A Small-Print Page on ρ_d, $d > 1$.

Definition. $\langle f(z_i), h(\zeta_i)\rangle_{\{z_i\}} := f(\partial_{\zeta_i})h\big|_{\zeta=0}$, so $\langle p^2 x^2, e^{g\pi\xi}\rangle = 2g^2$.

Baby Theorem. There exist (non unique) power series $r^\pm(p_1, p_2, x_1, x_2) = \sum_d \epsilon^d r_d^\pm(p_1, p_2, x_1, x_2) \in \mathbb{Q}[T^{\pm 1}, p_1, p_2, x_1, x_2][[\epsilon]]$ with $\deg r_d^\pm \leq 2d+2$ ("docile") such that the power series $Z^b = \sum \rho_d^b \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i, p_j, x_i, x_j)\right), \exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}\pi_\alpha\xi_\beta\right)\right\rangle_{\{p_\alpha, x_\beta\}}$$

is a bnot invariant. Beyond the once-and-for-all computation of $g_{\alpha\beta}$ (a matrix inversion), Z^b is computable in $O(n^d)$ operations in the ring $\mathbb{Q}[T^{\pm 1}]$.

(Bnots are knot diagrams modulo the braid-like Reidemeister moves, but not the cyclic ones).

Theorem. There also exist docile power series $\gamma^\varphi(\bar{p}, \bar{x}) = \sum_d \epsilon^d \gamma_d^\varphi \in \mathbb{Q}[T^{\pm 1}, \bar{p}, \bar{x}][[\epsilon]]$ such that the power series $Z = \sum \rho_d \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i, p_j, x_i, x_j) + \sum_k \gamma^{\varphi_k}(\bar{p}_k, \bar{x}_k)\right),\right.$$

$$\left. \exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}(\pi_\alpha + \bar{\pi}_\alpha)(\xi_\beta + \bar{\xi}_\beta) + \sum_\alpha \pi_\alpha\bar{\xi}_\alpha\right)\right\rangle_{\{p_\alpha, \bar{p}_\alpha, x_\beta, \bar{x}_\beta\}}$$

is a knot invariant, as easily computable as Z^b.

