Dror Bar-Natan — Handout Portfolio

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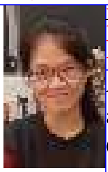
Warning. This is a reduced quality version with some content removed to control file size. My full handout portfolio is at the URL above.

Thanks for allowing me at USC!

Shifted Partial Quadratics, their Pushforwards, and Signature Invariants for Tangles

http://drorbn.net/usc24

Abstract. Following a general discussion of the computation of zombians of unfinished columbaria (with examples), I will tell you about my recent joint work w/ Jessica Liu on what we feel is the "textbook" extension of knot signatures to tangles, which for unknown reasons, is not in any of the textbooks that we know.


Jessica Liu

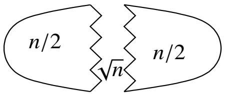

Columbaria in an East Sydney Cemetery


Image: Freepik.com

Jacobian, Hamiltonian, Zombian

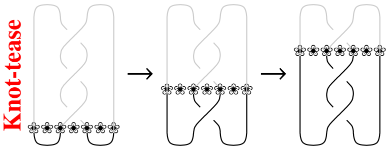
Prior Art on signatures for tangles / braids. Gambaudo and Ghys [GG], Cimasoni and Conway [CC], Conway [Co], Merz [Me]. All define signatures of tangles / braids by first closing them to links and then work hard to derive composition properties.

Why Tangles? • Faster!



$2^{n/2}+2^{n/2}+2^{\sqrt{n}} \ll 2^n$

- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:
  - ○ The Jones Polynomial ⇝ The Temperley-Lieb Algebra.
  - ○ Khovanov Homology ⇝ "Unfinished complexes", complexes in a category.
  - ○ The Kontsevich Integral ⇝ Associators.
  - ○ HFK ⇝ OMG, type D, type A, $\mathcal{A}_\infty$, ...



Computing Zombians of Unfinished Columbaria.

- Must be no slower than for finished ones.
- Future zombies must be able to complete the computation.
- Future zombies must not even know the size of the task that today's zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!

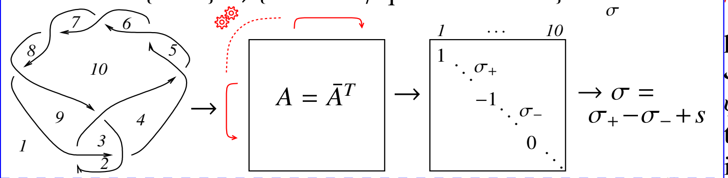

Zombies: Freepik.com

Columbarium near Assen

Example / Exercise. Compute the determinant of a $1,000 \times 1,000$ matrix in which 50 entries are not yet given.

Homework / Research Projects. • What with ZPUCs? • Use this to get an Alexander tangle invariant.

Reminders. {links} ⇉ {matrices / quadratic forms} $\xrightarrow[\sigma]{\text{signature}} \mathbb{Z}$:

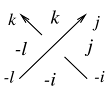


$\to \sigma = \sigma_+ - \sigma_- + s$

With $|\omega| = 1$, $t = 1-\omega$, $r = t+\bar{t}$, $v = \mathrm{Re}(\omega)$, and $u = \mathrm{Re}(\omega^{1/2})$:

Tristram-Levine *(TL)* Kashaev *(Kas)*

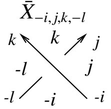
$X_{-i,j,k,-l}$



$$A \mathrel{+}= \begin{pmatrix} -r & -t & 2t & \bar{t} \\ -\bar{t} & 0 & \bar{t} & 0 \\ 2\bar{t} & t & -r & -\bar{t} \\ t & 0 & -t & 0 \end{pmatrix} \begin{matrix} i \\ j \\ k \\ l \end{matrix} \qquad s = 0$$

$$A \mathrel{+}= \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix} \begin{matrix} i \\ j \\ k \\ l \end{matrix} \qquad s \mathrel{-}= 1$$

$\bar{X}_{-i,j,k,-l}$



$$A \mathrel{+}= \begin{pmatrix} r & -t & -2\bar{t} & \bar{t} \\ -\bar{t} & 0 & \bar{t} & 0 \\ -2t & t & r & -\bar{t} \\ t & 0 & -t & 0 \end{pmatrix} \begin{matrix} i \\ j \\ k \\ l \end{matrix} \qquad s = 0$$

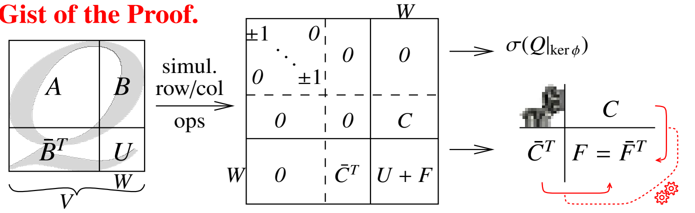
$$A \mathrel{-}= \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix} \begin{matrix} i \\ j \\ k \\ l \end{matrix} \qquad s \mathrel{+}= 1$$

~~Kashaev's Conjecture~~ [Ka] For links, $\sigma_{Kas} = 2\sigma_{TL}$.
Liu's Theorem [Li].

A *Partial Quadratic (PQ)* on V is a quadratic Q defined only on a subspace $\mathcal{D}_Q \subset V$. We add PQs with $\mathcal{D}_{Q_1+Q_2} := \mathcal{D}_{Q_1} \cap \mathcal{D}_{Q_2}$. Given a linear $\psi\colon V \to W$ and a PQ Q on W, there is an obvious *pullback* ψ^*Q, a PQ on V.

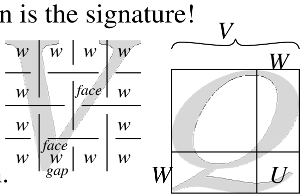
Theorem 1. Given a linear $\phi\colon V \to W$ and a PQ Q on V, there is a unique *pushforward* PQ ϕ_*Q on W such that for every *PQ* U on W, $\sigma_V(Q+\phi^*U) = \sigma_{\ker\phi}(Q|_{\ker\phi}) + \sigma_W(U+\phi_*Q)$.
(If you must, $\mathcal{D}(\phi_*Q) = \phi(\mathrm{ann}_Q(\mathcal{D}(Q) \cap \ker\phi))$ and $(\phi_*Q)(w) = Q(v)$, where v is s.t. $\phi(v) = w$ and $Q(v, \mathrm{rad}\, Q|_{\ker\phi}) = 0$).

Gist of the Proof.



... and the quadratic $F =: \phi_*Q$ is well-defined only on $D := \ker C$.

Exactly what we want, if the Zombian is the signature!



V: The full space of *faces*.
W: The boundary, made of *gaps*.
Q: The known parts.
U: The part yet unknown.
$\sigma_V(Q+\phi^*(U))$: The overall Zombian.
$\sigma(Q|_{\ker\phi})$: An internal bit. $U+\phi_*Q$: A boundary bit.
And so our ZPUC is the pair $S = (\sigma(Q|_{\ker\phi}), \phi_*Q)$.

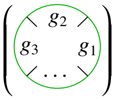
A *Shifted Partial Quadratic (SPQ)* on V is a pair $S = (s \in \mathbb{Z}, Q$ a PQ on $V)$. addition also adds the shifts, pullbacks keep the shifts, yet $\phi_*S := (s+\sigma_{\ker\phi}(Q|_{\ker\phi}), \phi_*Q)$ and $\sigma(S) := s+\sigma(Q)$.

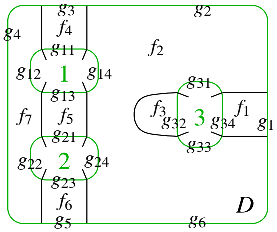
Theorem 1' *(Reciprocity)*. Given $\phi\colon V \to W$, for SPQs S on V and U on W we have $\sigma_V(S+\phi^*U) = \sigma_W(U+\phi_*S)$ (and this characterizes ϕ_*S). Note. ψ^* is additive but ϕ_* is not.

Theorem 2. ψ^* and ϕ_* are functorial.

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \\ V \arrow[r, "\beta"'] \arrow[ur, dotted] & Z \end{tikzcd}$$

Theorem 3. "The pullback of a pushforward scene is a pushforward scene": If, on the right, β and δ are arbitrary, $Y = \mathrm{EQ}(\beta,\gamma) = V \oplus_Z W = \{(v,w)\colon \beta v = \gamma w\}$ and μ and ν are the obvious projections, then $\gamma^*\beta_* = \nu_*\mu^*$.

Definition. $\mathcal{S}\Bigg($  $\Bigg) := \left\{\begin{matrix}\text{SPQ } S \\ \text{on } \langle g_i \rangle\end{matrix}\right\}$.

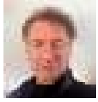


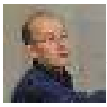
Connection Diagram

Theorem 4. $\{\mathcal{S}(\text{cyclic sets})\}$ is a planar algebra, with compositions $\mathcal{S}(D)((S_i)) := \phi^D_*(\psi^*_D(\bigoplus_i S_i))$, where $\psi_D\colon \langle f_i \rangle \to \langle g_{\alpha i} \rangle$ maps every face of D to the sum of the input gaps adjacent to it and $\phi^D\colon \langle f_i \rangle \to \langle g_i \rangle$ maps every face to the sum of the output gaps adjacent to it. So for our D, ψ_D: $f_1 \mapsto g_{34}$, $f_2 \mapsto g_{31}+g_{14}+g_{24}+g_{33}$, $f_3 \mapsto g_{32}$, $f_4 \mapsto g_{11}$, $f_5 \mapsto g_{13}+g_{21}$, $f_6 \mapsto g_{23}$, $f_7 \mapsto g_{12}+g_{22}$ and ϕ^D: $f_1 \mapsto g_1$, $f_2 \mapsto g_2+g_6$, $f_3 \mapsto 0$, $f_4 \mapsto g_3$, $f_5 \mapsto 0$, $f_6 \mapsto g_5$, $f_7 \mapsto g_4$.

Theorem 5. *TL* and *Kas*, defined on X and $\bar{X}$ as before, extend to planar algebra morphisms {tangles} → $\{\mathcal{S}\}$. Restricted to links, $TL = \sigma_{TL}$ and $Kas = \sigma_{Kas}$.


Levine


Tristram


Kashaev

Video: `http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201`. Handout: `http://www.math.toronto.edu/~drorbn/Talks/USC-240205`.

Implementation (sources: http://drorbn.net/icerm23/ap). I like it most when the implementation matches the math perfectly. We failed here.

```
Once[<< KnotTheory`];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

Utilities. The step function, algebraic numbers, canonical forms.

```
θ[x_] /; NumericQ[x] := UnitStep[x]
```

```
ω2[v_][p_] := Module[{q = Expand[p], n, c},
   If[q === 0, 0,
    c = Coefficient[q, ω, n = Exponent[q, ω]];
    c v^n + ω2[v][q - c (ω + ω^-1)^n]]];
```

```
sign[ℰ_] := Module[{n, d, v, p, rs, e, k},
  {n, d} = NumeratorDenominator[ℰ];
  {n, d} /= ω^(Exponent[n,ω]/2+Exponent[n,ω,Min]/2);
  p = Factor[ω2[v]@n * ω2[v]@d /. v → 4 u^2 - 2];
  rs = Solve[p == 0, u, Reals];
  If[rs === {}, Sign[p /. u → 0],
   rs = Union@(u /. rs);
   Sign[(-1)^(e=Exponent[p,u]) Coefficient[p, u, e]] + Sum[
     k = 0;
     While[(d = RootReduce[∂_{u,++k} p /. u → r]) == 0];
     If[EvenQ[k], 0, 2 Sign[d]] * θ[u - r],
     {r, rs}]
  ]
 ]
```

```
SetAttributes[B, Orderless];
CF[b_B] := RotateLeft[#, First@Ordering[#] - 1] & /@
  DeleteCases[b, {}]
```

```
CF[ℰ_] := Module[{γs = Union@Cases[ℰ, γ_ | γ̄_, ∞]},
  Total[CoefficientRules[ℰ, γs] /.
    (ps_ → c_) :→ Factor[c] × Times @@ γs^ps] ]
```

```
CF[{}] = {};
CF[𝒞_List] :=
 Module[{γs = Union@Cases[𝒞, γ_, ∞], γ},
  CF /@ DeleteCases[0][
    RowReduce[Table[∂_γ r, {r, 𝒞}, {γ, γs}]].γs] ]
```

```
(ℰ_)* := ℰ /. {γ̄ → γ, γ → γ̄, ω → ω^-1, c_Complex :→ c*};
r_Rule⁺ := {r, r*}
```

```
RulesOf[γi_ + rest_.] := (γi → -rest)⁺;
CF[PQ[𝒞_, q_]] := Module[{n𝒞 = CF[𝒞]},
  PQ[n𝒞, CF[q /. Union @@ RulesOf /@ n𝒞]] ]
```

```
CF[Σb_[σ_, pq_]] := Σ_CF[b][σ, CF[pq]]
```

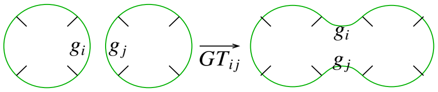
Pretty-Printing.

