 Thanks for inviting me to UTokyo!

Rooting the BKT for FTI

ωεβ:=http://drorbn.net/tok2309

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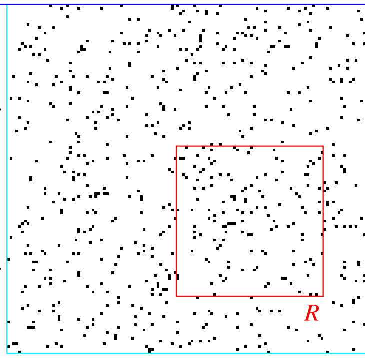
Abstract. Following joint work with Itai Bar-Natan, Iva Halacheva, and Nancy Scherich, I will show that the Best Known Time (BKT) to compute a typical Finite Type Invariant (FTI) of type d on a typical knot with n crossings is roughly equal to $n^{d/2}$, which is roughly the square root of what I believe was the standard belief before, namely about n^d.

Conventions. • $\underline{n} := \{1, 2, \ldots, n\}$. • For complexity estimates we ignore constant and logarithmic terms: $n^3 \sim 2023d!(\log n)^d n^3$.

A Key Preliminary. Let $Q \subset \underline{n}^l$ be an enumerated subset, with $1 \ll q = |Q| \ll n^l$. In time $\sim q$ we can set up a lookup table of size $\sim q$ so that we will be able to compute $|Q \cap R|$ in time ~ 1, for any rectangle $R \subset \underline{n}^l$.

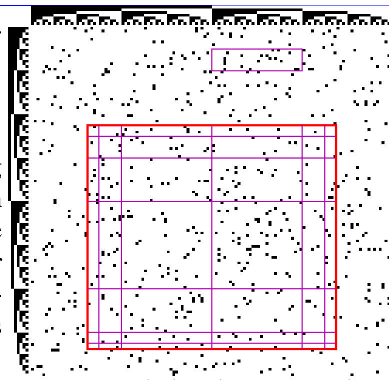
Fails. • Count after R is presented. • Make a lookup table of $|Q \cap R|$ counts for all R's.



Unfail. Make a restricted lookup table of the form

$$\left\{ \underset{\text{dyadic}}{R} \to \underset{>0}{|Q \cap R|} \right\}.$$

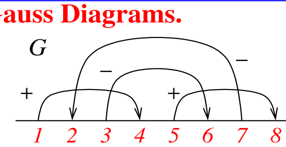
• Make the table by running through $x \in Q$, and for each one increment by 1 only the entries for dyadic $R \ni x$ (or create such an entry, if it didn't exist already). This takes $q \cdot (\log_2 n)^l \sim q$ ops.



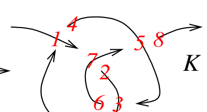
• Entries for empty dyadic R's are not needed and not created.
• Using standard sorting techniques, access takes $\log_2 q \sim 1$ ops.
• A general R is a union of at most $(2\log_2 n)^l \sim 1$ dyadic ones, so counting $|Q \cap R|$ takes ~ 1 ops.

Generalization. Without changing the conclusion, replace counts $|Q \cap R|$ with summations $\sum_R \theta$, where $\theta\colon \underline{n}^l \to V$ is supported on a sparse Q, takes values in a vector space V with $\dim V \sim 1$, and in some basis, all of its coefficients are "easy".

Gauss Diagrams.

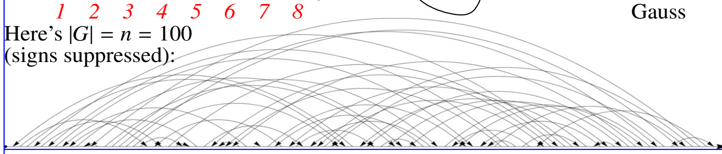


$\longleftrightarrow$



Gauss

Here's $|G| = n = 100$ (signs suppressed):

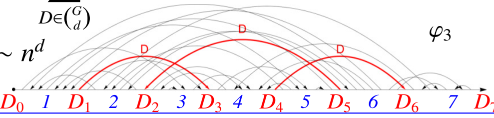


Definitions. Let $\mathcal{G} := \mathbb{Q}\langle\text{Gauss Diagrams}\rangle$, with $\mathcal{G}_d$ / $\mathcal{G}_{\le d}$ the diagrams with exactly / at most d arrows. Let $\varphi_d\colon \mathcal{G} \to \mathcal{G}_d$ be

$$\varphi_d\colon G \mapsto \sum_{D \subset G,\ |D|=d} D = \sum_{D \in \binom{G}{d}} D,$$

and let $\varphi_{\le d} = \sum_{e \le d} \varphi_e$.