Implementation. Data, then program (with output using the Conway variable $z = \sqrt{T} - 1/\sqrt{T}$), and then a demo. See Rho.nb of $\omega\epsilon\beta$/ap.

```
V@γ_{1,φ_}[k_] = φ (1/2 - p̄_k x̄_k); V@γ_{2,φ_}[k_] = -φ² p̄_k x̄_k / 2;
V@γ_{3,φ_}[k_] := -φ³ p̄_k x̄_k / 6
```

```
V@r_{1,s_}[i_, j_] :=
 s (-1 + 2 p_i x_i - 2 p_j x_i + (-1 + T^s) p_i p_j x_i² + (1 - T^s) p_j² x_i² - 2 p_i p_j x_i x_j + 2 p_j² x_i x_j) / 2
```

```
V@r_{2,1}[i_, j_] :=
 (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i² + 3 (-1 + 3 T) p_j² x_i² + 4 (-1 + T) p_i² p_j x_i³ -
  2 (-1 + T) (5 + T) p_i p_j² x_i³ + 2 (-1 + T) (3 + T) p_j³ x_i³ + 18 p_i p_j x_i x_j -
  18 p_j² x_i x_j - 6 p_i² p_j x_i² x_j + 6 (2 + T) p_i p_j² x_i² x_j - 6 (1 + T) p_j³ x_i² x_j -
  6 p_i p_j² x_i x_j² + 6 p_j³ x_i x_j²) / 12
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```
V@r_{2,-1}[i_, j_] :=
 (-6 T² p_i x_i + 6 T² p_j x_i + 3 (-3 + T) T p_i p_j x_i² - 3 (-3 + T) T p_j² x_i² -
  4 (-1 + T) T p_i² p_j x_i³ + 2 (-1 + T) (1 + 5 T) p_i p_j² x_i³ - 2 (-1 + T) (1 + 3 T) p_j³ x_i³ +
  18 T² p_i p_j x_i x_j - 18 T² p_j² x_i x_j - 6 T² p_i² p_j x_i² x_j + 6 T (1 + 2 T) p_i p_j² x_i² x_j -
  6 T (1 + T) p_j³ x_i² x_j - 6 T² p_i p_j² x_i x_j² + 6 T² p_j³ x_i x_j²) / (12 T²)
```

```
V@r_{3,1}[i_, j_] :=
 (4 p_i x_i - 4 p_j x_i + 2 (5 + 7 T) p_i p_j x_i² - 2 (5 + 7 T) p_j² x_i² - 4 (-5 + 6 T) p_i² p_j x_i³ +
  4 (-16 + 17 T + 2 T²) p_i p_j² x_i³ - 4 (-11 + 11 T + 2 T²) p_j³ x_i³ + 3 (-1 + T) p_i³ p_j x_i⁴ -
  3 (-1 + T) (4 + 3 T) p_i² p_j² x_i⁴ + (-1 + T) (13 + 22 T + T²) p_i p_j³ x_i⁴ -
  (-1 + T) (4 + 13 T + T²) p_j⁴ x_i⁴ - 28 p_i p_j x_i x_j + 28 p_j² x_i x_j + 36 p_i² p_j x_i² x_j -
  12 (9 + 2 T) p_i p_j² x_i² x_j + 24 (3 + T) p_j³ x_i² x_j - 4 p_i³ p_j x_i³ x_j + 28 T p_i² p_j² x_i³ x_j -
  4 (-6 + 17 T + T²) p_i p_j³ x_i³ x_j + 4 (-5 + 10 T + T²) p_j⁴ x_i³ x_j + 24 p_i p_j² x_i x_j² -
  24 p_j³ x_i x_j² - 24 p_i² p_j² x_i² x_j² + 6 (10 + T) p_i p_j³ x_i² x_j² - 6 (6 + T) p_j⁴ x_i² x_j² -
  4 p_i p_j³ x_i x_j³ + 4 p_j⁴ x_i x_j³) / 24
```

```
V@r_{3,-1}[i_, j_] :=
 (-4 T³ p_i x_i + 4 T³ p_j x_i - 2 T² (7 + 5 T) p_i p_j x_i² + 2 T² (7 + 5 T) p_j² x_i² -
  4 T² (-6 + 5 T) p_i² p_j x_i³ + 4 T (-2 - 17 T + 16 T²) p_i p_j² x_i³ -
  4 T (-2 - 11 T + 11 T²) p_j³ x_i³ + 3 (-1 + T)² p_i³ p_j x_i⁴ - 3 (-1 + T)² (3 + 4 T) p_i² p_j² x_i⁴ +
  (-1 + T) (1 + 22 T + 13 T²) p_i p_j³ x_i⁴ - (-1 + T) (1 + 13 T + 4 T²) p_j⁴ x_i⁴ +
  28 T³ p_i p_j x_i x_j - 28 T³ p_j² x_i x_j - 36 T³ p_i² p_j x_i² x_j + 12 T² (2 + 9 T) p_i p_j² x_i² x_j -
  24 T² (1 + 3 T) p_j³ x_i² x_j + 4 T³ p_i³ p_j x_i³ x_j - 28 T² p_i² p_j² x_i³ x_j -
  4 T (-1 - 17 T + 6 T²) p_i p_j³ x_i³ x_j + 4 T (-1 - 10 T + 5 T²) p_j⁴ x_i³ x_j -
  24 T³ p_i p_j² x_i x_j² + 24 T³ p_j³ x_i x_j² + 24 T³ p_i² p_j² x_i² x_j² - 6 T² (1 + 10 T) p_i p_j³ x_i² x_j² +
  6 T² (1 + 6 T) p_j⁴ x_i² x_j² + 4 T³ p_i p_j³ x_i x_j³ - 4 T³ p_j⁴ x_i x_j³) / (24 T³)
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{p*, x*, p̄*, x̄*} = {π, ξ, π̄, ξ̄}; (z_{-i__})* := (z*)_i;
Zip_{}[ε_] := ε;
Zip_{z_, zs___}[ε_] :=
 (Collect[ε // Zip_{zs}, z] /. f_. z^d_. :> (D[f, {z*, d}])) /. z* → 0

gPair[fs_, w_] :=
 gPair[fs, w] =
  Collect[Zip_{Join@@Table[{p_α, p̄_α, x_α, x̄_α}, {α, w}]}[
    (Times @@ (V /@ fs))
    Exp[Sum[g_{α,β} (π_α + π̄_α) (ξ_β + ξ̄_β), {α, w}, {β, w}] - Sum[ξ̄_α π_α, {α, w}]]],
   g__, Factor]

T2z[p_] := Module[{q = Expand[p], n, c},