```
Format[Σ_b_B[σ_, PQ[𝒞_, q_]]] := Module[{γs},
   γs = γ# & /@ Join @@ b;
   Column[{TraditionalForm@σ,
     TableForm[Join[
       Prepend[""] /@ Table[TraditionalForm[∂_c r],
         {r, 𝒞}, {c, γs}],
       {Prepend[""][
         Join @@
           (b /. {l_, m___, r_} :→
              {DisplayForm@RowBox[{"(", l}],
               m, DisplayForm@RowBox[{r, ")"}]}) /.
          i_Integer :→ γi ]},
       MapThread[Prepend,
        {Table[TraditionalForm[∂_{r,c} q], {r, γs*},
          {c, γs}], γs*}]
      ], TableAlignments → Center]
    }, Center] ];
```

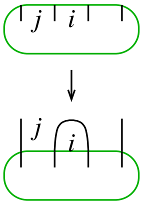
The Face-Centric Core.

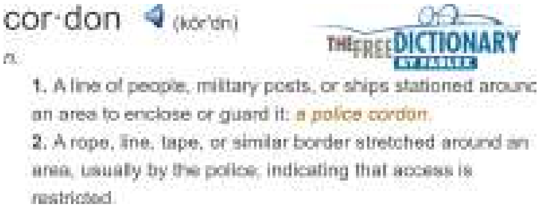
```
Σb1_[σ1_, PQ[𝒞1_, q1_]] ⊕ Σb2_[σ2_, PQ[𝒞2_, q2_]] ^:=
  CF@Σ_Join[b1,b2][σ1 + σ2, PQ[𝒞1 ⋃ 𝒞2, q1 + q2]];
```

GT for Gap Touch:



```
GT_{i_,j_}@Σ_B[{li___,i_,ri___},{lj___,j_,rj___},bs___][σ_,
   PQ[𝒞_, q_]] :=
 CF@Σ_B[{ri,li,j,rj,lj,i},bs][σ, PQ[𝒞 ⋃ {γi - γj}, q]]
```





cor·don (kôr′dn)

n.

1. A line of people, military posts, or ships stationed aroun
an area to enclose or guard it: a police cordon.

2. A rope, line, tape, or similar border stretched around an area, usually by the police, indicating that access is restricted.

$$\begin{array}{c} s\\ i\\ \\ \end{array}\left(\begin{array}{c|cc} 0 & \phi & C_{\text{rest}}\\ \hline \bar{\phi}^T & \lambda & \theta\\ \bar{C}^T_{\text{rest}} & \bar{\theta}^T & A_{\text{rest}} \end{array}\right) \to \begin{cases} \exists p\,\phi_p \neq 0 & \text{use } \phi_p \text{ to kill its row and column, drop a } \begin{pmatrix}01\\10\end{pmatrix} \text{ summand}\\ \phi=0, \lambda\neq 0 & \text{use } \lambda \text{ to kill } \theta\text{, let } s \mathrel{+}= \text{sign}(\lambda)\\ \phi=0, \lambda=0 & \text{append } \theta \text{ to } C_{\text{rest}}. \end{cases}$$

```
Cordon_i_@Σ_B[{li___,i_,ri___},bs___][σ_, PQ[𝒞_, q_]] :=
 Module[{ϕ = ∂_γi 𝒞, λ = ∂_{γ̄i,γi} q, nσ = σ, n𝒞, nq, p},
  {p} = FirstPosition[(# =!= 0) & /@ ϕ, True, {0}];
  {n𝒞, nq} = Which[
    p > 0, {𝒞, q} /. (γi → -𝒞[[p]] / ϕ[[p]])⁺ /. (γi → 0)⁺,
    λ =!= 0, (nσ += sign[λ];
     {𝒞, q /. (γi → -(∂_γ̄i q) / λ)⁺ /. (γi → 0)⁺}),
    λ === 0, {𝒞 ⋃ {∂_γ̄i q}, q /. (γi → 0)⁺}];
  CF@Σ_B[Most@{ri,li},bs][nσ,
    PQ[n𝒞, nq] /. (γ_Last@{ri,li} → γ_First@{ri,li})⁺] ]
```

Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.

Strand Operations. c for contract, mc for magnetic contract:

```
c_{i_,j_}@t : Σ_{B[{li___,i_,ri___},{___,j_,___},___]}[__] :=
 t // GT_{j,First@{ri,li}} // Cordon_j
```

```
c_{i_,j_}@t : Σ_{B[{___,i_,j_,___},___]}[__] := Cordon_j@t
c_{i_,j_}@t : Σ_{B[{j_,___,i_},___]}[__] := Cordon_j@t
c_{i_,j_}@t : Σ_{B[{___,j_,i_,___},___]}[__] := Cordon_i@t
c_{i_,j_}@t : Σ_{B[{i_,___,j_},___]}[__] := Cordon_i@t
```

```
mc[ℰ_] := ℰ //.
  t : Σ_{B[{___,i_,___},{___,j_,___},___]}[__] |
      Σ_{B[{___,i_,j_,___},___]}[__] | Σ_{B[{j_,___,i_},___]}[__] /;
    i + j == 0 :→ c_{i,j}@t
```

The Crossings (and empty strands).

```
Kas@P_{i_,j_} := CF@Σ_{B[{i,j}]}[0, PQ[{}, 0]];
TL@P_{i_,j_} := CF@Σ_{B[{i,j}]}[0, PQ[{}, 0]]
```

```
Kas[x : X[i_, j_, k_, l_]] :=
  Kas@If[PositiveQ[x], X_{-i,j,k,-l}, X̄_{-j,k,l,-i}];
Kas[(x : X | X̄)_{fs__}] := Module[{v = 2 u^2 - 1, p, γs, m},
  γs = γ_# & /@ {fs}; p = (x === X);
                 (v u 1 u)      (v u 1 u)
  m = If[p,      (u 1 u 1) , -  (u 1 u 1) ];
                 (1 u v u)      (1 u v u)
                 (u 1 u 1)      (u 1 u 1)
  CF@Σ_{B[{fs}]}[If[p, -1, 1], PQ[{}, γs*.m.γs]]]
```

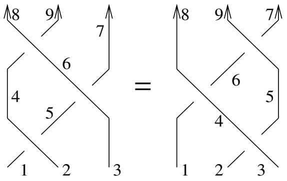
```
TL[x : X[i_, j_, k_, l_]] :=
  TL@If[PositiveQ[x], X_{-i,j,k,-l}, X̄_{-j,k,l,-i}];
TL[(x : X | X̄)_{fs__}] := Module[{t = 1 - ω, r, γs, m},
  r = t + t*; γs = γ_# & /@ {fs};
  m = If[x === X,
     ( -r   -t   2 t   t* )   (  r    -t  -2 t*   t* )
     ( -t*   0   t*    0  ) , ( -t*    0   t*     0  ) ];
     ( 2 t*  t   -r   -t* )   ( -2 t   t   r     -t* )
     (  t    0   -t    0  )   (  t     0  -t      0  )
  CF@Σ_{B[{fs}]}[0, PQ[{}, γs*.m.γs]]]
```

Evaluation on Tangles and Knots.

```
Kas[K_] := Fold[mc[#1⊕#2] &, Σ_{B[]}[0, PQ[{}, 0]],
   List @@ (Kas /@ PD@K)];
KasSig[K_] := Expand[Kas[K]⟦1⟧ / 2]
```

```
TL[K_] :=
  Fold[mc[#1⊕#2] &, Σ_{B[]}[0, PQ[{}, 0]],
    List @@ (TL /@ PD@K)] /.
   θ[c_ + u] /; Abs[c] ≥ 1 :→ θ[c];
TLSig[K_] := TL[K]⟦1⟧
```

Reidemeister 3.



```
R3L = PD[X_{-2,5,4,-1}, X_{-3,7,6,-5},
   X_{-6,9,8,-4}];
R3R = PD[X_{-3,5,4,-2}, X_{-4,6,8,-1},
   X_{-5,7,9,-6}];
{TL@R3L == TL@R3R, Kas@R3L == Kas@R3R}
```

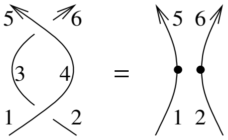
```
{True, True}
```

```
Kas@R3L
```

$$2\,\theta\left(u-\tfrac{1}{2}\right)-2\,\theta\left(u+\tfrac{1}{2}\right)-2$$

$$\begin{array}{ccccccc}
 & (\gamma_{-3} & \gamma_7 & \gamma_9 & \gamma_8 & \gamma_{-1} & \gamma_{-2}) \\
\bar\gamma_{-3} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} \\
\bar\gamma_{7} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2(2u^2-1)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} \\
\bar\gamma_{9} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} \\
\bar\gamma_{8} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} \\
\bar\gamma_{-1} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2(2u^2-1)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} \\
\bar\gamma_{-2} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}
\end{array}$$

Reidemeister 2.



```
TL@PD[X_{-2,4,3,-1}, X̄_{-4,6,5,-3}]
```

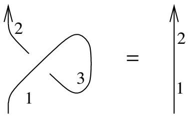
$$0$$

$$\begin{array}{ccccc}
 & 1 & 0 & -1 & 0 \\
 & (\gamma_{-2} & \gamma_6 & \gamma_5 & \gamma_{-1}) \\
\bar\gamma_{-2} & 0 & 0 & 0 & 0 \\
\bar\gamma_{6} & 0 & 0 & 0 & 0 \\
\bar\gamma_{5} & 0 & 0 & 0 & 0 \\
\bar\gamma_{-1} & 0 & 0 & 0 & 0
\end{array}$$

```
{TL@PD[X_{-2,4,3,-1}, X̄_{-4,6,5,-3}] == GT_{5,-2}@TL@PD[P_{-1,5}, P_{-2,6}],
 Kas@PD[X_{-2,4,3,-1}, X̄_{-4,6,5,-3}] == GT_{5,-2}@Kas@PD[P_{-1,5}, P_{-2,6}]}
```

```
{True, True}
```

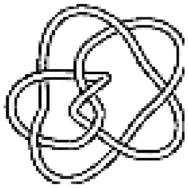
Reidemeister 1.



```
{TL@PD[X_{-3,3,2,-1}] == TL@P_{-1,2},
 Kas@PD[X_{-3,3,2,-1}] == Kas@P_{-1,2}}
```

```
{True, True}
```

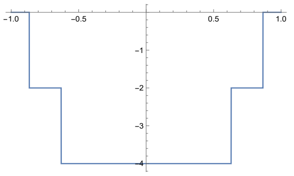
A Knot.



```
f = TLSig[Knot[8, 5]]
```

$$2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]-2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]-2\,\theta\left[u-\boxed{-0.630\ldots}\right]+2\,\theta\left[u-\boxed{0.630\ldots}\right]$$

