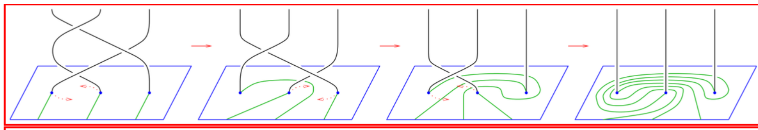
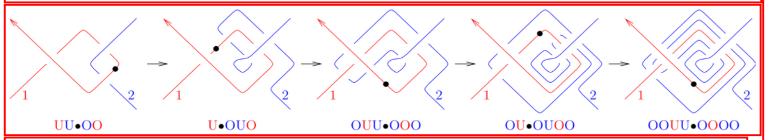
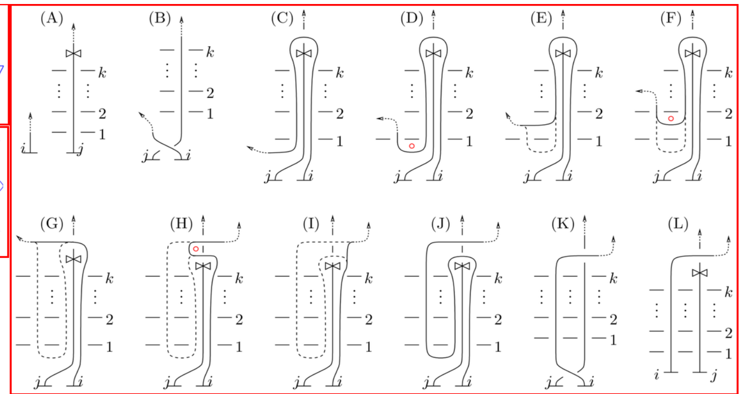
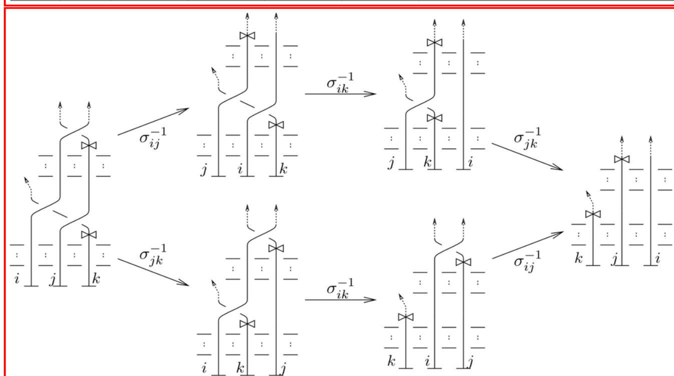


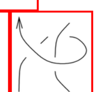




n-strand (pure) virtual braids with ($\leqslant m$)-xing:

$m\backslash n$	2	3	4	5	6	General n
0	1	1	1	1	1	1
1	5	13	25	41	61	$2n^2-2n+1$
2	17	145	529	1361	2881	$2n^4+4n^3-18n^2+12n+1$
3	53	1561	10873	43121	127021	$\frac{1}{3}\left(4n^6+36n^5-2n^4-546n^3+1066n^2-558n+3\right)$
4	161	16717	222289	1351481	5484721	
5	485	178873	4540201			
6	1457	1913737				





OVER THEN UNDER TANGLES

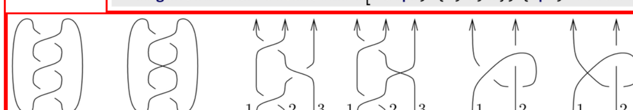
DROR BAR-NATAN, ZSUZSANNA DANCSO, AND ROLAND VAN DER VEEN

ABSTRACT. Brilliant wrong ideas should not be buried and forgotten. Instead, they should be mined for the gold that lies underneath the layer of wrong. In this paper we explain how "over then under tangles" lead to an easy classification of knots, and under the surface, also to some valid mathematics: a separation theorem for braids and virtual braids, a topological understanding of the Drinfel'd double construction of quantum group theory, and more.