  If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
   c z^{2n} + T2z[q - c (T^{1/2} - T^{-1/2})^{2n}]]];

Z_d_[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, d1, Z1, Z2, Z3},
  {Cs, φ} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
  {Δ, G} = Factor@{T^{(-Total[φ]-Total[Cs[[All,1]]])/2} Det@A, Inverse@A};
  Z1 =
   Exp[Total[Cases[Cs, {s_, i_, j_} :> Sum[ε^{d1} r_{d1,s}[i, j], {d1, d}]]] +
    Sum[ε^{d1} γ_{d1,φ[[k]]}[k], {k, 2 n}, {d1, d}] /. γ_{_,0}[_] → 0];
  Z2 = Expand[F[{}, {}] × Normal@Series[Z1, {ε, 0, d}]] //.
    F[fs_, {es___}] × (f : (r | γ)_{ps_}[is__])^{p_.} :>
     F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}];
  Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
     Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
     ] /. g_{α_,β_} :> G[es[[α]], es[[β]]]]];
  Collect[{Δ, Z3 /. ε^{p_.} → p! Δ^{2 p} ε^p}, ε, T2z]];
```

```
Z_2[GST48] (* takes a few minutes *)
{1 - 4 z² - 61 z⁴ - 207 z⁶ - 296 z⁸ - 210 z¹⁰ - 77 z¹² - 14 z¹⁴ - z¹⁶,
 1 + (38 z² + 255 z⁴ + 1696 z⁶ + 16 281 z⁸ + 86 952 z¹⁰ + 259 994 z¹² + 487 372 z¹⁴ + 615 066 z¹⁶ + 543 148 z¹⁸ + 341 714 z²⁰ +
    153 722 z²² + 48 983 z²⁴ + 10 776 z²⁶ + 1554 z²⁸ + 132 z³⁰ + 5 z³²) ε +
  (-8 - 484 z² + 9709 z⁴ + 165 952 z⁶ + 1 590 491 z⁸ + 16 256 508 z¹⁰ + 115 341 797 z¹² + 432 685 748 z¹⁴ + 395 838 354 z¹⁶ - 4 017 557 792 z¹⁸ - 23 300 064 167 z²⁰ -
    70 082 264 972 z²² - 142 572 271 191 z²⁴ - 209 475 503 700 z²⁶ - 221 616 295 209 z²⁸ - 151 502 648 428 z³⁰ - 23 700 199 243 z³² +
    99 462 146 328 z³⁴ + 164 920 463 074 z³⁶ + 162 550 825 432 z³⁸ + 119 164 552 296 z⁴⁰ + 69 153 062 608 z⁴² + 32 547 596 611 z⁴⁴ + 12 541 195 448 z⁴⁶ +
    3 961 384 155 z⁴⁸ + 1 021 219 696 z⁵⁰ + 212 773 106 z⁵² + 35 264 208 z⁵⁴ + 4 537 548 z⁵⁶ + 436 600 z⁵⁸ + 29 536 z⁶⁰ + 1252 z⁶² + 25 z⁶⁴) ε²}
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TableForm[Table[Join[{K[[1]]_{K[[2]]}}, Z_3[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center] (* takes a few minutes *)
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3_1	$1 + z^2$	$1 + (2z^2 + z^4)\epsilon + (2 - 4z^2 + 3z^4 + 4z^6 + z^8)\epsilon^2 + (-12 - 74z^2 - 27z^4 - 20z^6 + 8z^8 + 6z^{10} + z^{12})\epsilon^3$
4_1	$1 - z^2$	$1 + (-2 + 2z^4)\epsilon^2$
5_1	$1 + 3z^2 + z^4$	$1 + (10z^2 + 21z^4 + 12z^6 + 2z^8)\epsilon + (6 - 28z^2 + 33z^4 + 364z^6 + 655z^8 + 536z^{10} + 227z^{12} + 48z^{14} + 4z^{16})\epsilon^2 + (-60 + 970z^2 + 645z^4 - 3380z^6 - 3280z^8 + 7470z^{10} + 19475z^{12} + 20536z^{14} + 12564z^{16} + 4774z^{18} + 1109z^{20} + 144z^{22} + 8z^{24})\epsilon^3$
5_2	$1 + 2z^2$	$1 + (6z^2 + 5z^4)\epsilon + (4 - 20z^2 + 43z^4 + 64z^6 + 26z^8)\epsilon^2 + (-36 - 498z^2 - 883z^4 + 100z^6 + 816z^8 + 556z^{10} + 146z^{12})\epsilon^3$
6_1	$1 - 2z^2$	$1 + (-2z^2 + z^4)\epsilon + (-4 - 4z^2 + 25z^4 - 8z^6 + 2z^8)\epsilon^2 + (12 + 154z^2 - 223z^4 - 608z^6 + 100z^8 - 52z^{10} - 10z^{12})\epsilon^3$
6_2	$1 - z^2 - z^4$	$1 + (-2z^2 - 3z^4 + 2z^6 + z^8)\epsilon + (-2 - 4z^2 + 29z^4 + 28z^6 + 42z^8 - 8z^{10} - 2z^{12} + 4z^{14} + z^{16})\epsilon^2 + (12 + 166z^2 + 155z^4 - 194z^6 - 2453z^8 - 1622z^{10} - 1967z^{12} - 258z^{14} + 49z^{16} + z^{20} + 6z^{22} + z^{24})\epsilon^3$
6_3	$1 + z^2 + z^4$	$1 + (2 + 8z^2 - 16z^6 - 24z^8 - 16z^{10} - 2z^{12})\epsilon^2$

Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout:
http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.