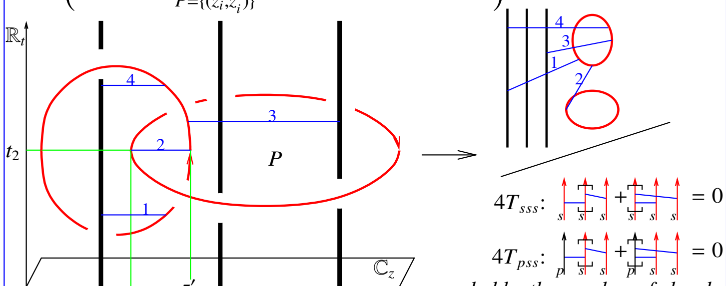
**Kontsevich in a Pole Dance Studio.** (w/o poles? See [Ko, BN])

$$Z=\left(\sum_{m=0}^{\infty}\frac{1}{(2\pi i)^m}\int_{\substack{t_1<\dots<t_m\\ P=\{(z_i,z_i')\}}}(-1)^{\#P_\downarrow}D_P\bigwedge_{i=1}^{m}\frac{dz_i-dz_i'}{z_i-z_i'}\right)^{\sim}\in\mathcal{A}$$



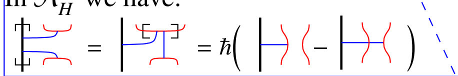

*graded by the number of chords*
*filtered by the number of ss chords*

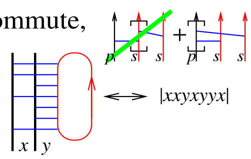
**Comments on the Kontsevich Integral.**

1. In the tangle case, the endpoints are fixed at top and bottom.
2. The $(\cdots)^{\sim}$ means "a correction is needed near the caps and the cups" (for the framed version, see [LM2, Da]).
3. There are never $pp$ chords, and no $4T_{pps}$ and $4T_{ppp}$ relations.
4. $Z$ is an "expansion".
5. $Z$ respects the $ss$ filtration and so descends to $Z^{/s}\colon\mathcal{K}^{/s}\to\mathcal{A}^{/s}$.

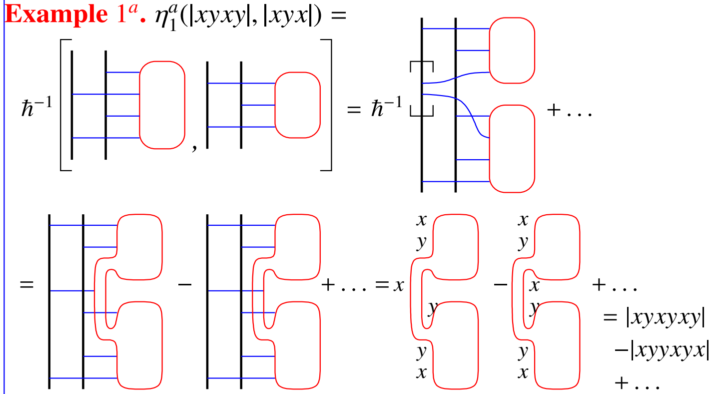
**Comments on $\mathcal{A}$.** In $\mathcal{A}^{/1}$ legs on poles commute, so $\mathcal{A}^{/1}(\bigcirc)=|A|$!

In $\mathcal{A}^{/2}_H$ we have:

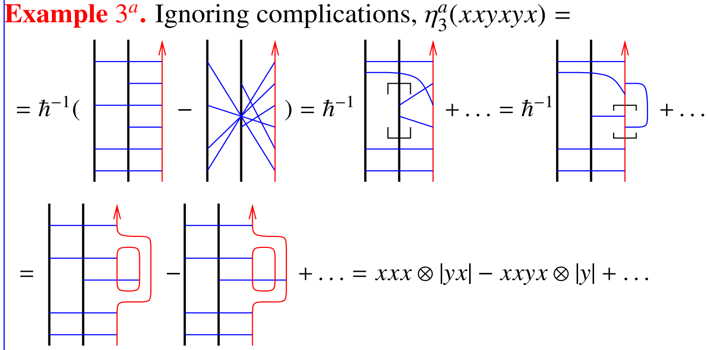



**Example 1$^a$.** $\eta_1^a(|xyxy|,|xyx|)=$




**Example 3$^a$.** Ignoring complications, $\eta_3^a(xxyxyx)=$



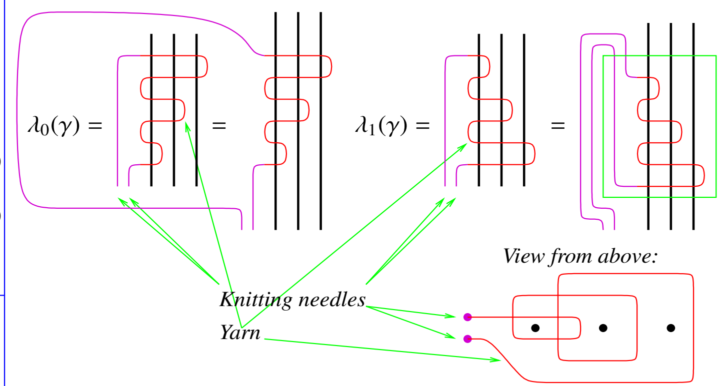

**Proof of Lemma 1.** We partially prove Theorem 2 instead:

**Theorem 2.** $\mathrm{gr}^\bullet\mathcal{K}_H\cong\mathbb{F}[\![\hbar]\!]\otimes(\mathcal{K}^{/1})_0$.

**Proof mod $\hbar^2$.** The map $\leftarrow$ is obvious. To go $\rightarrow$, map $\mathcal{K}_H\to\mathbb{F}[\![\hbar]\!]\otimes\mathcal{K}^{/1}$ using $\nearrow\!\!\!\!\nwarrow\mapsto\times+\frac{\hbar}{2})($ and $\nwarrow\!\!\!\!\nearrow\mapsto\times-\frac{\hbar}{2})($ and apply the functor $\mathrm{gr}^\bullet$.

**Unignoring the Complications.** We need $\lambda_0$ and $\lambda_1$ such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. Up to conjugation, $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma))=W(\gamma)$.




**Homework**

1. Is there more than Examples 1–4?
2. Derive the bialgebra axioms from this perspective.
3. What more do we get if we don't mod out by HOMFLY-PT?
4. What more do we get if we allow more than one strand-strand interaction?
5. In this language, recover Kashiwara-Vergne [AKKN1, AKKN2].
6. How is all this related to w-knots?
7. Do the same with associators. Use that to derive formulas for solutions of Kashiwara-Vergne.
8. What's the relationship with the Habiro-Massuyeau invariants of links in handlebodies [HM] (different filtration!).
9. Pole dance on other surfaces!
10. Explore the action of the mapping class group.

Kashiwara Vergne

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