Naively, it takes $\binom{n}{d} \sim n^d$ ops to compute φ_d.



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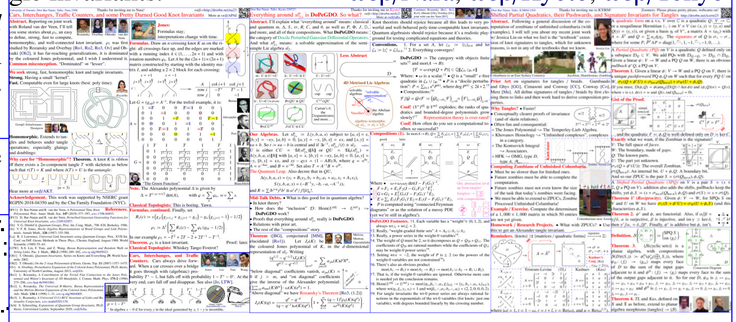
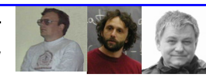
My Primary Interest. Strong, fast, homomorphic knot and tangle invariants. ωεβ/Nara, ωεβ/Kyoto, ωεβ/Tokyo



The [GPV] Theorem. A knot invariant is finite type of type d iff it is of the form $\omega \circ \varphi_{\le d}$ for some $\omega \in \mathcal{G}^*_{\le d}$.

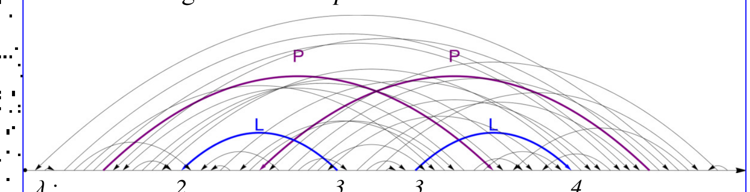


Goussarov-Polyak-Viro

• $\Leftarrow$ is easy; $\Rightarrow$ is hard and IMHO not well understood.
• $\varphi_{\le d}$ is not an invariants and not every ω gives an invariant!
• The theory of finite type invariants is very rich. Many knot invariants factor through finite type invariants, and it is possible that they separate knots.
• We need a fast algorithm to compute $\varphi_{\le d}$!

Our Main Theorem. On an n-arrow Gauss diagram, φ_d can be computed in time $\sim n^{\lceil d/2 \rceil}$.

Proof. With $d = p + l$ (p for "put", l for "lookup"), pick p arrows and look up in how many ways the remaining l can be placed in between the legs of the first p:



To reconstruct $D = P\#_\lambda L$ from P and L we need a non-decreasing "placement function" $\lambda\colon \underline{2l} \to \underline{2p+1}$.

$$\varphi_d(G) = \sum_{D \in \binom{G}{d}} D = \binom{d}{p}^{-1} \sum_{P \in \binom{G}{p}} \sum_{\substack{\text{non-decreasing} \\ \lambda\colon \underline{2l} \to \underline{2p+1}}} \sum_{\substack{L \in \binom{G}{l} \\ L_i \in (P_{\lambda(i)-1}, P_{\lambda(i)})}} P\#_\lambda L$$

Define $\theta_G\colon \underline{2n}^{2l} \to \mathcal{G}_l$ by

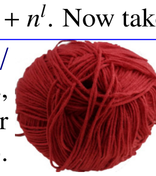
$$(L_1, \ldots, L_{2l}) \mapsto \begin{cases} L & \text{if } (L_1, \ldots, L_{2l}) \text{ are the ends of some } L \subset G \\ 0 & \text{otherwise} \end{cases}$$

and now

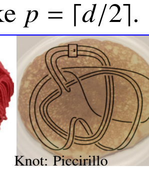
$$\varphi_d(G) = \binom{d}{p}^{-1} \sum_{P \in \binom{G}{p}} \sum_{\substack{\text{non-decreasing} \\ \lambda\colon \underline{2l} \to \underline{2p+1}}} P\#_\lambda \left(\sum_{\prod_i (P_{\lambda(i)-1}, P_{\lambda(i)})} \theta_G \right)$$

can be computed in time $\sim n^p + n^l$. Now take $p = \lceil d/2 \rceil$. □

Question ([BBHS], ωεβ/Fields). For computations, planar projections are better than braids (as likely $l \sim n^{3/2}$). But are yarn balls better than planar projections (here likely $n \sim L^{4/3}$)?



Length L



n crossings

length l

References.

[BBHS] D. Bar-Natan, I. Bar-Natan, I. Halacheva, and N. Scherich, *Yarn Ball Knots and Faster Computations,* J. of Appl. and Comp. Topology (to appear), arXiv:2108.10923.
[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Tokyo-230911/