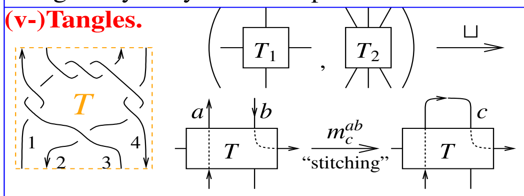
Algebraic Knot Theory

Abstract. This will be a very "light" talk: I will explain why about 13 years ago, in order to have a say on some problems in knot theory, I've set out to find tangle invariants with some nice compositional properties. In other talks in Sydney (ωεβ/talks) I have explained / will explain how such invariants were found - though they are yet to be explored and utilized.

(v-)Tangles.



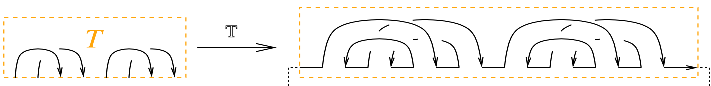
(meta-associativity: $m^{ab}_x /\!/ m^{xc}_y = m^{bc}_x /\!/ m^{ax}_y$) (tangles are generated by ⤫ and ⤬)

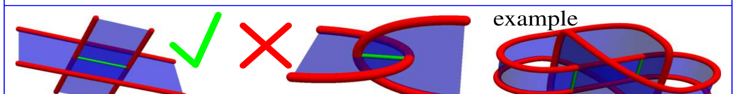
Strand doubling: $a \longrightarrow \xrightarrow{\Delta^a_{bc}} \; b, c$ Strand reversal: $a \xrightarrow{S_a} a$

Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.


Robert Engman's "The Loop"

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via





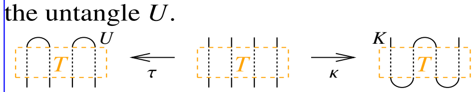
a ribbon singularity   a clasp singularity

A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

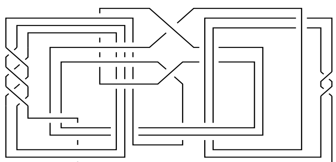
Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

Theorem. K is ribbon iff it is κT for a tangle T for which τT is the untangle U.

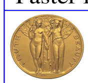



Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccccc} & U \in \mathcal{T}_n & & 1 \in \mathcal{A}_n & \\ & {}^{\tau}\nearrow & & {}^{\tau}\nearrow & \\ \mathcal{T}_{2n} & \xrightarrow{z} & \mathcal{A}_{2n} & & \text{with } \mathcal{R} := \kappa(\tau^{-1}(1)) \\ & {}_{\kappa}\searrow & & {}_{\kappa}\searrow & \\ & \text{ribbon } K \in \mathcal{T}_1 & & z(K) \in \mathcal{R} \subseteq \mathcal{A}_1 & \end{array}$$



Faster is better, leaner is meaner!



The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\}$ with $R_S := \mathbb{Z}(\{T_a : a \in S\})$:

$$\left(_a{\nwarrow\!\!\!\!\nearrow}_b, {}_b{\nwarrow\!\!\!\!\nearrow}_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a, T_b \to T_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array} \right)$$

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

Strand Doubling and Reversal.

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu := T_a - 1 \\ \nu := \alpha - \sigma_a \\ T_a \mapsto T_b T_c}]{\sigma\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b - 1)T_c\nu/\mu & (T_b - 1)T_c\theta/\mu \\ c & (T_c - 1)\nu/\mu & (\alpha - \sigma_a T_a - \nu T_c)/\mu & (T_c - 1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array} \right)$$

$$dS^a \Big\downarrow T_a \to T_a^{-1}$$

$$\left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array} \right)$$

Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left(_a{\nwarrow\!\!\!\!\nearrow}_b, {}_b{\nwarrow\!\!\!\!\nearrow}_a\right) = (a \to 1, b \to t_a^{\pm 1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma /\!/ m^{ab}_c = (\sigma \setminus \{a,b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a,t_b \to t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Implementation key idea: ωεβ/AlexDemo

$$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \textstyle\sum \alpha_{ab} t_a h_b)$$

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Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  ( α β θ        ( ∂_{ta,ha} λ  ∂_{ta,hb} λ  ∂_{ta} λ
    γ δ ϵ    =     ∂_{tb,ha} λ  ∂_{tb,hb} λ  ∂_{tb} λ     /. (t | h)_{a|b} → 0;
    ϕ ψ Ξ )        ∂_{ha} λ     ∂_{hb} λ     λ        )
  Γ[(μ = 1 - β) ω, {t_c, 1}.( γ + α δ / μ   ϵ + δ θ / μ
                              ϕ + α ψ / μ   Ξ + ψ θ / μ ).{h_c, 1}]
    /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.( 1  1 - T_a
                                0  T_a ).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1 / T_a;

ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_{a_} :→ a, ∞];
   M = Outer[Factor[∂_{h#1 t#2} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
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Meta-Associativity **Runs.**

```
ζ = Γ[ω, {t1, t2, t3, ts}.( α11 α12 α13 θ1
                             α21 α22 α23 θ2
                             α31 α32 α33 θ3
                             ϕ1  ϕ2  ϕ3  Ξ ).{h1, h2, h3, hs}];
(ζ // m_{12→1} // m_{13→1}) == (ζ // m_{23→2} // m_{12→1})
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True
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R3 … divide and conquer!

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{Rm51 Rm62 Rp34 // m_{14→1} // m_{25→2} // m_{36→3},
  Rp61 Rm24 Rm35 // m_{14→1} // m_{25→2} // m_{36→3}}
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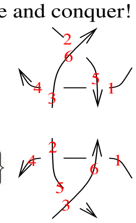
$$\left\{ \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix} \right\}$$

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z = Rm_{12,1} Rm27 Rm83 Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15};
Do[z = z // m_{1k→1}, {k, 2, 16}];
z
```
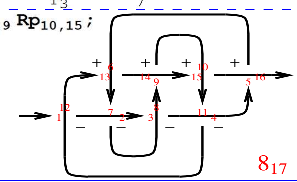

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$



Fact. Γ is better viewed as an invariant of a certain class of 2D knotted objects in $\mathbb{R}^4$ [BND, BN].


M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

Fact. Γ is the "0-loop" part of an invariant that generalizes to "n-loops" (1D tangles only, see further talks and future publications with van der Veen).

Speculation. Stepping stones to categorification?

Ask me about geography vs. identity!

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: w-Knots and the Alexander Polynomial,* Alg. and Geom. Top. **16-2** (2016) 1063–1133, arXiv:1405.1956, ωεβ/WKO1.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[Vo] H. Vo, *Alexander Invariants of Tangles via Expansions,* University of Toronto Ph.D. thesis, ωεβ/Vo.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)   www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Macquarie-191016/