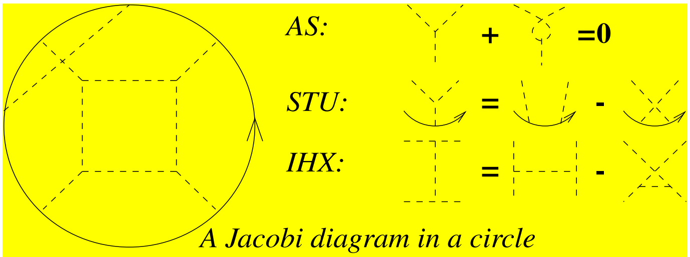
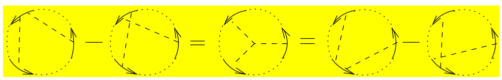


A Jacobi diagram in a circle

Theorem 4. [BN1] *The space $\mathcal{A}_m$ is isomorphic to the space $\mathcal{A}^t_m$ generated by "Jacobi diagrams in a circle" (chord diagrams that are also allowed to have oriented internal trivalent vertices) that have exactly $2m$ vertices, modulo the AS, STU and IHX relations. See the figure above.*

The key to the proof of Theorem 4 is

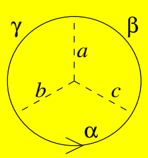


the figure above, which shows that the $4T$ relation is a consequence of two STU relations. The rest is more or less an exercise in induction.

Thinking of internal trivalent vertices as graphical analogs of the Lie bracket, the AS relation becomes the anti-commutativity of the bracket, STU becomes the equation $[x, y] = xy - yx$ and IHX becomes the Jacobi identity. This analogy is made concrete within the following construction, originally due to Penrose [Pe] and to Cvitanović [Cv]. Given a finite dimensional metrized Lie algebra $\mathfrak{g}$ (e.g., any semi-simple Lie algebra) and a finite-dimensional representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of $\mathfrak{g}$, choose an orthonormal basis[4] $\{X_a\}_{a=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$ and some basis $\{v_\alpha\}_{\alpha=1}^{\dim V}$ of V, let f_{abc} and $r^\gamma_{\alpha\beta}$ be the "structure constants" defined by

$$f_{abc} := \langle [X_a, X_b], X_c \rangle \qquad \text{and} \qquad \rho(X_a)(v_\beta) = \sum_\gamma r^\gamma_{a\beta} v_\gamma.$$

Now given a Jacobi diagram D label its circle-arcs with Greek letters $\alpha, \beta, \ldots$, and its chords with Latin letters a, $b, \ldots$, and map it to a sum as suggested by the following example:



$$\longrightarrow \sum_{a,b,c,\alpha,\beta,\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

$\left(\begin{array}{l}\text{internal vertices go to } f\text{'s,}\\ \text{circle-vertices to } r\text{'s}\end{array}\right)$

Theorem 5. *This construction is well defined, and the basic properties of Lie algebras imply that it respects the AS, STU, and IHX relations. Therefore it defines a linear functional $W_{\mathfrak{g},\rho} : \mathcal{A}_m \to \mathbb{Q}$, for any m.*

The last assertion along with Theorem 3 show that associated with any $\mathfrak{g}$, ρ and m there is a weight system and hence a knot invariant. Thus knots are indeed linked with Lie algebras.

The above is of course merely a sketch of the beginning of a long story. You can read the details, and some of the rest, in [Book].

What I like about [Book]. Detailed, well thought out, and carefully written. Lots of pictures! Many excellent exercises! A complete discussion of "the algebra of chord diagrams". A nice discussion of the pairing of diagrams with Lie algebras, including examples aplenty. The discussion of the Kontsevich integral (meaning, the proof of the hard side of Theorem 3) is terrific — detailed and complete and full of pictures and examples, adding a great deal to the original sources. The subject of "associators" is huge and worthy of its own book(s); yet in as much as they are related to Vassiliev invariants, the discussion in [Book] is excellent. A great many further topics are touched — multiple ζ-values, the relationship of the Hopf link with the Duflo isomorphism, intersection graphs and other combinatorial aspects of chord diagrams, Rozansky's rationality conjecture, the Melvin-Morton conjecture, braids, n-equivalence, etc.

For all these, I'd certainly recommend [Book] to any newcomer to the subject of knot theory, starting with my own students.

However, some proofs other than that of Theorem 3 are repeated as they appear in original articles with only a superficial touch-up, or are omitted altogether, thus missing an opportunity to clarify some mysterious points. This includes Vogel's construction of a non-Lie-algebra weight system and the Goussarov-Polyak-Viro proof of the existence of "Gauss diagram formulas".

What I wish there was in the book, but there isn't. The relationship with Chern-Simons theory, Feynman diagrams, and configuration space integrals, culminating in an alternative (and more "3D") proof of the Fundamental Theorem. This is a major omission.

Why I hope there will be a continuation book, one day. There's much more to the story! There are finite type invariants of 3-manifolds, and of certain classes of 2-dimensional knots in $\mathbb{R}^4$, and of "virtual knots", and they each have their lovely yet non-obvious theories, and these theories link with each other and with other branches of Lie theory, algebra, topology, and quantum field theory. Volume 2 is sorely needed.

References

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[4] This requirement can easily be relaxed.