```
Plot[f, {u, -1, 1}]
```



Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.

The Conway-Kinoshita-Terasaka Tangles.

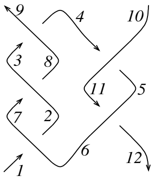
Conway

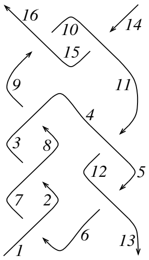
Kinoshita

Terasaka

$\mathrm{T1} = \mathrm{PD}\left[\bar{X}_{-6,2,7,-1}, \bar{X}_{-2,8,3,-7}, \bar{X}_{-8,4,9,-3}, X_{-11,6,12,-5}, X_{-4,11,5,-10}\right];$

$\mathrm{T2} = \mathrm{PD}\left[X_{-6,2,7,-1}, X_{-2,8,3,-7}, X_{-8,4,9,-3}, \bar{X}_{-12,6,13,-5}, \bar{X}_{-4,12,5,-11}, \bar{X}_{-10,15,11,-14}, \bar{X}_{-15,10,16,-9}\right];$





`Column@{TL[T1], Kas[T1]}`

$$-2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-1$$

$$\begin{array}{ccccc}
 & (\gamma_{-10} & \gamma_{9} & \gamma_{-1} & \gamma_{12}) \\
\bar\gamma_{-10} & 0 & 1-\omega & 0 & \omega-1 \\
\bar\gamma_{9} & \frac{\omega-1}{\omega} & \frac{2\,\omega}{\omega^2-\omega+1} & -\frac{\omega-1}{\omega} & -\frac{2\,\omega}{\omega^2-\omega+1} \\
\bar\gamma_{-1} & 0 & \omega-1 & 0 & 1-\omega \\
\bar\gamma_{12} & -\frac{\omega-1}{\omega} & -\frac{2\,\omega}{\omega^2-\omega+1} & \frac{\omega-1}{\omega} & \frac{2\,\omega}{\omega^2-\omega+1}
\end{array}$$

$$-2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-1$$

$$\begin{array}{ccccc}
 & (\gamma_{-10} & \gamma_{9} & \gamma_{-1} & \gamma_{12}) \\
\bar\gamma_{-10} & 2\,(u-1)\,(u+1)\,\left(4\,u^2-3\right) & 0 & -2\,(u-1)\,(u+1)\,\left(4\,u^2-3\right) & 0 \\
\bar\gamma_{9} & 0 & \frac{1}{2\,\left(4\,u^2-3\right)} & 0 & -\frac{1}{2\,\left(4\,u^2-3\right)} \\
\bar\gamma_{-1} & -2\,(u-1)\,(u+1)\,\left(4\,u^2-3\right) & 0 & 2\,(u-1)\,(u+1)\,\left(4\,u^2-3\right) & 0 \\
\bar\gamma_{12} & 0 & -\frac{1}{2\,\left(4\,u^2-3\right)} & 0 & \frac{1}{2\,\left(4\,u^2-3\right)}
\end{array}$$

`Column@{TL[T2], Kas[T2]}`

$$0$$

$$\begin{array}{ccccc}
 & (\gamma_{-14} & \gamma_{16} & \gamma_{-1} & \gamma_{13}) \\
\bar\gamma_{-14} & 0 & 1-\omega & 0 & \omega-1 \\
\bar\gamma_{16} & \frac{\omega-1}{\omega} & -\frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1} & -\frac{\omega-1}{\omega} & \frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1} \\
\bar\gamma_{-1} & 0 & \omega-1 & 0 & 1-\omega \\
\bar\gamma_{13} & -\frac{\omega-1}{\omega} & \frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1} & \frac{\omega-1}{\omega} & -\frac{2\,(\omega-1)^2\,\omega}{\omega^4-3\,\omega^3+5\,\omega^2-3\,\omega+1}
\end{array}$$

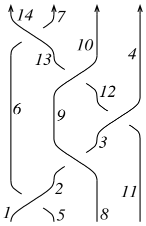
$$1$$

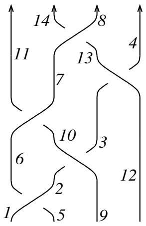
$$\begin{array}{ccccc}
 & (\gamma_{-14} & \gamma_{16} & \gamma_{-1} & \gamma_{13}) \\
\bar\gamma_{-14} & \frac{1}{2}\left(-16\,u^4+28\,u^2-13\right) & 0 & \frac{1}{2}\left(16\,u^4-28\,u^2+13\right) & 0 \\
\bar\gamma_{16} & 0 & -\frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13} & 0 & \frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13} \\
\bar\gamma_{-1} & \frac{1}{2}\left(16\,u^4-28\,u^2+13\right) & 0 & \frac{1}{2}\left(-16\,u^4+28\,u^2-13\right) & 0 \\
\bar\gamma_{13} & 0 & \frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13} & 0 & -\frac{2\,(u-1)\,(u+1)}{16\,u^4-28\,u^2+13}
\end{array}$$

Examples with non-trivial codimension.

$\mathrm{B1} = \mathrm{PD}\left[X_{-5,2,6,-1}, \bar{X}_{-8,3,9,-2}, X_{-11,4,12,-3}, X_{-12,10,13,-9}, \bar{X}_{-13,7,14,-6}\right];$

$\mathrm{B2} = \mathrm{PD}\left[X_{-5,2,6,-1}, \bar{X}_{-9,3,10,-2}, X_{-10,7,11,-6}, \bar{X}_{-12,4,13,-3}, X_{-13,8,14,-7}\right];$





`Column@{TL[B1], Kas[B1]}`

$$0$$

$$\begin{array}{ccccccccc}
 & 1 & 0 & -1 & 0 & \frac{1}{\omega} & 0 & -\frac{1}{\omega} & 0 \\
 & 0 & 0 & 0 & -1 & \frac{1}{\omega} & 0 & -\frac{1}{\omega} & 1 \\
 & (\gamma_{-11} & \gamma_{4} & \gamma_{10} & \gamma_{7} & \gamma_{14} & \gamma_{-1} & \gamma_{-5} & \gamma_{-8}) \\
\bar\gamma_{-11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{4} & 0 & 0 & 0 & 0 & \frac{\omega-1}{\omega^2} & 0 & -\frac{\omega-1}{\omega^2} & 0 \\
\bar\gamma_{10} & 0 & 0 & 0 & 0 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 \\
\bar\gamma_{7} & 0 & 0 & 0 & 0 & \frac{(\omega-1)^2}{\omega^2} & 0 & -\frac{(\omega-1)^2}{\omega^2} & 0 \\
\bar\gamma_{14} & 0 & -((\omega-1)\,\omega) & \omega-1 & (\omega-1)^2 & 0 & -\frac{\omega-1}{\omega} & \frac{\omega-1}{\omega} & 0 \\
\bar\gamma_{-1} & 0 & 0 & 0 & 0 & \omega-1 & 0 & 1-\omega & 0 \\
\bar\gamma_{-5} & 0 & (\omega-1)\,\omega & 1-\omega & -(\omega-1)^2 & 1-\omega & \frac{\omega-1}{\omega} & \frac{(\omega-1)^2}{\omega} & 0 \\
\bar\gamma_{-8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$$0$$

$$\begin{array}{ccccccccc}
 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\
 & (\gamma_{-11} & \gamma_{4} & \gamma_{10} & \gamma_{7} & \gamma_{14} & \gamma_{-1} & \gamma_{-5} & \gamma_{-8}) \\
\bar\gamma_{-11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{4} & 0 & 0 & 0 & -1 & -u & 0 & u & 1 \\
\bar\gamma_{10} & 0 & 0 & 0 & -u & 1-2\,u^2 & 0 & 2\,u^2-1 & u \\
\bar\gamma_{7} & 0 & -1 & -u & 2\,u^2-3 & -u & -1 & 0 & 1 \\
\bar\gamma_{14} & 0 & -u & 1-2\,u^2 & -u & -1 & -u & -2\,(u-1)\,(u+1) & u \\
\bar\gamma_{-1} & 0 & 0 & 0 & -1 & -u & 0 & u & 1 \\
\bar\gamma_{-5} & 0 & u & 2\,u^2-1 & 0 & -2\,(u-1)\,(u+1) & u & 4\,u^2-3 & 0 \\
\bar\gamma_{-8} & 0 & 1 & u & 1 & u & 1 & 0 & 1-2\,u^2
\end{array}$$

`Column@{TL[B2], Kas[B2]}`

$$0$$

$$\begin{array}{ccccccccc}
 & (\gamma_{-12} & \gamma_{4} & \gamma_{8} & \gamma_{14} & \gamma_{11} & \gamma_{-1} & \gamma_{-5} & \gamma_{-9}) \\
\bar\gamma_{-12} & \frac{(\omega-1)^2}{\omega} & \omega-1 & -2\,(\omega-1) & \frac{2\,(\omega-1)^2}{\omega} & \frac{2\,(\omega-1)}{\omega^2} & 0 & -\frac{2\,(\omega-1)}{\omega^2} & -\frac{(\omega-1)\,(2\,\omega-3)}{\omega} \\
\bar\gamma_{4} & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{8} & \frac{2\,(\omega-1)}{\omega} & 1-\omega & \frac{(\omega-1)^2}{\omega} & -\frac{(\omega-1)\,(2\,\omega-3)}{\omega} & -\frac{2\,(\omega-1)}{\omega^2} & 0 & \frac{2\,(\omega-1)}{\omega^2} & \frac{2\,(\omega-2)\,(\omega-1)}{\omega} \\
\bar\gamma_{14} & \frac{2\,(\omega-1)^2}{\omega} & 0 & -\frac{(\omega-1)\,(3\,\omega-2)}{\omega} & \frac{3\,(\omega-1)^2}{\omega} & -\frac{(\omega-2)\,(\omega-1)}{\omega^2} & 0 & -\frac{2\,(\omega-1)}{\omega^2} & -\frac{2\,(\omega-2)\,(\omega-1)}{\omega} \\
\bar\gamma_{11} & -2\,(\omega-1)\,\omega & 0 & 2\,(\omega-1)\,\omega & -((\omega-1)\,(2\,\omega-1)) & \frac{(\omega-1)^2}{\omega} & -\frac{\omega-1}{\omega} & \frac{2\,(\omega-1)}{\omega} & 2\,(\omega-1)^2 \\
\bar\gamma_{-1} & 0 & 0 & 0 & 0 & \omega-1 & 0 & 1-\omega & 0 \\
\bar\gamma_{-5} & 2\,(\omega-1)\,\omega & 0 & -2\,(\omega-1)\,\omega & 2\,(\omega-1)\,\omega & -2\,(\omega-1) & \frac{\omega-1}{\omega} & \frac{(\omega-1)^2}{\omega} & -((\omega-1)\,(2\,\omega-1)) \\
\bar\gamma_{-9} & -\frac{(\omega-1)\,(3\,\omega-2)}{\omega} & 0 & \frac{2\,(\omega-1)\,(2\,\omega-1)}{\omega} & -\frac{2\,(\omega-1)\,(2\,\omega-1)}{\omega} & \frac{2\,(\omega-1)^2}{\omega^2} & 0 & -\frac{(\omega-2)\,(\omega-1)}{\omega^2} & \frac{3\,(\omega-1)^2}{\omega}
\end{array}$$

$$2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)-2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)$$

$$\begin{array}{ccccccccc}
 & 1 & \frac{1}{2\,u} & 0 & -\frac{1}{2\,u} & -1 & -\frac{1}{2\,u} & 0 & \frac{1}{2\,u} \\
 & (\gamma_{-12} & \gamma_{4} & \gamma_{8} & \gamma_{14} & \gamma_{11} & \gamma_{-1} & \gamma_{-5} & \gamma_{-9}) \\
\bar\gamma_{-12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{4} & 0 & -\frac{(2\,u-1)\,(2\,u+1)\,\left(2\,u^2-1\right)}{4\,u^2\,\left(4\,u^2-3\right)} & -\frac{2\,u^2-1}{2\,u} & \frac{1}{4\,u^2\,\left(4\,u^2-3\right)} & 0 & -\frac{(2\,u-1)\,(2\,u+1)}{4\,u^2\,\left(4\,u^2-3\right)} & -\frac{1}{2\,u\,\left(4\,u^2-3\right)} & \frac{8\,u^4-6\,u^2-1}{4\,u^2\,\left(4\,u^2-3\right)} \\
\bar\gamma_{8} & 0 & -\frac{2\,u^2-1}{2\,u} & -2\,(u-1)\,(u+1) & \frac{2\,u^2-1}{2\,u} & 0 & -\frac{1}{2\,u} & 0 & \frac{1}{2\,u} \\
\bar\gamma_{14} & 0 & \frac{1}{4\,u^2\,\left(4\,u^2-3\right)} & \frac{2\,u^2-1}{2\,u} & \frac{\left(2\,u^2-1\right)\,\left(16\,u^4-16\,u^2+1\right)}{4\,u^2\,\left(4\,u^2-3\right)} & 0 & -\frac{8\,u^4-10\,u^2+1}{4\,u^2\,\left(4\,u^2-3\right)} & \frac{1}{2\,u\,\left(4\,u^2-3\right)} & \frac{1}{4\,u^2\,\left(4\,u^2-3\right)} \\
\bar\gamma_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{-1} & 0 & -\frac{(2\,u-1)\,(2\,u+1)}{4\,u^2\,\left(4\,u^2-3\right)} & -\frac{1}{2\,u} & -\frac{8\,u^4-10\,u^2+1}{4\,u^2\,\left(4\,u^2-3\right)} & 0 & \frac{8\,u^4-10\,u^2-1}{4\,u^2\,\left(4\,u^2-3\right)} & \frac{8\,u^4-10\,u^2+1}{2\,u\,\left(4\,u^2-3\right)} & \frac{16\,u^4-16\,u^2+1}{4\,u^2\,\left(4\,u^2-3\right)} \\
\bar\gamma_{-5} & 0 & -\frac{1}{2\,u\,\left(4\,u^2-3\right)} & 0 & \frac{1}{2\,u\,\left(4\,u^2-3\right)} & 0 & \frac{8\,u^4-10\,u^2+1}{2\,u\,\left(4\,u^2-3\right)} & \frac{2\,(u-1)\,(u+1)\,(2\,u-1)\,(2\,u+1)}{4\,u^2-3} & \frac{8\,u^4-6\,u^2-1}{2\,u\,\left(4\,u^2-3\right)} \\
\bar\gamma_{-9} & 0 & \frac{8\,u^4-6\,u^2-1}{4\,u^2\,\left(4\,u^2-3\right)} & \frac{1}{2\,u} & \frac{1}{4\,u^2\,\left(4\,u^2-3\right)} & 0 & \frac{16\,u^4-16\,u^2+1}{4\,u^2\,\left(4\,u^2-3\right)} & \frac{8\,u^4-6\,u^2-1}{2\,u\,\left(4\,u^2-3\right)} & -\frac{32\,u^6-64\,u^4+30\,u^2+1}{4\,u^2\,\left(4\,u^2-3\right)}
\end{array}$$

$$\begin{pmatrix} A & B \\ C & U \end{pmatrix} \xrightarrow{\det(A)} \begin{pmatrix} I & A^{-1}B \\ C & U \end{pmatrix} \xrightarrow{1} \begin{pmatrix} I & A^{-1}B \\ 0 & U-CA^{-1}B \end{pmatrix},$$

so $\det\begin{pmatrix} A & B \\ C & U \end{pmatrix} = \det(A)\det(U-CA^{-1}B)$.

Roughly, $\det(A)$ is “det on ker”, $-CA^{-1}B$ is “a pushforward of $\begin{pmatrix} A & B \\ C & U \end{pmatrix}$”. (what if $\nexists A^{-1}$?)

Questions. 1. Does this have a topological meaning? 2. Is there a version of the Kashaev Conjecture for tangles? 3. Find all solutions of R123 in our “algebra”. 4. Braids and the Burau representation. 5. Recover the work in “Prior Art”. 6. Are there any concordance properties? 7. What is the “SPQ group”? 8. The jumping points of signatures are the roots of the Alexander polynomial. Does this generalize to tangles? 9. Which of the three Cordon cases is the most common? 10. Are there interesting examples of tangles for which rels is non-trivial? 11. Is the *pq* part determined by Γ-calculus? 12. Is the *pq* part determined by finite type invariants? 13. Does it work with closed components / links? 14. Strand-doubling formulas? 15. A multivariable version? 16. Mutation invariance? 17. Ribbon knots? 18. Are there “face-virtual knots”? 19. Does the pushforward story extend to ranks? To formal Gaussian measures? To super Gaussian measures?

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Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.

Some Rigor. (Exercises hints and partial solutions at end)

Exercise 1. Show that if two SPQ's S_1 and S_2 on V satisfy $\sigma(S_1+U) = \sigma(S_2+U)$ for every quadratic U on V, then they have the same shifts and the same domains.

Exercise 2. Show that if two full quadratics Q_1 and Q_2 satisfy $\sigma(Q_1+U) = \sigma(Q_2+U)$ for every U, then $Q_1 = Q_2$.

Proof of Theorem 1'. Fix W and consider triples $(V, S, \phi\colon V \to W)$ where $S = (s, D, Q)$ is an SPQ on V. Say that two triples are "push-equivalent", $(V_1, S_1, \phi_1) \sim (V_2, S_2, \phi_2)$ if for every quadratic U on W,

$$\sigma_{V_1}(S_1 + \phi_1^* U) = \sigma_{V_2}(S_2 + \phi_2^* U).$$

Given our (V, S, ϕ), we need to show:

1. There is an SPQ S' on W such that $(V, S, \phi) \sim (W, S', I)$.
2. If $(W, S', I) \sim (W, S'', I)$ then $S' = S''$.

Property 2 is easy (Exercises 1, 2). Property 1 follows from the following three claims, each of which is easy.

Claim 1. If $v \in \ker\phi \cap D(S)$, and $\lambda := Q(v,v) \neq 0$, then $(V, S, \phi) \sim$

$$\Big(V/\langle v\rangle, \big(s+\operatorname{sign}(\lambda), D(S)/\langle v\rangle, Q-\lambda^{-1}Q(-,v)\otimes Q(v,-)\big), \phi/\langle v\rangle\Big).$$

So wlog $Q|_{\ker\phi} = 0$ (meaning, $Q|_{\ker\phi\otimes\ker\phi} = 0$). □

Claim 2. If $Q|_{\ker\phi} = 0$ and $v \in \ker\phi \cap D(S)$, let $V' = \ker Q(v,-)$ and then $(V, S, \phi) \sim (V', S|_{V'}, \phi|_{V'})$ so wlog $Q|_{V\otimes\ker\phi+\ker\phi\otimes V} = 0$. □

Claim 3. If $Q|_{V\otimes\ker\phi+\ker\phi\otimes V} = 0$ then $S = \phi^* S'$ for some SPQ S' on $\operatorname{im}\phi$ and then $(V, S, \phi) \sim (W, S', I)$. □ □

Proof of Theorem 2. The functoriality of pullbacks needs no proof. Now assume $V_0 \xrightarrow{\alpha} V_1 \xrightarrow{\beta} V_2$ and that S is an SPQ on V_0. Then for every SPQ U on V_2 we have, using reciprocity three times, that $\sigma(\beta_*\alpha_*S + U) = \sigma(\alpha_*S + \beta^*U) = \sigma(S + \alpha^*\beta^*U) = \sigma(S + (\beta\alpha)^*U) = \sigma((\beta\alpha)_*S + U)$. Hence $\beta_*\alpha_*S = (\beta\alpha)_*S$. □

Definition. A commutative square as on the right is called *admissible* if $\gamma^*\beta_* = \nu_*\mu^*$.

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \\ V \arrow[r, "\beta"'] \arrow[ur, dotted] & Z \end{tikzcd}$$

Lemma 1. If $V = W = Y = Z$ and $\beta = \gamma = \mu = \nu = I$, the square is admissible. □

Lemma 2. The following are equivalent:

1. A square as above is admissible.
2. The *Pairing Condition* holds. Namely, if S_1 is an SPQ on V (write $S_1 \vdash V$) and $S_2 \vdash W$, then $\sigma(\mu^*S_1 + \nu^*S_2) = \sigma(\beta_*S_1 + \gamma_*S_2)$.
3. The square is mirror admissible: $\beta^*\gamma_* = \mu_*\nu^*$.

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \dashv S_2 \arrow[d, "\gamma"] \\ S_1 \vdash V \arrow[r, "\beta"'] & Z \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \arrow[dl, dotted] \\ V \arrow[r, "\beta"'] & Z \end{tikzcd}$$

Proof. Using Exercises 1 and 2 below, and then using reciprocity on both sides, we have $\forall S_1\, \gamma^*\beta_*S_1 = \nu_*\mu^*S_1 \Leftrightarrow \forall S_1 \forall S_2\, \sigma(\gamma^*\beta_*S_1+S_2) = \sigma(\nu_*\mu^*S_1+S_2) \Leftrightarrow \forall S_1\forall S_2\, \sigma(\beta_*S_1+\gamma_*S_2) = \sigma(\mu^*S_1 + \nu^*S_2)$, and thus 1 $\Leftrightarrow$ 2. But the condition in 2 is symmetric under $\beta \leftrightarrow \gamma$, $\mu \leftrightarrow \nu$, so also 2 $\Leftrightarrow$ 3. □

Lemma 3. If the first diagram below is admissible, then so is the second.

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \\ V \arrow[r, "\beta"'] \arrow[ur, dotted] & Z \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma\oplus 0"] \\ V \arrow[r, "\beta\oplus 0"'] \arrow[ur, dotted] & Z\oplus F \end{tikzcd}$$

□

Lemma 4. A pushforward by an inclusion is the do nothing operation (though note that the pushforward via an inclusion of a fully defined quadratic retains its domain of definition, which now may become partial). □

Lemma 5. For any linear $\phi\colon V \to W$, the diagram on the right is admissible, where ι denotes the inclusion maps.

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[d, "\phi"'] & V\oplus C \arrow[d, "\phi\oplus I"] \\ W \arrow[r, "\iota"'] \arrow[ur, dotted] & W\oplus C \end{tikzcd}$$

Proof. Follows easily from Lemma 4. □

Definition. If S is an SPQ with domain D and quadratic Q, the radical of S is the radical of Q considered as a fully-defined quadratic on D. Namely, $\operatorname{rad} S := \{u \in D\colon \forall v \in D,\ Q(u,v) = 0\}$.

Lemma 6. Always, $\phi(\operatorname{rad} S) \subset \operatorname{rad} \phi_*S$.

Proof. Pick $w \in \phi(\operatorname{rad} S)$ and repeat the proof of Theorem 1' but now considering quadruples (V, S, ϕ, v), where (V, S, ϕ) are as before and $v \in \operatorname{rad} S$ satisfies $\phi(v) = w$. Clearly our initial triple (V, S, ϕ) can be extended to such a quadruple, and it is easy to repeat the steps of the proof of Theorem 1' extending everything to such quadruples. □

We have to acknowledge that our proof of Lemma 6 is ugly. We wish we had a cleaner one.

Exercise 3. Show that if two SPQ's S_1 and S_2 on $V \oplus A$ satisfy $A \subset \operatorname{rad} S_i$ and $\sigma(S_1 + \pi^*U) = \sigma(S_2 + \pi^*U)$ for every quadratic U on V, where $\pi\colon V\oplus A \to V$ is the projection, then $S_1 = S_2$.

Exercise 4. Show that if $\phi\colon V \to W$ is surjective and Q is a quadratic on W, then $\sigma(Q) = \sigma(\phi^*Q)$.

Exercise 5. Show that always, $\phi_*\phi^*S = S|_{\operatorname{im}\phi}$.

Lemma 7. For any linear $\phi\colon V \to W$, the diagram on the right is admissible, where $\phi^+ := \phi\oplus I$ and α and β denote the projection maps.

$$\begin{tikzcd} V\oplus C \arrow[r, "\phi^+"] \arrow[d, "\alpha"'] & W\oplus C \arrow[d, "\beta"] \\ V \arrow[r, "\phi"'] \arrow[ur, dotted] & W \end{tikzcd}$$

Proof. Let S be an SPQ on V. Clearly $C \subset \beta^*\phi_*S$. Also, $C \subset \operatorname{rad}\alpha^*S$ so by Lemma 6, $C = \phi^+(C) \subset \phi^+(\operatorname{rad}\alpha^*S) \subset \operatorname{rad}\phi^+_*\alpha^*S$. Hence using Exercise 3, it is enough to show that $\sigma(\phi^+_*\alpha^*S + \beta^*U) = \sigma(\beta^*\phi_*S + \beta^*U)$ for every U on W. Indeed, $\sigma(\phi^+_*\alpha^*S + \beta^*U) \overset{(1)}{=} \sigma(\beta_*\phi^+_*\alpha^*S + U) \overset{(2)}{=} \sigma(\phi_*\alpha_*\alpha^*S + U) \overset{(3)}{=} \sigma(\phi_*S + U) \overset{(4)}{=} \sigma(\beta^*(\phi_*S + U)) \overset{(5)}{=} \sigma(\beta^*\phi_*S + \beta^*U)$, using (1) reciprocity, (2) the commutativity of the diagram and the functoriality of pushing, (3) Exercise 5, (4) Exercise 4, and (5) the additivity of pullbacks. □

Lemma 8. If the first diagram below is admissible, then so are the other two.

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \\ V \arrow[r, "\beta"'] \arrow[ur, dotted] & Z \end{tikzcd} \qquad \begin{tikzcd} Y\oplus E \arrow[r, "\nu\oplus 0"] \arrow[d, "\mu\oplus I"'] & W \arrow[d, "\gamma"] \\ V\oplus E \arrow[r, "\beta\oplus 0"'] \arrow[ur, dotted] & Z \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\nu\oplus 0"] \arrow[d, "\mu"'] & W\oplus F \arrow[d, "\gamma\oplus I"] \\ V \arrow[r, "\beta\oplus 0"'] \arrow[ur, dotted] & Z\oplus F \end{tikzcd}$$

Proof. In the diagram

$$\begin{tikzcd} Y\oplus E \arrow[r, "\pi"] \arrow[d, "\mu\oplus I"'] & Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[r, "\iota"] \arrow[d, "\gamma"] & W\oplus F \arrow[d, "\gamma\oplus I"] \\ V\oplus E \arrow[r, "\pi"'] \arrow[ur, dotted] & V \arrow[r, "\beta"'] \arrow[ur, dotted] & Z \arrow[r, "\iota"'] \arrow[ur, dotted] & Z\oplus F, \end{tikzcd}$$