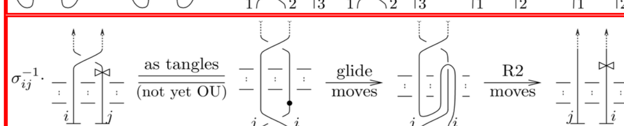
```
SetAttributes[VD, Orderless]

Tidy[vd_VD] := Module[{ps = Union @@ (List @@@ vd)},
  Replace[vd, Thread[ps → Range@Length@ps], {2}] ]

R12Reduce1[vd_VD] := Tidy@Module[{R2s, R2}, Which[
    Length[R2s = Cases[vd, X_s_[i_, j_] :→ X_-s[i + 1, j + 1]] ⋂ (List @@ vd)] > 0,
    Complement[vd, VD[R2 = First@R2s, R2 /. X_s_[i_, j_] :→ X_-s[i - 1, j - 1]]],
    Length[R2s = Cases[vd, X_s_[i_, j_] :→ X_-s[i + 1, j - 1]] ⋂ (List @@ vd)] > 0,
    Complement[vd, VD[R2 = First@R2s, R2 /. X_s_[i_, j_] :→ X_-s[i - 1, j + 1]]],
    True, DeleteCases[vd, X_[i_, j_] /; Abs[i - j] == 1] ]]

R12Reduce[vd_VD] := FixedPoint[R12Reduce1, vd]

γ[vd_VD] := Module[{js, s1, i1, j1, s2, i2, j2},
  js = Cases[vd, X_[_, j_] :→ j] ⋂ Cases[vd, X_[i_, _] :→ i - 1];
  If[Length[js] == 0, vd,
   j1 = RandomChoice[js]; i2 = j1 + 1;
   Cases[vd, X_s_[i_, j1] :→ (s1 = s; i1 = i)];
   Cases[vd, X_s_[i2, j_] :→ (s2 = s; j2 = j)];
   Tidy@Join[Complement[vd, VD[X_s1[i1, j1], X_s2[i2, j2]]],
     VD[X_s2[j1, j2], X_s1[i1, i2], X_s1 s2[i1 - s1/3, j2 + s2/3],
      X_-s1 s2[i1 + s1/3, j2 - s2/3]]
    ] ]]

Γ[vd_VD] := FixedPoint[γ, vd, 2^8]

Γ[T_] /; Head[T] =!= VD := Γ[VD[T]]
```

- Enriquez' universal quantization of Lie bi-algebra.
- All else about quantization of Lie bi-algebra.
- PBW / normal ordering.
- Audoux-Meilhan "Characterization of the Reduced Peripheral System of Links".
- B-N's "Balloons and Hoops" paper.

