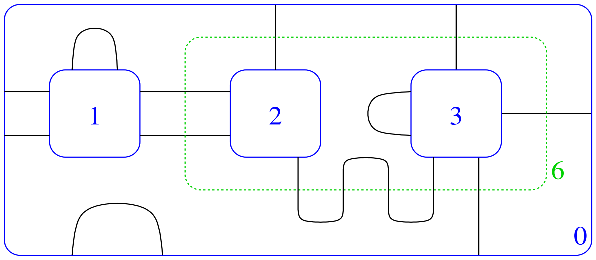


In this example, if you ignore the dotted green line (marked "6"), you see the planar connection diagram $D_B$, which has three inputs (1,2,3) and a single output, the cycle 0. If you only look inside the green line, you see $D_I$, with inputs 2 and 3 and an output cycle 6. If you ignore the inside of 6 you see $D_O$, with inputs 1 and 6 and output cycle 0.

Let $F_B$ (Big Faces) denote the vector space whose basis are the faces of $D_B$, let $F_I$ (Inner Faces) be the space of faces of $D_I$, and let $F_O$ (Outer Faces) be the space of faces of $D_O$. Let $G_1$, $G_2$, $G_3$, $G_6$, and $G_0$ be the spaces of gaps (edges) along the cycles 1,2,3,6, and 0, respectively. Let $\psi := \psi_{D_B}$ and $\phi := \phi^{D_B}$ be the maps defining $\mathcal{S}(D_B)$ and let $\gamma := \psi_{D_O}$ and $\delta := \phi^{D_O}$ be the maps defining $\mathcal{S}(D_O)$. Further, let $\alpha := \psi_{D_I}\colon F_I \to G_2 \oplus G_3$ and $\beta := \phi^{D_I}\colon F_I \to G_6$ be the maps defining $\mathcal{S}(D_I)$, and let $\alpha_+ := I \oplus \alpha$ and $\beta^+ := I \oplus \beta$ be the extensions of $\alpha$ and $\beta$ by an identity on an extra factor of $G_1$, so that $\beta^+_*\alpha^*_+ = I_{G_1} \oplus \mathcal{S}(D_I)$. Let $\mu$ map any big face to the sum of $G_1$ gaps around it, plus the sum of the inner faces it contains. Let $\nu$ map any big face to the sum of the outer faces it contains. It is easy to see that the master diagram $(MD)$ shown on the right, made of all of these spaces and maps, is commutative.

$$(MD)\qquad \begin{array}{ccccc} & & & & G_0 \\ & & & \overset{\phi}{\nearrow} & \uparrow\delta \\ & & F_B & \xrightarrow{\nu} & F_O \\ & \overset{\psi}{\swarrow} & \downarrow\mu & & \downarrow\gamma \\ G_1 \oplus G_2 \oplus G_3 & \xleftarrow[\alpha_+]{} & G_1 \oplus F_I & \xrightarrow[\beta^+]{} & G_1 \oplus G_6 \end{array}$$

**Claim.** The bottom right square of $(MD)$ is an equalizer square, namely $F_B \simeq EQ(\beta^+,\gamma)$. Hence $\nu_*\mu^* = \gamma^*\beta^+_*$.

$$\begin{array}{ccc} F_B & \xrightarrow{\nu} & F_O \\ \downarrow\mu & & \downarrow\gamma \\ G_1 \oplus F_I & \xrightarrow[\beta^+]{} & G_1 \oplus G_6 \end{array}$$

**Proof.** A big face (an element of $F_B$) is a sum of outer faces $f_o$ and a sum of inner faces $f_i$, and it has a boundary $g_1$ on input cycle 1, such that the boundary of the outer pieces $f_o$ is equal to the boundary of the inner pieces $f_i$ plus $g_1$. That matches perfectly with the definition of the equalizer: $EQ(\beta^+,\gamma) = \{(g_1, f_i, f_o)\colon \beta^+(g_1, f_i) = \gamma(f_o)\} = \{(g_1, f_i, f_o)\colon \gamma(f_o) = (g_1, \beta(f_i))\}$. □

**Proof of Theorem 4.** With notation as above, with the example above (which is general enough), and with the claim above, and also using functoriality, we have $\mathcal{S}(D_B) = \phi_*\psi^* = \delta_*\nu_*\mu^*\alpha^*_+ = \delta_*\gamma^*\beta^+_*\alpha^*_+ = \mathcal{S}(D_O) \circ (I_{G_1} \oplus \mathcal{S}(D_I))$, as required. □

**Proof of Theorem 5.** We need to verify the Reidemeister moves and that was done in the computational section, and the statement about the restriction to links, which is easy: simply assemble an $n$-crossing knot using an $n$-input planar connection diagram, and the formulas clearly match. □

**Further Homework.**

**Exercise 6.** By taking $U = 0$ in the reciprocity statement, prove that always $\sigma(\phi_*S) = \sigma(S)$. But that seems wrong, if $\phi = 0$. What saves the day?