with π marking projections and ι inclusions, the left square is admissible by Lemma 7, the middle square by assumption, and the right square by Lemma 5. Along with the functoriality of pushforwards this shows the admissibility of both the left and the right 1×2 subrectangles, and these are the diagrams we wanted. □

Proof of Theorem 3. Decompose $Z = A\oplus B\oplus C\oplus D$, where $A = \operatorname{im}\beta\cap\operatorname{im}\gamma$, $\operatorname{im}\beta = A\oplus B$, and $\operatorname{im}\gamma = A\oplus C$. Write $V \simeq A\oplus B\oplus E$ with $\beta = I$ on $A\oplus B$ yet $\beta = 0$ on E, and write $W \simeq A\oplus C\oplus F$ with $\gamma = I$ on $A\oplus C$ yet $\gamma = 0$ on F. Then $Y = V\oplus_Z W \simeq A\oplus E\oplus F$ and our square is as shown on the right, with all maps equal to I on like-named summands and equal to 0 on non-like-named summands. But this diagram is admissible: build it up using Lemma 1 for the A's, and then Lemma 8 for E and C, and then again Lemma 8 along with the mirror property of Lemma 2 for B and F, and then Lemma 3 for D. □

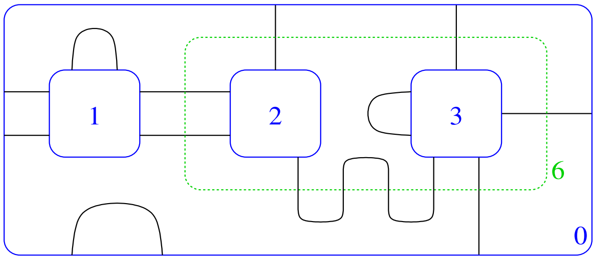
$$\begin{tikzcd} A\oplus E\oplus F \arrow[r] \arrow[d] & A\oplus C\oplus F \arrow[d] \\ A\oplus B\oplus E \arrow[r] \arrow[ur, dotted] & A\oplus B\oplus C\oplus D \end{tikzcd}$$

To prove Theorem 4, given three[1] SPQ's S_1, S_2, and S_3, we need to show that planar-multiplying them in two steps, first using a planar connection diagram D_I (I for Inner) to yield $S_6 = \mathcal{S}(D_I)(S_2, S_3)$ and then using a second planar connection diagram D_O (O for Outer) to yield $\mathcal{S}(D_O)(S_1, S_6)$, gives the same answer as multiplying them all at once using the composition planar connection diagram $D_B = D_O \circ_6 D_I$ (B for Big) to yield $\mathcal{S}(D_B)(S_1, S_2, S_3)$.[2] An example should help:

---

[1] Truly, we need the same for any number of input SPQ's that are divided into two groups, "multiply in the first step" and "multiply in the second step". But there's no added difficulty here, only an added notational complexity.

[2] Aren't we sassy? We picked "6" for the name of the product of "2" and "3".



In this example, if you ignore the dotted green line (marked "6"), you see the planar connection diagram D_B, which has three inputs (1,2,3) and a single output, the cycle 0. If you only look inside the green line, you see D_I, with inputs 2 and 3 and an output cycle 6. If you ignore the inside of 6 you see D_O, with inputs 1 and 6 and output cycle 0.

$$(MD)\qquad \begin{tikzcd} & & & G_0 \\ & & F_B \arrow[ur, "\phi"] \arrow[r, "\nu"] \arrow[dll, "\psi"'] \arrow[d, "\mu"] & F_O \arrow[u, "\delta"'] \arrow[d, "\gamma"] \\ G_1\oplus G_2\oplus G_3 & & G_1\oplus F_I \arrow[ll, "\alpha_+"] \arrow[r, "\beta^+"'] & G_1\oplus G_6 \end{tikzcd}$$

Let F_B (Big Faces) denote the vector space whose basis are the faces of D_B, let F_I (Inner Faces) be the space of faces of D_I, and let F_O (Outer Faces) be the space of faces of D_O. Let G_1, G_2, G_3, G_6, and G_0 be the spaces of gaps (edges) along the cycles 1,2,3,6, and 0, respectively. Let $\psi := \psi_{D_B}$ and $\phi := \phi^{D_B}$ be the maps defining $\mathcal{S}(D_B)$ and let $\gamma := \psi_{D_O}$ and $\delta := \phi^{D_O}$ be the maps defining $\mathcal{S}(D_O)$. Further, let $\alpha := \psi_{D_I}\colon F_I \to G_2\oplus G_3$ and $\beta := \phi^{D_I}\colon F_I \to G_6$ be the maps defining $\mathcal{S}(D_I)$, and let $\alpha_+ := I\oplus\alpha$ and $\beta^+ := I\oplus\beta$ be the extensions of α and β by an identity on an extra factor of G_1, so that $\beta^+_*\alpha^*_+ = I_{G_1}\oplus\mathcal{S}(D_I)$. Let μ map any big face to the sum of G_1 gaps around it, plus the sum of the inner faces it contains. Let ν map any big face to the sum of the outer faces it contains. It is easy to see that the master diagram (MD) shown on the right, made of all of these spaces and maps, is commutative.

$$\begin{tikzcd} F_B \arrow[r, "\nu"] \arrow[d, "\mu"] & F_O \arrow[d, "\gamma"] \\ G_1\oplus F_I \arrow[r, "\beta^+"'] & G_1\oplus G_6 \end{tikzcd}$$

Claim. The bottom right square of (MD) is an equalizer square, namely $F_B \simeq EQ(\beta^+,\gamma)$. Hence $\nu_*\mu^* = \gamma^*\beta^+_*$.

Proof. A big face (an element of F_B) is a sum of outer faces f_o and a sum of inner faces f_i, and it has a boundary g_1 on input cycle 1, such that the boundary of the outer pieces f_o is equal to the boundary of the inner pieces f_i plus g_1. That matches perfectly with the definition of the equalizer: $EQ(\beta^+,\gamma) = \{(g_1,f_i,f_o)\colon \beta^+(g_1,f_i) = \gamma(f_o)\} = \{(g_1,f_i,f_o)\colon \gamma(f_o) = (g_1,\beta(f_i))\}$. □

Proof of Theorem 4. With notation as above, with the example above (which is general enough), and with the claim above, and also using functoriality, we have $\mathcal{S}(D_B) = \phi_*\psi^* = \delta_*\nu_*\mu^*\alpha^*_+ = \delta_*\gamma^*\beta^+_*\alpha^*_+ = \mathcal{S}(D_O)\circ(I_{G_1}\oplus\mathcal{S}(D_I))$, as required. □

Proof of Theorem 5. We need to verify the Reidemeister moves and that was done in the computational section, and the statement about the restriction to links, which is easy: simply assemble an n-crossing knot using an n-input planar connection diagram, and the formulas clearly match. □

Further Homework.

Exercise 6. By taking $U = 0$ in the reciprocity statement, prove that always $\sigma(\phi_*S) = \sigma(S)$. But that seems wrong, if $\phi = 0$. What saves the day?

Exercise 7. By taking $S = 0$ in the reciprocity statement, frove that always $\sigma(\phi^*U) = \sigma(U)$. But wait, this is nonsense! What went wrong?

Exercise 8. Given $\phi\colon V\to W$ and a subspace $D\subset V$, show that there is a unique subspace $\phi_*D\subset W$ such that for every quadratic Q on W, $\sigma(\phi^*Q|_D) = \sigma(Q|_{\phi_*D})$.

Exercise 9. When are diagrams as on the right equalizer diagrams? What then do we learn from Theorem 3?

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ V \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & W \arrow[d] \\ V \arrow[r] & 0 \end{tikzcd}$$

Exercise 10. There are 11 types or irreducible commutative squares:

$$\begin{tikzcd}[sep=small] 1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}, \begin{tikzcd}[sep=small] 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}, \begin{tikzcd}[sep=small] 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 1 \arrow[r] & 0 \end{tikzcd}, \begin{tikzcd}[sep=small] 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 1 \end{tikzcd}, \begin{tikzcd}[sep=small] 1 \arrow[r, "1"'] \arrow[d] & 1 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}, \begin{tikzcd}[sep=small] 0 \arrow[r] \arrow[d] & 1 \arrow[d, "1"'] \\ 0 \arrow[r] & 1 \end{tikzcd}, \begin{tikzcd}[sep=small] 0 \arrow[r] \arrow[d] & 1 \arrow[d, "1"'] \\ 0 \arrow[r] & 1 \end{tikzcd}, \begin{tikzcd}[sep=small] 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 1 \arrow[r, "1"] & 1 \end{tikzcd},$$

$$\begin{tikzcd}[sep=small] 0 \arrow[r] \arrow[d] & 1 \arrow[d, "1"'] \\ 1 \arrow[r, "1"] & 1 \end{tikzcd}, \begin{tikzcd}[sep=small] 1 \arrow[r, "1"] \arrow[d, "1"] & 1 \arrow[d] \\ 1 \arrow[r] & 0 \end{tikzcd}, \text{ and } \begin{tikzcd}[sep=small] 1 \arrow[r, "1"] \arrow[d, "1"] & 1 \arrow[d, "1"'] \\ 1 \arrow[r, "1"] & 1 \end{tikzcd}.$$

Show that pushing commutes with pulling for all but four of them. Compare with the statement of Theorem 3.

Exercise 11. Prove that a square is admissible iff it is an equalizer square, with an additional direct summand A added to the Y term, and with the maps μ and ν extended by 0 on A.

Exercise 12. Prove that the direct sum of two admissible squares is admissible. *Warning:* Harder than it seems! Not all quadratics on $V_1\oplus V_2$ are direct sums of quadratics on V_1 and on V_2.

Exercise 13. Given a quadratic Q on a space V, let π be the projection $V\to V/\operatorname{rad}(Q)$ and show that $\pi_*Q = Q/\operatorname{rad}(Q)$, with the obvious definition for the latter.

Exercise 14. Show that for any partial quadratic Q on a space W there exists a space A and a fully-defined quadratic F on $W\oplus A$ such that $\pi_*F = Q$, where $\pi\colon W\oplus A\to W$ is the projection (these are not unique). Furthermore, if $\phi\colon V\to W$, then $\phi^*Q = \pi_*\phi^*_+F$, where $\phi_+ = \phi\oplus I\colon V\oplus A\to W\oplus A$ and π also denotes the projection $V\oplus A\to V$.

Solutions / Hints.

Hint for 1. On a vector in the domain of one but not the other, take an outrageous value for U, that will raise or lower the signature.

Hint for 2. WLOG, Q_1 is diagonal and $Q_1 = 0$.

Hint for 5. It's enough to test that against U with $\mathcal{D}(U) = \operatorname{im}\phi$.

Hint for 6. The "shift" part of 0_*S is $\sigma(S)$.

Hint for 7. ϕ_*S isn't 0, it's the *partial* quadratic "0 on $\operatorname{im}\phi$" (and indeed, $\sigma(\phi^*(U)) = \sigma(U)$ if ϕ is surjective).

Hint for 10. The exceptions are $\begin{smallmatrix}10\\00\end{smallmatrix}$, $\begin{smallmatrix}00\\01\end{smallmatrix}$, $\begin{smallmatrix}01\\11\end{smallmatrix}$, and $\begin{smallmatrix}11\\01\end{smallmatrix}$.

Hint for 12. Use Exercise 11.

Video: `http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201`. Handout: `http://www.math.toronto.edu/~drorbn/Talks/USC-240205`.

Dror Bar-Natan: Talks: Tokyo-230911: Thanks for inviting me to UTokyo!

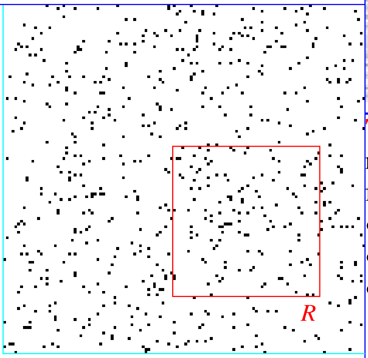
Rooting the BKT for FTI

ωεβ:=http://drorbn.net/tok2309

Abstract. Following joint work with Itai Bar-Natan, Iva Halacheva, and Nancy Scherich, I will show that the Best Known Time (BKT) to compute a typical Finite Type Invariant (FTI) of type d on a typical knot with n crossings is roughly equal to $n^{d/2}$, which is roughly the square root of what I believe was the standard belief before, namely about n^d.

Conventions. • $\underline{n} := \{1, 2, \ldots, n\}$. • For complexity estimates we ignore constant and logarithmic terms: $n^3 \sim 2023d!(\log n)^d n^3$.

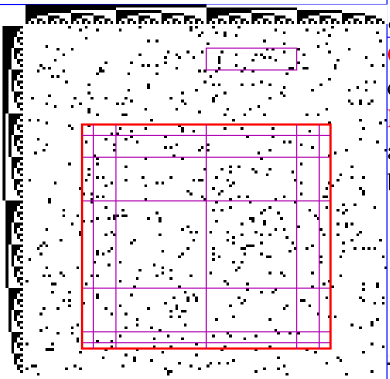
A Key Preliminary. Let $Q \subset \underline{n}^l$ be an enumerated subset, with $1 \ll q = |Q| \ll n^l$. In time $\sim q$ we can set up a lookup table of size $\sim q$ so that we will be able to compute $|Q \cap R|$ in time ~ 1, for any rectangle $R \subset \underline{n}^l$.



Fails. • Count after R is presented. • Make a lookup table of $|Q \cap R|$ counts for all R's.

Unfail. Make a restricted lookup table of the form

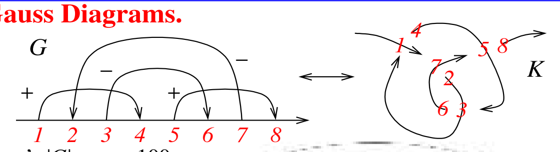
$$\left\{ \underset{\text{dyadic}}{R} \to \underset{>0}{|Q \cap R|} \right\}.$$



- Make the table by running through $x \in Q$, and for each one increment by 1 only the entries for dyadic $R \ni x$ (or create such an entry, if it didn't exist already). This takes $q \cdot (\log_2 n)^l \sim q$ ops.
- Entries for empty dyadic R's are not needed and not created.
- Using standard sorting techniques, access takes $\log_2 q \sim 1$ ops.
- A general R is a union of at most $(2 \log_2 n)^l \sim 1$ dyadic ones, so counting $|Q \cap R|$ takes ~ 1 ops.

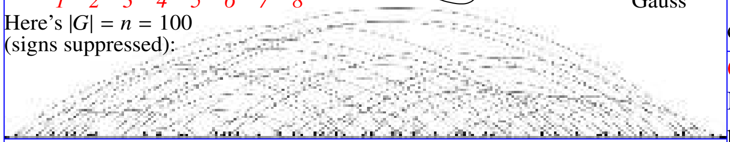
Generalization. Without changing the conclusion, replace counts $|Q \cap R|$ with summations $\sum_R \theta$, where $\theta \colon \underline{n}^l \to V$ is supported on a sparse Q, takes values in a vector space V with $\dim V \sim 1$, and in some basis, all of its coefficients are "easy".

Gauss Diagrams.



Gauss

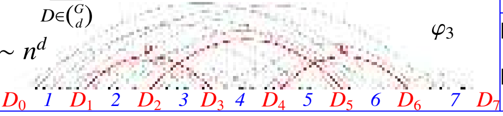
Here's $|G| = n = 100$ (signs suppressed):



Definitions. Let $\mathcal{G} := \mathbb{Q}\langle\text{Gauss Diagrams}\rangle$, with $\mathcal{G}_d$ / $\mathcal{G}_{\leq d}$ the diagrams with exactly / at most d arrows. Let $\varphi_d \colon \mathcal{G} \to \mathcal{G}_d$ be

$$\varphi_d \colon G \mapsto \sum_{D \subset G,\ |D|=d} D = \sum_{D \in \binom{G}{d}} D,$$ and let $\varphi_{\leq d} = \sum_{e \leq d} \varphi_e$.

Naively, it takes $\binom{n}{d} \sim n^d$ ops to compute φ_d.



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My Primary Interest. Strong, fast, homomorphic knot and tangle invariants. ωεβ/Nara, ωεβ/Kyoto, ωεβ/Tokyo

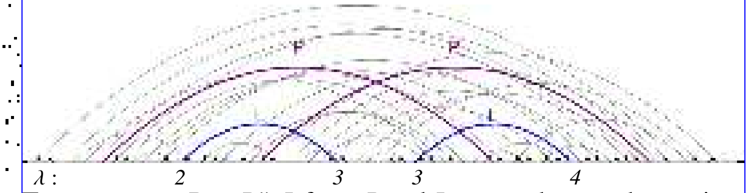
The [GPV] Theorem. A knot invariant is finite type of type d iff it is of the form $\omega \circ \varphi_{\leq d}$ for some $\omega \in \mathcal{G}^*_{\leq d}$.

Goussarov-Polyak-Viro

- $\Leftarrow$ is easy; $\Rightarrow$ is hard and IMHO not well understood.
- $\varphi_{\leq d}$ is not an invariants and not every ω gives an invariant!
- The theory of finite type invariants is very rich. Many knot invariants factor through finite type invariants, and it is possible that they separate knots.
- We need a fast algorithm to compute $\varphi_{\leq d}$!

Our Main Theorem. On an n-arrow Gauss diagram, φ_d can be computed in time $\sim n^{\lceil d/2 \rceil}$.

Proof. With $d = p + l$ (p for "put", l for "lookup"), pick p arrows and look up in how many ways the remaining l can be placed in between the legs of the first p:



To reconstruct $D = P\#_\lambda L$ from P and L we need a non-decreasing "placement function" $\lambda \colon \underline{2l} \to \underline{2p+1}$.

$$\varphi_d(G) = \sum_{D \in \binom{G}{d}} D = \binom{d}{p}^{-1} \sum_{P \in \binom{G}{p}} \sum_{\substack{\text{non-decreasing} \\ \lambda \colon \underline{2l} \to \underline{2p+1}}} \sum_{\substack{L \in \binom{G}{l} \\ L_i \in (P_{\lambda(i)-1}, P_{\lambda(i)})}} P\#_\lambda L$$

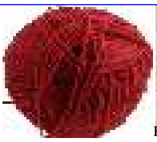
Define $\theta_G \colon \underline{2n}^{2l} \to \mathcal{G}_l$ by

$$(L_1, \ldots, L_{2l}) \mapsto \begin{cases} L & \text{if } (L_1, \ldots, L_{2l}) \text{ are the ends of some } L \subset G \\ 0 & \text{otherwise} \end{cases}$$

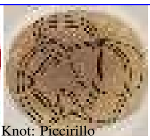
and now $$\varphi_d(G) = \binom{d}{p}^{-1} \sum_{P \in \binom{G}{p}} \sum_{\substack{\text{non-decreasing} \\ \lambda \colon \underline{2l} \to \underline{2p+1}}} P\#_\lambda \left(\sum_{\prod_i (P_{\lambda(i)-1}, P_{\lambda(i)})} \theta_G \right)$$

can be computed in time $\sim n^p + n^l$. Now take $p = \lceil d/2 \rceil$. □

Question ([BBHS], ωεβ/Fields). For computations, planar projections are better than braids (as likely $l \sim n^{3/2}$). But are yarn balls better than planar projections (here likely $n \sim L^{4/3}$)?



Length L



Knot: Piccirillo

n crossings

length l

References.

[BBHS] D. Bar-Natan, I. Bar-Natan, I. Halacheva, and N. Scherich, *Yarn Ball Knots and Faster Computations,* J. of Appl. and Comp. Topology (to appear), arXiv:2108.10923.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Tokyo-230911/`

University of Toronto: Dror Bar-Natan: Talks: Nara-2308: Thanks for inviting me to Nara! ωεβ:=http://drorbn.net/na23

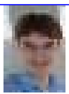
Cars, Interchanges, Traffic Counters, and some Pretty Darned Good Knot Invariants

More at ωεβ/APAI


Rozansky


Overbay


Ohtsuki


van der Veen

Abstract. Reporting on joint work with Roland van der Veen, I'll tell you some stories about ρ_1, an easy to define, strong, fast to compute, homomorphic, and well-connected knot invariant. ρ_1 was first studied by Rozansky and Overbay [Ro1, Ro2, Ro3, Ov] and Ohtsuki [Oh2], it has far-reaching generalizations, it is elementary and dominated by the coloured Jones polynomial, and I wish I understood it.

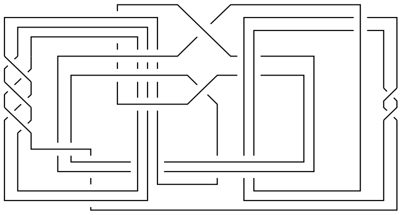
Common misconception. Dominated, elementary ⇒ lesser.

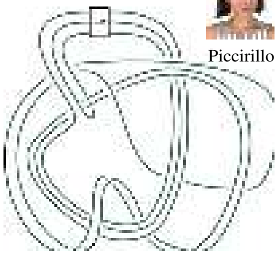
We seek strong, fast, homomorphic knot and tangle invariants.

Strong. Having a small "kernel".

Fast. Computable even for large knots (best: poly time).

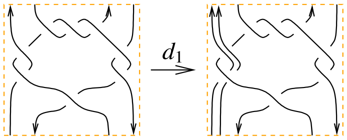

Piccirillo



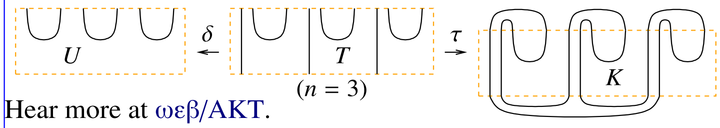


Gompf–Scharlemann–Thompson

Homomorphic. Extends to tangles and behaves under tangle operations; especially gluings and doublings:



Why care for "Homomorphic"? Theorem. A knot K is *ribbon* iff there exists a $2n$-component tangle T with skeleton as below such that $\tau(T) = K$ and where $\delta(T) = U$ is the *untangle*:



Hear more at ωεβ/AKT.

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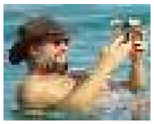
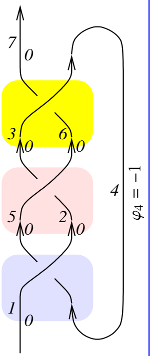
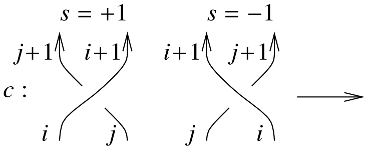


Jones:

Formulas stay;
interpretations change with time.



Formulas. Draw an n-crossing knot K as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \dots, 2n+1\}$ and with rotation numbers φ_k. Let A be the $(2n+1)\times(2n+1)$ matrix constructed by starting with the identity matrix I, and adding a 2×2 block for each crossing:



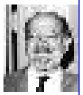
A	col $i+1$	col $j+1$
row i	$-T^s$	T^s-1
row j	0	-1

Let $G = (g_{\alpha\beta}) = A^{-1}$. For the trefoil example, it is:


Burau

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$


Alexander


Fox

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$


Wirtinger


"The Green Function"


Blanchfield

Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A), \qquad \text{with } \varphi = \sum_k \varphi_k,\ w = \sum_c s.$$

Classical Topologists: This is boring. Yawn.

Formulas, continued. Finally, set

$$R_1(c) := s\left(g_{ji}\left(g_{j+1,j} + g_{j,j+1} - g_{ij}\right) - g_{ii}\left(g_{j,j+1} - 1\right) - 1/2\right)$$

$$\rho_1 := \Delta^2\left(\sum_c R_1(c) - \sum_k \varphi_k\,(g_{kk} - 1/2)\right).$$

In our example $\rho_1 = -T^2 + 2T - 2 + 2T^{-1} - T^{-2}$.

Theorem. ρ_1 is a knot invariant. Proof: later.

Classical Topologists: Whiskey Tango Foxtrot?


Jones

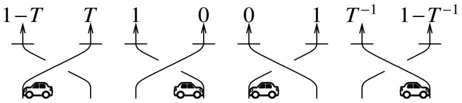

Lin


Tian


Wang

Cars, Interchanges, and Traffic Counters. Cars always drive forward. When a car crosses over a bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0^*$. At the very end, cars fall off and disappear. See also [Jo, LTW].


$p = 1 - T^s$



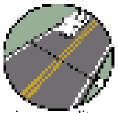


image credits: diamondtraffic.com

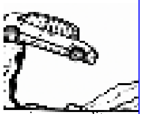


image credits: Dall-E

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* In algebra $x \sim 0$ if for every y in the ideal generated by x, $1 - y$ is invertible.

Video: `http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210`. Handout: `http://www.math.toronto.edu/~drorbn/Talks/Nara-2308`.

Preliminaries

This is Rho.nb of http://drorbn.net/oa22/ap.

