The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $(p,x)^* = (\pi,\xi)$

The Generating Series $\mathcal{G}$: $\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \to \mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \xrightarrow[\mathcal{G}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}L(z_A^n) = L\left(e^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p\in\mathbb{Q}[z_A].$$

Claim. If $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $M\in\operatorname{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Examples. • $\mathcal{G}(id\colon \mathbb{Q}[p,x]\to\mathbb{Q}[p,x]) = e^{\pi p+\xi x}$.

• Consider $R_{ij}\in(\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!]\cong\operatorname{Hom}\left(\mathbb{Q}[]\to\mathbb{Q}[p_i,x_i,p_j,x_j]\right)[\![t]\!]$. Then $\mathcal{G}(R_{ij}) = e^{(e^t-1)(p_i-p_j)x_j} = e^{(T-1)(p_i-p_j)x_j}$.

Heisenberg Algebras. Let $\mathfrak{h} = A\langle p,x\rangle/([p,x]=1)$, let $\mathbb{O}_i\colon\mathbb{Q}[p_i,x_i]\to\mathfrak{h}_i$ is the "p before x" PBW normal ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[p_i,x_i,p_j,x_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathfrak{h}_i\otimes\mathfrak{h}_j \xrightarrow{m_k^{ij}} \mathfrak{h}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[p_k,x_k].$$

Then $\mathcal{G}(hm_k^{ij}) = e^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}$.

Proof. Recall the "Weyl CCR" $e^{\xi x}e^{\pi p} = e^{-\xi\pi}e^{\pi p}e^{\xi x}$, and find

$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= e^{\pi_ip_i+\xi_ix_i+\pi_jp_j+\xi_jx_j}/\!\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1}\\ &= e^{\pi_ip_i}e^{\xi_ix_i}e^{\pi_jp_j}e^{\xi_jx_j}/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} = e^{\pi_ip_k}e^{\xi_ix_k}e^{\pi_jp_k}e^{\xi_jx_k}/\!\!/\mathbb{O}_k^{-1}\\ &= e^{-\xi_i\pi_j}e^{(\pi_i+\pi_j)p_k}e^{(\xi_i+\xi_j)x_k}/\!\!/\mathbb{O}_k^{-1} = e^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}.\end{aligned}$$

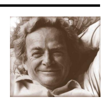
GDO := The category with objects finite sets and

$$\operatorname{mor}(A\to B) = \left\{\mathcal{L} = \omega e^Q\right\}\subset\mathbb{Q}[\![\zeta_A, z_B]\!],$$

where: • ω is a scalar. • Q is a "small" quadratic in $\zeta_A\cup z_B$.
• Compositions: $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

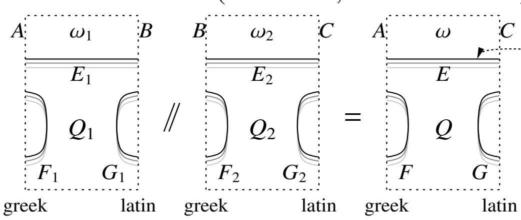
Compositions. In $\operatorname{mor}(A\to B)$,

$$Q = \sum_{i\in A, j\in B}E_{ij}\zeta_iz_j + \frac12\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j + \frac12\sum_{i,j\in B}G_{ij}z_iz_j,$$



R. Feynman

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



$$E_1E_2 + E_1F_2G_1E_2 + E_1F_2G_1F_2G_1E_2 + \dots = \sum_{r=0}^{\infty}E_1(F_2G_1)^rE_2$$

where • $E = E_1(I-F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$
• $G = G_2 + E_2^TG_1(I-F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2\det(I-F_2G_1)^{-1/2}$

Proof of Claim in Example 2. Let $\Phi_1 := e^{t(p_i-p_j)x_j}$ and $\Phi_2 := \mathbb{O}_{p_jx_j}\left(e^{(e^t-1)(p_i-p_j)x_j}\right) =: \mathbb{O}(\Psi)$. We show that $\Phi_1=\Phi_2$ in $(\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!]$ by showing that both solve the ODE $\partial_t\Phi = (p_i-p_j)x_j\Phi$ with $\Phi|_{t=0}=1$. For Φ_1 this is trivial. $\Phi_2|_{t=0}=1$ is trivial, and

$$\partial_t\Phi_2 = \mathbb{O}(\partial_t\Psi) = \mathbb{O}(e^t(p_i-p_j)x_j\Psi)$$

$$\begin{aligned}(p_i-p_j)x_j\Phi_2 &= (p_i-p_j)x_j\mathbb{O}(\Psi) = (p_i-p_j)\mathbb{O}(x_j\Psi - \partial_{p_j}\Psi)\\ &= \mathbb{O}\left((p_i-p_j)(x_j\Psi + (e^t-1)x_j\Psi)\right) = \mathbb{O}(e^t(p_i-p_j)x_j\Psi)\end{aligned}$$