**Exercise 7.** By taking $S = 0$ in the reciprocity statement, frove that always $\sigma(\phi^*U) = \sigma(U)$. But wait, this is nonsense! What went wrong?

**Exercise 8.** Given $\phi\colon V \to W$ and a subspace $D \subset V$, show that there is a unique subspace $\phi_*D \subset W$ such that for every quadratic $Q$ on $W$, $\sigma(\phi^*Q|_D) = \sigma(Q|_{\phi_*D})$.

**Exercise 9.** When are diagrams as on the right equalizer diagrams? What then do we learn from Theorem 3?

$$\begin{array}{ccc} Y & \to & 0 \\ \downarrow & & \downarrow \\ V & \to & Z \end{array} \qquad \begin{array}{ccc} Y & \to & W \\ \downarrow & & \downarrow \\ V & \to & 0 \end{array}$$

**Exercise 10.** There are 11 types or irreducible commutative squares:

$$\begin{smallmatrix}1&\to&0\\\downarrow&&\downarrow\\0&\to&0\end{smallmatrix},\ \begin{smallmatrix}0&\to&1\\\downarrow&&\downarrow\\0&\to&0\end{smallmatrix},\ \begin{smallmatrix}0&\to&0\\\downarrow&&\downarrow\\1&\to&0\end{smallmatrix},\ \begin{smallmatrix}0&\to&0\\\downarrow&&\downarrow\\0&\to&1\end{smallmatrix},\ \begin{smallmatrix}1&\xrightarrow[1]{}&1\\\downarrow&&\downarrow\\0&\to&0\end{smallmatrix},\ \begin{smallmatrix}0&\to&1\\\downarrow&&\downarrow 1\\0&\to&1\end{smallmatrix},\ \begin{smallmatrix}0&\to&1\\\downarrow&&\downarrow 1\\0&\to&1\end{smallmatrix},\ \begin{smallmatrix}0&\to&0\\\downarrow&&\downarrow\\1&\xrightarrow{1}&1\end{smallmatrix},$$

$$\begin{smallmatrix}0&\to&1\\\downarrow&&\downarrow 1\\1&\xrightarrow{1}&1\end{smallmatrix},\ \begin{smallmatrix}1&\xrightarrow[1]{}&1\\1\downarrow&&\downarrow\\1&\to&0\end{smallmatrix},\ \text{and}\ \begin{smallmatrix}1&\xrightarrow[1]{}&1\\1\downarrow&&\downarrow 1\\1&\xrightarrow{1}&1\end{smallmatrix}.$$

Show that pushing commutes with pulling for all but four of them. Compare with the statement of Theorem 3.

**Exercise 11.** Prove that a square is admissible iff it is an equalizer square, with an additional direct summand $A$ added to the $Y$ term, and with the maps $\mu$ and $\nu$ extended by 0 on $A$.

**Exercise 12.** Prove that the direct sum of two admissible squares is admissible. *Warning:* Harder than it seems! Not all quadratics on $V_1 \oplus V_2$ are direct sums of quadratics on $V_1$ and on $V_2$.

**Exercise 13.** Given a quadratic $Q$ on a space $V$, let $\pi$ be the projection $V \to V/\operatorname{rad}(Q)$ and show that $\pi_*Q = Q/\operatorname{rad}(Q)$, with the obvious definition for the latter.

**Exercise 14.** Show that for any partial quadratic $Q$ on a space $W$ there exists a space $A$ and a fully-defined quadratic $F$ on $W \oplus A$ such that $\pi_*F = Q$, where $\pi\colon W \oplus A \to W$ is the projection (these are not unique). Furthermore, if $\phi\colon V \to W$, then $\phi^*Q = \pi_*\phi^*_+F$, where $\phi_+ = \phi \oplus I\colon V \oplus A \to W \oplus A$ and $\pi$ also denotes the projection $V \oplus A \to V$.

**Solutions / Hints.**

Hint for 1. On a vector in the domain of one but not the other, take an outrageous value for $U$, that will raise or lower the signature.

Hint for 2. WLOG, $Q_1$ is diagonal and $Q_1 = 0$.

Hint for 5. It's enough to test that against $U$ with $\mathcal{D}(U) = \operatorname{im}\phi$.

Hint for 6. The "shift" part of $0_*S$ is $\sigma(S)$.

Hint for 7. $\phi_*S$ isn't 0, it's the partial quadratic "0 on $\operatorname{im}\phi$" (and indeed, $\sigma(\phi^*U) = \sigma(U)$ if $\phi$ is surjective).

Hint for 10. The exceptions are $\begin{smallmatrix}10\\00\end{smallmatrix}$, $\begin{smallmatrix}00\\01\end{smallmatrix}$, $\begin{smallmatrix}01\\11\end{smallmatrix}$, and $\begin{smallmatrix}11\\01\end{smallmatrix}$.

Hint for 12. Use Exercise 11.

Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.