```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/la22/ap
  to compute rotation numbers.
```

The Program

```
R_1[s_, i_, j_] :=
  s (g_{ji} (g_{j+,j} + g_{j,j+} - g_{ij}) - g_{ii} (g_{j,j+} - 1) - 1/2);
Z[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴
      (A〚{i, j}, {i + 1, j + 1}〛 += ( -T^s  T^s - 1
                                          0     -1   ))];
   Δ = T^((-Total[φ]-Total[Cs〚All,1〛])/2) Det[A];
   G = Inverse[A];
   ρ1 = Σ_{k=1}^{n} R_1 @@ Cs〚k〛 - Σ_{k=1}^{2 n} φ〚k〛 (g_{kk} - 1/2);
   Factor@
    {Δ, Δ^2 ρ1 /. α_^+ ⧴ α + 1 /. g_{α_,β_} ⧴ G〚α, β〛}];
```

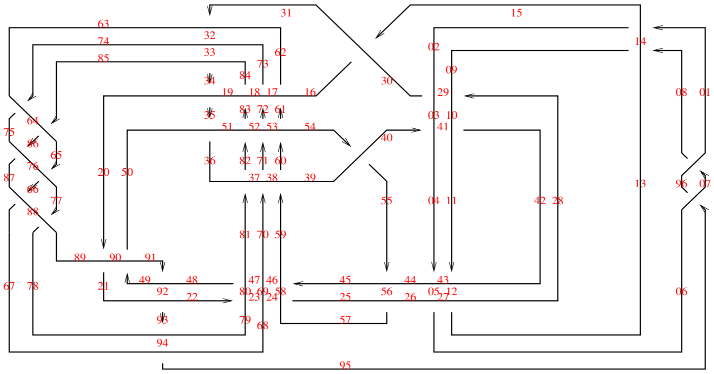
The First Few Knots