□

Implementation. Without, don't trust!

```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
```

```
𝔼A1→B1[ω1_, Q1_] 𝔼A2→B2[ω2_, Q2_] ^:= 𝔼A1∪A2→B1∪B2[ω1 ω2, Q1 + Q2]
```

```
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂i,jQ1, {i, A1}, {j, B1}]; E2 = M[∂i,jQ2, {i, A2}, {j, B2}];
  F1 = M[∂i,jQ1, {i, A1}, {j, A1}]; F2 = M[∂i,jQ2, {i, A2}, {j, A2}];
  G1 = M[∂i,jQ1, {i, B1}, {j, B1}]; G2 = M[∂i,jQ2, {i, B2}, {j, B2}];
  𝔼A1→B2[CF[ω1 ω2 Det[I - F2.G1]^(1/2)], CF@Plus[
    If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
    If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).A1],
    If[B2 === {}, 0, 1/2 B2.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).B2]]]]
```

```
A_ \ B_ := Complement[A, B];
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* =!= A2) :=
  𝔼A1∪(A2\B1*)→B1∪A2*[ω1, Q1 + Sum[ξ* ξ, {ξ, A2 \ B1*}]] //
   𝔼B1*∪A2→B2∪(B1\A2*)[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)i;
l_List* := #* & /@ l;
```

```
Ri_,j_ := 𝔼{}→{pi,xi,pj,xj}[T^(-1/2), (1 - T) pj xj + (T - 1) pi xj];
R̄i_,j_ := 𝔼{}→{pi,xi,pj,xj}[T^(1/2), (1 - T^-1) pj xj + (T^-1 - 1) pi xj];
Ci_ := 𝔼{}→{pi,xi}[T^(-1/2), 0];
C̄i_ := 𝔼{}→{pi,xi}[T^(1/2), 0];
```

```
hmi_,j_→k_ := 𝔼{πi,ξi,πj,ξj}→{pk,xk}[1, -ξi πj + (πi + πj) pk + (ξi + ξj) xk]
```

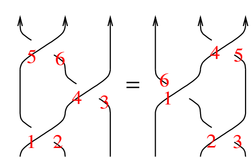
```
𝔼{}→vs_[ωi_, Q_]h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[ωi, 1 + Length@ps, 1 + Length@xs];
  M⟦2 ;;, 2 ;;⟧ = Table[CF[∂i,jQ], {i, ps}, {j, xs}];
  M⟦2 ;;, 1⟧ = ps; M⟦1, 2 ;;⟧ = xs;
  MatrixForm[M]h]
```

Proof of Reidemeister 3.

```
(R1,2 R4,3 R5,6 // hm1,4→1 hm2,5→2 hm3,6→3) ==
 (R2,3 R1,6 R4,5 // hm1,4→1 hm2,5→2 hm3,6→3)
```

```
True
```

□



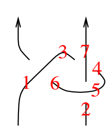
The "First Tangle".

```
Factor /@
 (z = R1,6 C̄3 R̄7,4 R̄5,2 // hm1,3→1 // hm1,4→1 // hm1,5→1 // hm1,6→1 // hm2,7→2)
```

$$\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{-1+2T}{T}, \frac{(-1+T)(p_1-p_2)(Tx_1-x_2)}{-1+2T}\right]$$

```
zh
```

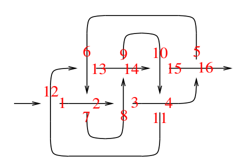
$\frac{-1+2T}{T}$	x_1	x_2
p_1	$\frac{-T+T^2}{-1+2T}$	$\frac{1-T}{-1+2T}$
p_2	$\frac{T-T^2}{-1+2T}$	$\frac{-1+T}{-1+2T}$



The knot 8_{17}.

```
z = R̄12,1 R̄27 R̄83 R̄4,11 R16,5 R6,13 R14,9 R10,15;
Table[z = z // hm1k→1, {k, 2, 16}] // Last
```

$$\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3}, 0\right]$$



Proof of Theorem 3, (3).

$$\left\{\left(\gamma1 = \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2,p_3,x_3\}}\left[\omega, \{p_1,p_2,p_3\}.\begin{pmatrix}\alpha&\beta&\theta\\ \gamma&\delta&\epsilon\\ \phi&\psi&\Xi\end{pmatrix}.\{x_1,x_2,x_3\}\right]\right)_h, \left(\gamma1 \,/\!/\, hm_{1,2\to0}\right)_h\right\}$$

$$\left\{\begin{pmatrix}\omega & x_1 & x_2 & x_3\\ p_1 & \alpha & \beta & \theta\\ p_2 & \gamma & \delta & \epsilon\\ p_3 & \phi & \psi & \Xi\end{pmatrix}_h, \begin{pmatrix}\omega+\gamma\,\omega & x_0 & x_3\\ p_0 & \frac{\alpha+\beta+\gamma+\beta\,\gamma+\delta-\alpha\,\delta}{1+\gamma} & \frac{\epsilon-\alpha\,\epsilon+\theta+\gamma\,\theta}{1+\gamma}\\ p_3 & \frac{\phi-\delta\,\phi+\psi+\gamma\,\psi}{1+\gamma} & \frac{\Xi+\gamma\,\Xi-\epsilon\,\phi}{1+\gamma}\end{pmatrix}_h\right\}$$

□

References. On ωεβ=http://drorbn.net/cat20

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LearningSeminarOnCategorification-2006/