{y_i,b_i,t_i,a_i,x_i\}_{i\in S}$), where $\mathrm{wt}(\eta_i,\xi_i,y_i,x_i)=1$ and $\mathrm{wt}(\beta_i,\tau_i,\alpha_i;b_i,t_i,a_i)=(2,2,0;0,0,2)$.

†9. For tangle invariants the wt-0 power series are always rational functions in the exponentials of the wt-0 variables (for knots: just one variable), with degrees bounded linearly by the crossing number.

Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907, http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.