```
TableForm[Table[Join[{K〚1〛_{K〚2〛}}, Z[K]],
  {K, AllKnots[{3, 6}]}], TableAlignments → Center]
```

3_1	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2 \left(1+T^2\right)}{T^2}$
4_1	$-\frac{1-3 T+T^2}{T}$	0
5_1	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2 \left(1+T^2\right) \left(2+T^2+2 T^4\right)}{T^4}$
5_2	$\frac{2-3 T+2 T^2}{T}$	$\frac{(-1+T)^2 \left(5-4 T+5 T^2\right)}{T^2}$
6_1	$-\frac{(-2+T) (-1+2 T)}{T}$	$\frac{(-1+T)^2 \left(1-4 T+T^2\right)}{T^2}$
6_2	$-\frac{1-3 T+3 T^2-3 T^3+T^4}{T^2}$	$\frac{(-1+T)^2 \left(1-4 T+4 T^2-4 T^3+4 T^4-4 T^5+T^6\right)}{T^4}$
6_3	$\frac{1-3 T+5 T^2-3 T^3+T^4}{T^2}$	0

$p = 1 - T^s$

Fast!



```
Timing@
 Z[GST48 = EPD[X_{14,1}, X̄_{2,29}, X_{3,40}, X_{43,4}, X̄_{26,5}, X_{6,95},
    X_{96,7}, X_{13,8}, X̄_{9,28}, X_{10,41}, X_{42,11}, X̄_{27,12}, X_{30,15},
    X̄_{16,61}, X̄_{17,72}, X̄_{18,83}, X_{19,34}, X̄_{89,20}, X̄_{21,92},
    X̄_{79,22}, X̄_{68,23}, X̄_{57,24}, X̄_{25,56}, X_{62,31}, X_{73,32},
    X_{84,33}, X̄_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, X̄_{39,54}, X_{44,55},
    X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71},
    X_{53,60}, X̄_{63,74}, X̄_{64,85}, X̄_{76,65}, X̄_{87,66}, X̄_{67,94},
    X̄_{75,86}, X̄_{88,77}, X̄_{78,93}]]
```

$$\left\{170.313, \left\{-\frac{1}{T^8}\left(-1+2 T-T^2-T^3+2 T^4-T^5+T^8\right)\left(-1+T^3-2 T^4+T^5+T^6-2 T^7+T^8\right), \frac{1}{T^{16}}(-1+T)^2\left(5-18 T+33 T^2-32 T^3+2 T^4+42 T^5-62 T^6-8 T^7+166 T^8-242 T^9+108 T^{10}+132 T^{11}-226 T^{12}+148 T^{13}-11 T^{14}-36 T^{15}-11 T^{16}+148 T^{17}-226 T^{18}+132 T^{19}+108 T^{20}-242 T^{21}+166 T^{22}-8 T^{23}-62 T^{24}+42 T^{25}+2 T^{26}-32 T^{27}+33 T^{28}-18 T^{29}+5 T^{30}\right)\right\}\right\}$$

Strong!

```
{NumberOfKnots[{3, 12}],
 Length@
  Union@Table[Z[K], {K, AllKnots[{3, 12}]}],
 Length@
  Union@Table[{HOMFLYPT[K], Kh[K]},
    {K, AllKnots[{3, 12}]}]}
```

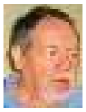
```
{2977, 2882, 2785}
```

So the pair (Δ, ρ_1) attains 2,882 distinct values on the 2,977 prime knots with up to 12 crossings (a deficit of 95), whereas the pair (HOMFLYPT, Khovanov Homology) attains only 2,785 distinct values on the same knots (a deficit of 192).

Hoste

Ocneanu

Millett

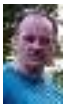


Freyd

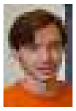
Lickorish

Yetter

Przytycki



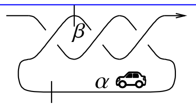
Traczyk



Khovanov

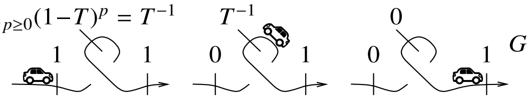
Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout: http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.

Theorem. The Green function $g_{\alpha\beta}$ is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point).

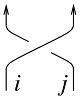


Example.

$\sum_{p\geq 0}(1-T)^p = T^{-1}$



$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



Proof. Near a crossing c with sign s, incoming upper edge i and incoming lower edge j, both sides satisfy the *g-rules*:

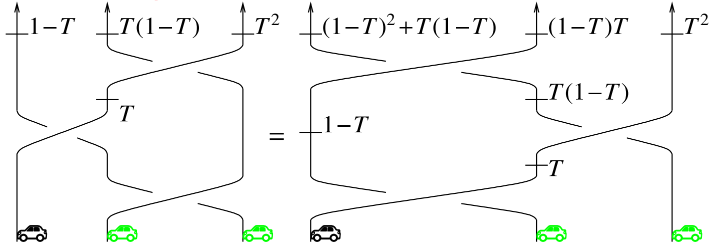
$$g_{i\beta} = \delta_{i\beta} + T^s g_{i+1,\beta} + (1-T^s) g_{j+1,\beta}, \quad g_{j\beta} = \delta_{j\beta} + g_{j+1,\beta},$$

and always, $g_{\alpha,2n+1} = 1$: use common sense and $AG = I\ (= GA)$.

Bonus. Near c, both sides satisfy the further *g-rules*:

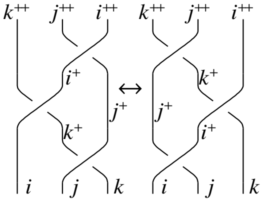
$$g_{\alpha i} = T^{-s}(g_{\alpha,i+1} - \delta_{\alpha,i+1}), \quad g_{\alpha j} = g_{\alpha,j+1} - (1-T^s)g_{\alpha i} - \delta_{\alpha,j+1}.$$

Invariance of ρ_1. We start with the hardest, Reidemeister 3:



⇒ Overall traffic patterns are unaffected by Reid3!

⇒ Green's $g_{\alpha\beta}$ is unchanged by Reid3, provided the cars injection site α and the traffic counters β are away.



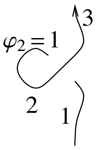
⇒ Only the contribution from the R_1 terms within the Reid3 move matters, and using g-rules the relevant $g_{\alpha\beta}$'s can be pushed outside of the Reid3 area:

```
δ_{i_,j_} := If[i === j, 1, 0];
gRules_{s_,i_,j_} :=
  {g_{iβ_} :→ δ_{iβ} + T^s g_{i+,β} + (1 - T^s) g_{j+,β}, g_{jβ_} :→ δ_{jβ} + g_{j+,β},
   g_{α_,i} :→ T^-s (g_{α,i+} - δ_{α,i+}),
   g_{α_j} :→ g_{α,j+} - (1 - T^s) g_{αi} - δ_{α,j+}}
lhs = R_1[1, j, k] + R_1[1, i, k^+] + R_1[1, i^+, j^+] //.
    gRules_{1,j,k} ⋃ gRules_{1,i,k+} ⋃ gRules_{1,i+,j+};
rhs = R_1[1, i, j] + R_1[1, i^+, k] + R_1[1, j^+, k^+] //.
    gRules_{1,i,j} ⋃ gRules_{1,i+,k} ⋃ gRules_{1,j+,k+};
Simplify[lhs == rhs]
True
```

Next comes Reid1, where we use results from an earlier example:

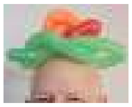
```
R_1[1, 2, 1] - 1 (g_22 - 1 / 2) /. g_{α_,β_} :→ (1 T^-1 1
                                                0 T^-1 1
                                                0  0   1)[[α, β]]
```

$$\frac{1}{T^2} - \frac{1}{T} - \frac{-1 + \frac{1}{T}}{T}$$



Invariance under the other moves is proven similarly.

Wearing my Topology hat the formula for R_1, and even the idea to look for R_1, remain a complete mystery to me.

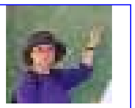


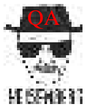

Heisenberg


Yang


Baxter


Drinfel'd


Lawrence



Wearing my Quantum Algebra hat, I spy a Heisenberg algebra $\mathbb{H} = A\langle p, x\rangle/([p,x]=1)$:

$$\text{cars} \leftrightarrow p \qquad \text{traffic counters} \leftrightarrow x$$

Where did it come from? Consider $\mathfrak{g}_\epsilon := sl_{2+}^\epsilon := L\langle y, b, a, x\rangle$ with relations

$$[b,x] = \epsilon x, \quad [b,y] = -\epsilon y, \quad [b,a] = 0,$$
$$[a,x] = x, \quad [a,y] = -y, \quad [x,y] = b + \epsilon a.$$

At invertible ϵ, it is isomorphic to sl_2 plus a central factor, and it can be quantized à la Drinfel'd [Dr] much like sl_2 to get an algebra $QU = A\langle y, b, a, x\rangle$ subject to (with $q = \mathbb{e}^{\hbar\epsilon}$):

$$[b,a] = 0, \quad [b,x] = \epsilon x, \quad [b,y] = -\epsilon y,$$
$$[a,x] = x, \quad [a,y] = -y, \quad xy - qyx = \frac{1 - \mathbb{e}^{-\hbar(b+\epsilon a)}}{\hbar}.$$

Now QU has an R-matrix solving Yang-Baxter (meaning Reid3),

$$R = \sum_{m,n\geq 0} \frac{y^n b^m \otimes (\hbar a)^m (\hbar x)^n}{m![n]_q!}, \qquad ([n]_q! \text{ is a "quantum factorial"})$$

and so it has an associated "universal quantum invariant" à la Lawrence and Ohtsuki [La, Oh1], $Z_\epsilon(K) \in QU$.

Now $QU \cong \mathcal{U}(\mathfrak{g}_\epsilon)$ (only as algebras!) and $\mathcal{U}(\mathfrak{g}_\epsilon)$ represents into $\mathbb{H}$ via

$$y \to -tp - \epsilon\cdot xp^2, \quad b \to t + \epsilon\cdot xp, \quad a \to xp, \quad x \to x,$$

(abstractly, $\mathfrak{g}_\epsilon$ acts on its Verma module

$$\mathcal{U}(\mathfrak{g}_\epsilon)/(\mathcal{U}(\mathfrak{g}_\epsilon)\langle y, a, b - \epsilon a - t\rangle) \cong \mathbb{Q}[x]$$

by differential operators, namely via $\mathbb{H}$), so R can be pushed to $\mathcal{R} \in \mathbb{H}\otimes\mathbb{H}$.

Everything still makes sense at $\epsilon = 0$ and can be expanded near $\epsilon = 0$ resulting with $\mathcal{R} = \mathcal{R}_0(1 + \epsilon\mathcal{R}_1 + \cdots)$, with $\mathcal{R}_0 = \mathbb{e}^{t(xp\otimes 1 - x\otimes p)}$ and $\mathcal{R}_1$ a quartic polynomial in p and x. So p's and x's get created along K and need to be pushed around to a standard location ("normal ordering"). This is done using

$$(p\otimes 1)\mathcal{R}_0 = \mathcal{R}_0(T(p\otimes 1) + (1-T)(1\otimes p)),$$
$$(1\otimes p)\mathcal{R}_0 = \mathcal{R}_0(1\otimes p),$$

and when the dust settles, we get our formulas for ρ_1. But QU is a quasi-triangular Hopf algebra, and hence **ρ_1 is homomorphic**. Read more at [BV1, BV2] and hear more at ωεβ/SolvApp, ωεβ/Dogma, ωεβ/DoPeGDO, ωεβ/FDA, ωεβ/AQDW.


Schaveling

Also, we can (and know how to) look at higher powers of ϵ and we can (and more or less know how to) replace sl_2 by arbitrary semi-simple Lie algebra (e.g., [Sch]). So **ρ_1 is not alone!**

These constructions are very similar to Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] and hence to the "loop expansion" of the Kontsevich integral and the coloured Jones polynomial [Oh2].

If this all reads like **insanity** to you, it should (and you haven't seen half of it). Simple things should have simple explanations. Hence, **Homework.** Explain ρ_1 with no reference to quantum voodoo and find it a topology home (large enough to house generalizations!). Make explicit the homomorphic properties of ρ_1. Use them to do topology!

P.S. As a friend of Δ, ρ_1 gives a genus bound, sometimes better than Δ's. How much further does this friendship extend?

Video: `http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210`. Handout: `http://www.math.toronto.edu/~drorbn/Talks/Nara-2308`.

A Small-Print Page on ρ_d, $d > 1$.

Definition. $\langle f(z_i), h(\zeta_i)\rangle_{\{z_i\}} := f(\partial_{\zeta_i})h\big|_{\zeta_i=0}$, so $\langle p^2x^2, \mathbb{e}^{g\pi\xi}\rangle = 2g^2$.

Baby Theorem. There exist (non unique) power series $r^\pm(p_1,p_2,x_1,x_2) = \sum_d \epsilon^d r_d^\pm(p_1,p_2,x_1,x_2) \in \mathbb{Q}[T^{\pm 1}, p_1, p_2, x_1, x_2][\![\epsilon]\!]$ with $\deg r_d^\pm \leq 2d+2$ ("docile") such that the power series $Z^b = \sum \rho_d^b \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i,p_j,x_i,x_j)\right), \exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}\pi_\alpha\xi_\beta\right)\right\rangle_{\{p_\alpha,x_\beta\}}$$

is a bnot invariant. Beyond the once-and-for-all computation of $g_{\alpha\beta}$ (a matrix inversion), Z^b is computable in $O(n^d)$ operations in the ring $\mathbb{Q}[T^{\pm 1}]$.

(Bnots are knot diagrams modulo the braid-like Reidemeister moves, but not the cyclic ones).

Theorem. There also exist docile power series $\gamma^\varphi(\bar p, \bar x) = \sum_d \epsilon^d \gamma_d^\varphi \in \mathbb{Q}[T^{\pm 1}, \bar p, \bar x][\![\epsilon]\!]$ such that the power series $Z = \sum \rho_d \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i,p_j,x_i,x_j) + \sum_k \gamma^{\varphi_k}(\bar p_k, \bar x_k)\right), \right.$$
$$\left. \exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}(\pi_\alpha+\bar\pi_\alpha)(\xi_\beta+\bar\xi_\beta) + \sum_\alpha \pi_\alpha\bar\xi_\alpha\right)\right\rangle_{\{p_\alpha,\bar p_\alpha,x_\beta,\bar x_\beta\}}$$

is a knot invariant, as easily computable as Z^b.

Implementation. Data, then program (with output using the Conway variable $z = \sqrt{T} - 1/\sqrt{T}$), and then a demo. See `Rho.nb` of ωεβ/ap.