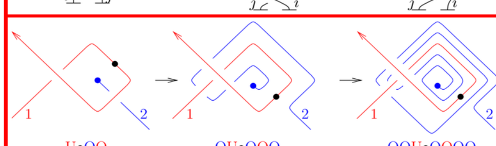
```
Γ̄[vd_VD] := FixedPoint[R12Reduce@*γ, vd, 2^8]

Γ̄[T_] /; Head[T] =!= VD := Γ̄[VD[T]]

VPB[n_, {σs___}] := VPB[n, σs];

VD /: vd1_VD ** vd2_VD := Module[{es1, es2, m2},
  es1 = Cases[vd1, EOS[i_] :→ i];
  m2 = Max[es2 = Cases[vd2, EOS[i_] :→ i]];
  Tidy[
   vd1 ⋃ Replace[DeleteCases[vd2, _EOS],
     i_ :→ i / m2 - 1 + es1[[1 + Count[es2, e_ /; i > e]]], {2}]]
 ]

VD[VPB[n_]] := VD @@ (EOS /@ Range[n]);
VD[VPB[n_, σ_i_,j_]] := Tidy@Append[VD @@ (EOS /@ Range[n]), X_+1[i - 0.5, j - 0.5]];
VD[VPB[n_, σ̄_i_,j_]] := Tidy@Append[VD @@ (EOS /@ Range[n]), X_-1[i - 0.5, j - 0.5]];
VD[VPB[n_, σ_, σs__]] := VD[VPB[n, σ]] ** VD[VPB[n, σs]]

VPBGenerators[n_] :=
  VPBGenerators[n] =
   Flatten@Table[{σ_i,j, σ̄_i,j}, {i, n}, {j, DeleteCases[Range@n, i]}];

ProudFollowers[n_, σ_i_,j_] := ProudFollowers[n, σ_i,j] = Module[{p, q, s},
    Flatten@{σ_i,j, σ_j,i, σ̄_j,i,
      Table[{σ_p,q, σ_q,p, σ̄_p,q, σ̄_q,p}, {p, {i, j}}, {q, Complement[Range[n], {i, j}]}],
      Table[{σ_p,q, σ̄_p,q}, {p, Complement[Range[i + 1, n], {j}]},
       {q, Complement[Range[n], {i, j, p}]}]
     }];
ProudFollowers[n_, σ̄_i_,j_] :=
 ProudFollowers[n, σ̄_i,j] = ProudFollowers[n, σ_i,j] /. σ_i,j → σ̄_i,j

ProudVPBs[n_, 0] := {VPB[n]};
ProudVPBs[n_, 1] := VPB[n, #] & /@ VPBGenerators[n];
ProudVPBs[n_, m_] /; m > 1 :=
 Flatten[ProudVPBs[n, m - 1] /.
   VPB[n, σs___, σ_] :→ (VPB[n, σs, σ, #] & /@ ProudFollowers[n, σ])]

CountOUForms[n_, m_] := Module[{k},
  Length@Union@Flatten@Table[Γ̄@vpb, {k, 0, m}, {vpb, ProudVPBs[n, k]}]]
```

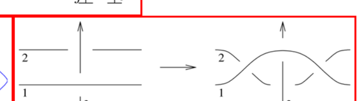


- Subsets.
- Supersets.
- Subsets of supersets.
- Completions.
- Images.
- Completions of sub-sets of supersets...

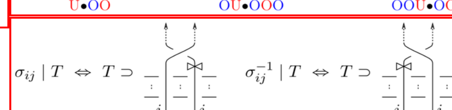


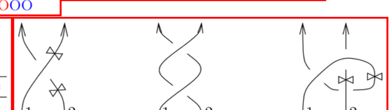
Credit to Manturov and Chterental!

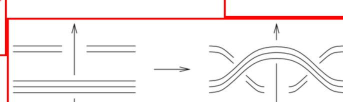


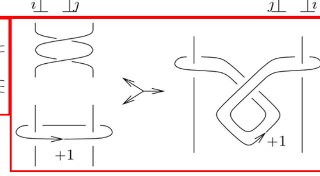


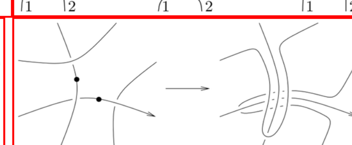
See http://drorbn.net/mo20.

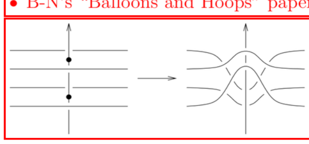


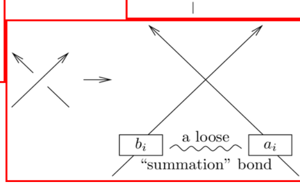


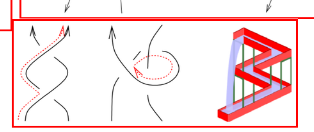












Audio and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2004//