```
V@γ1,φ_[k_] = φ (1/2 - p̄_k x̄_k); V@γ2,φ_[k_] = -φ^2 p̄_k x̄_k/2;
V@γ3,φ_[k_] := -φ^3 p̄_k x̄_k/6
```

```
V@r1,s_[i_, j_] :=
 s (-1 + 2 p_i x_i - 2 p_j x_i + (-1 + T^s) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j)/2
```

```
V@r2,1[i_, j_] :=
 (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
   2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j -
   18 p_j^2 x_i x_j - 6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j -
   6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2)/12
```

```
V@r2,-1[i_, j_] :=
 (-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 -
   4 (-1 + T) T p_i^2 p_j x_i^3 + 2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 +
   18 T^2 p_i p_j x_i x_j - 18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j -
   6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2)/(12 T^2)
```

```
V@r3,1[i_, j_] :=
 (4 p_i x_i - 4 p_j x_i + 2 (5 + 7 T) p_i p_j x_i^2 - 2 (5 + 7 T) p_j^2 x_i^2 - 4 (-5 + 6 T) p_i^2 p_j x_i^3 +
   4 (-16 + 17 T + 2 T^2) p_i p_j^2 x_i^3 - 4 (-11 + 11 T + 2 T^2) p_j^3 x_i^3 + 3 (-1 + T) p_i^3 p_j x_i^4 -
   3 (-1 + T) (4 + 3 T) p_i^2 p_j^2 x_i^4 + (-1 + T) (13 + 22 T + T^2) p_i p_j^3 x_i^4 -
   (-1 + T) (4 + 13 T + T^2) p_j^4 x_i^4 - 28 p_i p_j x_i x_j + 28 p_j^2 x_i x_j + 36 p_i^2 p_j x_i^2 x_j -
   12 (9 + 2 T) p_i p_j^2 x_i^2 x_j + 24 (3 + T) p_j^3 x_i^2 x_j - 4 p_i^3 p_j x_i^3 x_j + 28 T p_i^2 p_j^2 x_i^3 x_j -
   4 (-6 + 17 T + T^2) p_i p_j^3 x_i^3 x_j + 4 (-5 + 10 T + T^2) p_j^4 x_i^3 x_j + 24 p_i p_j^2 x_i x_j^2 -
   24 p_j^3 x_i x_j^2 - 24 p_i^2 p_j^2 x_i^2 x_j^2 + 6 (10 + T) p_i p_j^3 x_i^2 x_j^2 - 6 (6 + T) p_j^4 x_i^2 x_j^2 -
   4 p_i p_j^3 x_i x_j^3 + 4 p_j^4 x_i x_j^3)/24
```

```
V@r3,-1[i_, j_] :=
 (-4 T^3 p_i x_i + 4 T^3 p_j x_i - 2 T^2 (7 + 5 T) p_i p_j x_i^2 + 2 T^2 (7 + 5 T) p_j^2 x_i^2 -
   4 T^2 (-6 + 5 T) p_i^2 p_j x_i^3 + 4 T (-2 - 17 T + 16 T^2) p_i p_j^2 x_i^3 -
   4 T (-2 - 11 T + 11 T^2) p_j^3 x_i^3 + 3 (-1 + T) T^2 p_i^3 p_j x_i^4 - 3 (-1 + T) T (3 + 4 T) p_i^2 p_j^2 x_i^4 +
   (-1 + T) (1 + 22 T + 13 T^2) p_i p_j^3 x_i^4 - (-1 + T) (1 + 13 T + 4 T^2) p_j^4 x_i^4 +
   28 T^3 p_i p_j x_i x_j - 28 T^3 p_j^2 x_i x_j - 36 T^3 p_i^2 p_j x_i^2 x_j + 12 T^2 (2 + 9 T) p_i p_j^2 x_i^2 x_j -
   24 T^2 (1 + 3 T) p_j^3 x_i^2 x_j + 4 T^3 p_i^3 p_j x_i^3 x_j - 28 T^2 p_i^2 p_j^2 x_i^3 x_j -
   4 T (-1 - 17 T + 6 T^2) p_i p_j^3 x_i^3 x_j + 4 T (-1 - 10 T + 5 T^2) p_j^4 x_i^3 x_j -
   24 T^3 p_i p_j^2 x_i x_j^2 + 24 T^3 p_j^3 x_i x_j^2 + 24 T^3 p_i^2 p_j^2 x_i^2 x_j^2 - 6 T^2 (1 + 10 T) p_i p_j^3 x_i^2 x_j^2 +
   6 T^2 (1 + 6 T) p_j^4 x_i^2 x_j^2 + 4 T^3 p_i p_j^3 x_i x_j^3 - 4 T^3 p_j^4 x_i x_j^3)/(24 T^3)
```

```
{p*, x*, p̄*, x̄*} = {π, ξ, π̄, ξ̄}; (z_i__)* := (z*)_i;
Zip{}[ℰ_] := ℰ;
Zip{z_,zs___}[ℰ_] :=
 (Collect[ℰ // Zip{zs}, z] /. f_. z^d_. :> (D[f, {z*, d}])) /. z* → 0
```

```
gPair[fs_, w_] :=
 gPair[fs, w] =
  Collect[ZipJoin@@Table[{p_α,p̄_α,x_α,x̄_α},{α,w}][
    (Times @@ (V /@ fs))
     Exp[Sum[g_α,β (π_α + π̄_α) (ξ_β + ξ̄_β), {α, w}, {β, w}] - Sum[ξ̄_α π_α, {α, w}]]],
   g__, Factor]
```

```
T2z[p_] := Module[{q = Expand[p], n, c},
   If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
    c z^(2 n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2 n)]]];
```

```
Z_d_[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, d1, Z1, Z2, Z3},
   {Cs, φ} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 ))];
                                                              (  0     -1    )
   {Δ, G} = Factor@{T^((-Total[φ]-Total[Cs[[All,1]]])/2) Det@A, Inverse@A};
   Z1 =
    Exp[Total[Cases[Cs, {s_, i_, j_} :> Sum[ϵ^d1 r_d1,s[i, j], {d1, d}]]] +
       Sum[ϵ^d1 γ_d1,φ[[k]][k], {k, 2 n}, {d1, d}] /. γ_,0[_] → 0];
   Z2 = Expand[F[{}, {}] × Normal@Series[Z1, {ϵ, 0, d}]] //.
     F[fs_, {es___}] × (f : (r | γ)_ps__[is__])^p_. :>
      F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}];
   Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
          Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
         ] /. g_α_,β_ :> G[[es[[α]], es[[β]]]]] ];
   Collect[{Δ, Z3 /. ϵ^p_. → p! Δ^(2 p) ϵ^p}, ϵ, T2z] ];
```

```
Z_2[GST48] (* takes a few minutes *)
```

```
{1 - 4 z^2 - 61 z^4 - 207 z^6 - 296 z^8 - 210 z^10 - 77 z^12 - 14 z^14 - z^16,
 1 + (38 z^2 + 255 z^4 + 1696 z^6 + 16 281 z^8 + 86 952 z^10 + 259 994 z^12 + 487 372 z^14 + 615 066 z^16 + 543 148 z^18 + 341 714 z^20 +
      153 722 z^22 + 48 983 z^24 + 10 776 z^26 + 1554 z^28 + 132 z^30 + 5 z^32) ϵ +
   (-8 - 484 z^2 + 9709 z^4 + 165 952 z^6 + 1 590 491 z^8 + 16 256 508 z^10 + 115 341 797 z^12 + 432 685 748 z^14 + 395 838 354 z^16 - 4 017 557 792 z^18 - 23 300 064 167 z^20 -
      70 082 264 972 z^22 - 142 572 271 191 z^24 - 209 475 503 700 z^26 - 221 616 295 209 z^28 - 151 502 648 428 z^30 - 23 700 199 243 z^32 +
      99 462 146 328 z^34 + 164 920 463 074 z^36 + 162 550 825 432 z^38 + 119 164 552 296 z^40 + 69 153 062 608 z^42 + 32 547 596 611 z^44 + 12 541 195 448 z^46 +
      3 961 384 155 z^48 + 1 021 219 696 z^50 + 212 773 106 z^52 + 35 264 208 z^54 + 4 537 548 z^56 + 436 600 z^58 + 29 536 z^60 + 1252 z^62 + 25 z^64) ϵ^2}
```

```
TableForm[Table[Join[{K[[1]]_K[[2]]}, Z_3[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center] (* takes a few minutes *)
```

```
3_1    1 + z^2              1 + (2 z^2 + z^4) ϵ + (2 - 4 z^2 + 3 z^4 + 4 z^6 + z^8) ϵ^2 + (-12 + 74 z^2 - 27 z^4 - 20 z^6 + 8 z^8 + 6 z^10 + z^12) ϵ^3
4_1    1 - z^2              1 + (-2 + 2 z^4) ϵ^2
5_1    1 + 3 z^2 + z^4      1 + (10 z^2 + 21 z^4 + 12 z^6 + 2 z^8) ϵ + (6 - 28 z^2 + 33 z^4 + 364 z^6 + 655 z^8 + 536 z^10 + 227 z^12 + 48 z^14 + 4 z^16) ϵ^2 + (-60 + 970 z^2 + 645 z^4 - 3380 z^6 - 3280 z^8 + 7470 z^10 + 19 475 z^12 + 20 536 z^14 + 12 564 z^16 + 4774 z^18 + 1109 z^20 + 144 z^22 + 8 z^24) ϵ^3
5_2    1 + 2 z^2            1 + (6 z^2 + 5 z^4) ϵ + (4 - 20 z^2 + 43 z^4 + 64 z^6 + 26 z^8) ϵ^2 + (-36 + 498 z^2 - 883 z^4 + 100 z^6 + 816 z^8 + 556 z^10 + 146 z^12) ϵ^3
6_1    1 - 2 z^2            1 + (-2 z^2 + z^4) ϵ + (-4 + 4 z^2 + 25 z^4 - 8 z^6 + 2 z^8) ϵ^2 + (12 + 154 z^2 - 223 z^4 - 608 z^6 + 100 z^8 - 52 z^10 + 10 z^12) ϵ^3
6_2    1 - z^2 - z^4        1 + (-2 z^2 - 3 z^4 + 2 z^6 + z^8) ϵ + (-2 - 4 z^2 + 29 z^4 + 28 z^6 + 42 z^8 - 8 z^10 - 2 z^12 + 4 z^14 + z^16) ϵ^2 + (12 + 166 z^2 + 155 z^4 - 194 z^6 - 2453 z^8 - 1622 z^10 - 1967 z^12 - 258 z^14 + 49 z^16 - 30 z^18 + z^20 + 6 z^22 + z^24) ϵ^3
6_3    1 + z^2 + z^4        1 + (2 + 8 z^2 - 16 z^6 - 24 z^8 - 16 z^10 - 2 z^12) ϵ^2
```

Video: `http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210`. Handout: `http://www.math.toronto.edu/~drorbn/Talks/Nara-2308`.

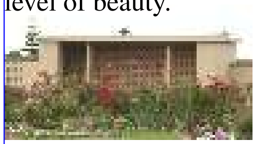
Computing the Zombian of an Unfinished Columbarium

Confession. It's about 50% of what I do.

Apology. It's a 20 minutes talk. Necessarily, it will be superficial.

Abstract. The zombies need to compute a quantity, the zombian, that pertains to some structure — say, a columbarium. But unfortunately (for them), a part of that structure will only be known in the future. What can they compute today with the parts they already have to hasten tomorrow's computation?

That's a common quest, and I will illustrate it with a few examples from knot theory and with two examples about matrices — determinants and signatures. I will also mention two of my dreams (perhaps delusions): that one day I will be able to reproduce, and extend, the Rolfsen table of knots using code of the highest level of beauty.




Columbaria in an East Sydney Cemetery

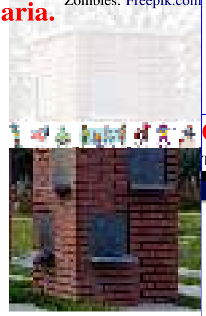
Image: Freepik.com

Jacobian, Hamiltonian, Zombian

Zombies: Freepik.com

Computing Zombians of Unfinished Columbaria.

- Future zombies must be able to complete the computation.
- Must be no slower than for finished ones.
- Future zombies must not even know the size of the task that today's zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!



Columbarium near Assen

Exercise 1. Compute the sum of 1,000 numbers, the last 50 of which are still unknown.

Exercise 2. Compute the determinant of a $1,000 \times 1,000$ matrix in which 50 entries are not yet given.

Example 3. Same, for signatures of matrices / quadratic forms.

A quadratic form on a v.s. V over $\mathbb{C}$ is a quadratic $Q\colon \underline{V} \to \mathbb{C}$, or a sesquilinear Hermitian $\langle\cdot,\cdot\rangle$ on $V\times V$ (so $\langle x,y\rangle = \overline{\langle y,x\rangle}$ and $Q(y) = \langle y,y\rangle$), or given a basis η_i of V^*, a matrix $A=(a_{ij})$ with $A=\bar{A}^T$ and $Q=\sum a_{ij}\bar\eta_i\eta_j$. The signature σ of Q is $\sigma_+ - \sigma_-$, where for some P, $\bar{P}^TAP = \operatorname{diag}(1,\overset{\sigma_+}{\ddots},1,-1,\overset{\sigma_-}{\ddots},-1,0,\ldots)$.

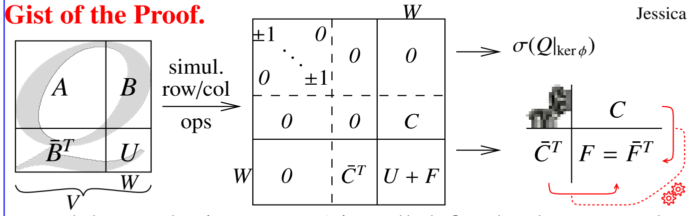
A *Partial Quadratic (PQ)* on V is a quadratic Q defined only on a subspace $\mathcal{D}_Q \subset V$. We add PQs with $\mathcal{D}_{Q_1+Q_2} := \mathcal{D}_{Q_1}\cap\mathcal{D}_{Q_2}$. Given a linear $\psi\colon V\to W$ and a PQ Q on W, there is an obvious *pullback* ψ^*Q, a PQ on V.

Jessica Liu

Theorem 1 (with Jessica Liu). Given a linear $\phi\colon V\to W$ and a PQ Q on V, there is a unique *pushforward* PQ ϕ_*Q on W such that for every *PQ* U on W,

$$\sigma_V(Q+\phi^*U) = \sigma_{\ker\phi}(Q|_{\ker\phi}) + \sigma_W(U+\phi_*Q).$$

Gist of the Proof.



...and the quadratic $F =: \phi_*Q$ is well-defined only on $D := \ker C$.

(more at ωεβ/icerm.)

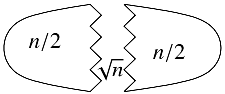
Acknowledgement. This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Knots and Tangles.


"Nautical Knots"

Image by macrovector on Freepik.com

Why Tangles? • As common as knots!



$2^{n/2}+2^{n/2}+2^{\sqrt{n}} \ll 2^n$

- Faster computations!
- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:

○ The Alexander polynomial ⇝ Zombian = det.

○ Knot signatures ⇝ Pushforwards of quadratic forms.

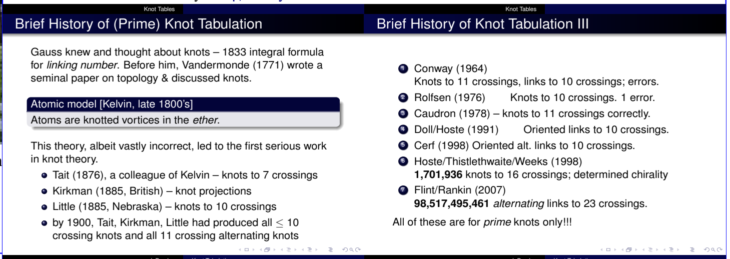
○ The Jones Polynomial ⇝ The Temperley-Lieb Algebra.

○ Khovanov Homology ⇝ "Unfinished complexes", complexes in a category.

○ The Kontsevich Integral ⇝ Drinfel'd Associators. ...

One more story is left to tell, of knot tabulation.

Two slides from R. Jason Parsley's ωεβ/history:



Knot Tables

Brief History of (Prime) Knot Tabulation

Gauss knew and thought about knots – 1833 integral formula for *linking number*. Before him, Vandermonde (1771) wrote a seminal paper on topology & discussed knots.

Atomic model [Kelvin, late 1800's]

Atoms are knotted vortices in the *ether*.

This theory, albeit vastly incorrect, led to the first serious work in knot theory.

- Tait (1876), a colleague of Kelvin – knots to 7 crossings
- Kirkman (1885, British) – knot projections
- Little (1885, Nebraska) – knots to 10 crossings
- by 1900, Tait, Kirkman, Little had produced all ≤ 10 crossing knots and all 11 crossing alternating knots

J. Parsley Knot Tabulation

Knot Tables

Brief History of Knot Tabulation III

1. Conway (1964) Knots to 11 crossings, links to 10 crossings; errors.
2. Rolfsen (1976) Knots to 10 crossings. 1 error.
3. Caudron (1978) – knots to 11 crossings correctly.
4. Doll/Hoste (1991) Oriented links to 10 crossings.
5. Cerf (1998) Oriented alt. links to 10 crossings.
6. Hoste/Thistlethwaite/Weeks (1998) **1,701,936** knots to 16 crossings; determined chirality
7. Flint/Rankin (2007) **98,517,495,461** *alternating* links to 23 crossings.

All of these are for *prime* knots only!!!

J. Parsley Knot Tabulation

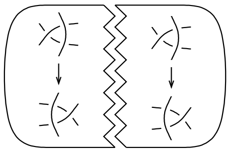
There's also Burton's tabulation to 19 crossings ωεβ/Burton, and Khesin's K250, arXiv:1705.10319.

Embarrassment 1 (personal). I don't know how to reproduce the Rolfsen table of knots! Many others can, yet I still take it on faith, contradicting one of the tenets of our practice, "thou shalt not use what thou canst not prove".

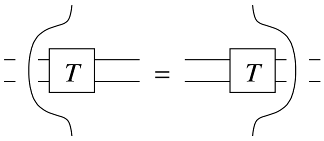
It's harder than it seems! Producing all knot diagrams is a mess, identifying all available Reidemeister moves is a mess, and you sometimes have to go up in crossing number before you can go down again.

Embarrassment 2 (communal). There isn't anywhere a tabulation of tangles! When you want to test your new discoveries, where do you go?

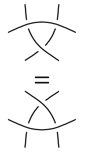
Dream. Conquer both embarrassments at once. Reproduce the Rolfsen table, and extend it to tangles, using code of the highest level of beauty. The algorithm should be so clear and simple that anyone should be able to easily implement it in an afternoon without messing with any technicalities.



We don't even need to look at all knot diagrams!



The dreaded slide moves, which go up in crossing number, are parametrized by tangles!



R-moves are tangle equalities!

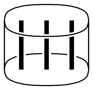
$\begin{pmatrix}A&B\\C&U\end{pmatrix}\xrightarrow{\det(A)}\begin{pmatrix}I&A^{-1}B\\C&U\end{pmatrix}\xrightarrow{1}\begin{pmatrix}I&A^{-1}B\\0&U-CA^{-1}B\end{pmatrix}$, so $\det\begin{pmatrix}A&B\\C&U\end{pmatrix}=\det(A)\det(U-CA^{-1}B)$.

Roughly, $\det(A)$ is "det on ker", $-CA^{-1}B$ is "a pushforward of $\begin{pmatrix}A&B\\C&U\end{pmatrix}$". (what if $\nexists A^{-1}$?)

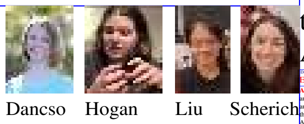
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ottawa-2306/

Dror Bar-Natan: Talks: LesDiablerets-2208: Thanks for inviting me to Les Diablerets! ωεβ:=http://drorbn.net/ld22/

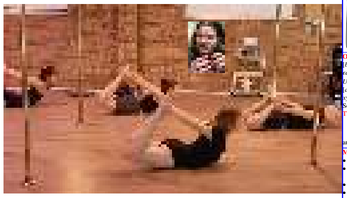
Tangles in a Pole Dance Studio: A Reading of Massuyeau, Alekseev, and Naef



Preliminary Definitions. Fix $p \in \mathbb{N}$ and $\mathbb{F} = \mathbb{Q}/\mathbb{C}$. Let $D_p := D^2\backslash(p \text{ pts})$, and let the Pole Dance Studio be $PDS_p := D_p \times I$.


Dancso Hogan Liu Scherich

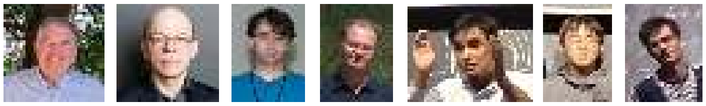
Abstract. I will report on joint work with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich. Little of what we do is original, and much of it is simply a reading of Massuyeau [Ma] and Alekseev and Naef [AN1].


Jessica, Nancy, Tamara, Zsuzsi, & Dror in PDS_4

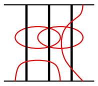
We study the pole-strand and strand-strand double filtration on the space of tangles in a pole dance studio (a punctured disk cross an interval), the corresponding homomorphic expansions, and a strand-only HOMFLY-PT relation. When the strands are transparent or nearly transparent to each other we recover and perhaps simplify substantial parts of the work of the aforementioned authors on expansions for the Goldman-Turaev Lie bi-algebra.

⟹ Expansions $W\colon FG\langle X_i\rangle \to FA\langle x_i\rangle$:
Magnus: $X_i \mapsto 1 + x_i$, $X_i^{-1} \mapsto 1 - x_i + x_i^2 - \ldots$
Exponential: $X_i^{\pm 1} \mapsto e^{\pm x_i}$

Definitions. Let $\pi := FG\langle X_1, \ldots, X_p\rangle$ be the free group (of deformation classes of based curves in D_p), $\bar\pi$ be the framed free group (deformation classes of based immersed curves), $|\pi|$ and $|\bar\pi|$ denote $\mathbb{F}$-linear combinations of cyclic words ($|x_i w| = |w x_i|$, unbased curves), $A := FA\langle x_1, \ldots, x_p\rangle$ be the free associative algebra, and let $|A| := A/(x_i w = w x_i)$ denote cyclic algebra words.



Theorem 1 (Goldman, Turaev, Massuyeau, Alekseev, Kawazumi, Kuno, Naef). $|\bar\pi|$ and $|A|$ are Lie bialgebras, and there is a "homomorphic expansion" $W\colon |\bar\pi| \to |A|$: a morphism of Lie bialgebras with $W(|X_i|) = 1 + |x_i| + \ldots$.

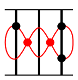


Further Definitions. • $\mathcal{K} = \mathcal{K}_0 = \mathcal{K}_0^0 = \mathcal{K}(S) := \mathbb{F}\langle\text{framed tangles in } PDS_p\rangle$.

- $\mathcal{K}_t^s$:=(the image via ⤫ → ↗↖ − ↖↗ of tangles in PDS_p that have t double points, of which s are strand-strand).

E.g.,

$$\mathcal{K}_5^2(\bigcirc) = \left\langle \right\rangle /.\ ⤫ \to ↗↖ - ↖↗$$



- $\mathcal{K}^{/s} := \mathcal{K}/\mathcal{K}^s$. Most important, $\mathcal{K}^{/1}(\bigcirc) = |\bar\pi|$, and there is $P\colon \mathcal{K}(\bigcirc) \to |\bar\pi|$.
- $\mathcal{A} := \prod \mathcal{K}_t/\mathcal{K}_{t+1}$, $\mathcal{A}^s := \prod \mathcal{K}_t^s/\mathcal{K}_{t+1}^s \subset \mathcal{A}$, $\mathcal{A}^{/s} := \mathcal{A}/\mathcal{A}^s$.

Fact 1. The Kontsevich Integral is an "expansion" $Z\colon \mathcal{K} \to \mathcal{A}$, compatible with several noteworthy structures.

Fact 2 (Le-Murakami, [LM1]). Z satisfies the strand-strand HOMFLY-PT relations: It descends to $Z_H\colon \mathcal{K}_H \to \mathcal{A}_H$, where

$$\mathcal{K}_H := \mathcal{K}\Big/\!\!\Big(↗↖ - ↖↗ = (e^{\hbar/2} - e^{-\hbar/2}) \cdot)(\Big)$$

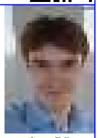
$$\mathcal{A}_H := \mathcal{A}/(\downarrow\!\!\vdash\!\!\dashv\!\!\uparrow = \hbar \asymp \text{ or } \uparrow\!\!\vdash\!\!\dashv\!\!\uparrow = \hbar \asymp)$$

and $\deg\hbar = (1, 1)$.

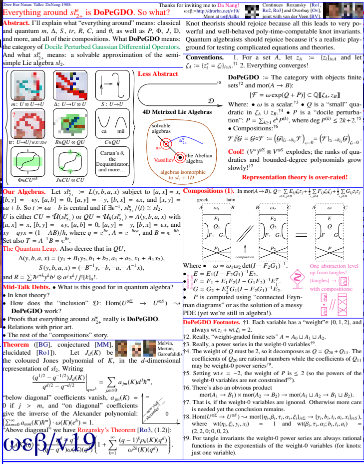
Proof of Fact 2. $Z(↗↖) - Z(↖↗) = ⤫ \cdot \left(e^{\vdash\!\dashv/2} - e^{-\vdash\!\dashv/2}\right)$

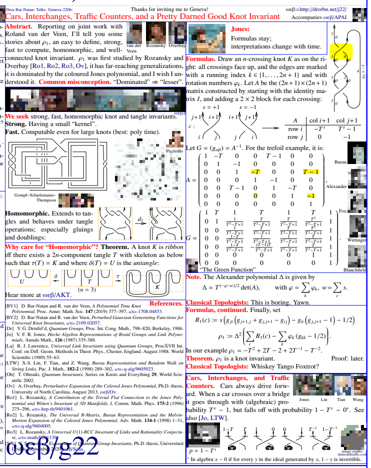
$$= ⤫ \cdot \left(e^{\hbar ⤫/2} - e^{-\hbar ⤫/2}\right) = \left(e^{\hbar/2} - e^{-\hbar/2}\right))(. \qquad \square$$


Le, Murakami


van der Veen

Other Passions. With Roland van der Veen, I use "solvable approximation" and "Perturbed Gaussian Differential Operators" to unveil simple, strong, fast to compute, and topologically meaningful knot invariants near the Alexander polynomial. ($\subset$ polymath!)





Key 1. $W\colon |\bar\pi| \to |A|$ is $Z_H^{/1}\colon \mathcal{K}_H^{/1}(\bigcirc) \to \mathcal{A}_H^{/1}(\bigcirc)$.

Key 2 (Schematic). Suppose $\lambda_0, \lambda_1\colon |\bar\pi| \to \mathcal{K}(\bigcirc)$ are two ways of lifting plane curves into knots in PDS_p (namely, $P \circ \lambda_i = I$). Then for $\gamma \in |\bar\pi|$,

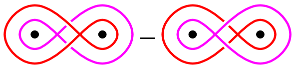
Lemma 1. "Division by $\hbar$" is well-defined.

$$\eta(\gamma) := (\lambda_0(\gamma) - \lambda_1(\gamma))/\hbar \in \mathcal{K}_H^{/1}(\bigcirc\bigcirc) = |\bar\pi| \otimes |\bar\pi|$$

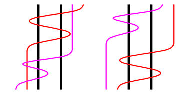
and we get an operation η on plane curves. If Kontsevich likes λ_0 and λ_1 (namely if there are λ_i^a with $Z^{/2}(\lambda_i(\gamma)) = \lambda_i^a(W(\gamma))$), then η will have a compatible algebraic companion η^a:

$$\eta^a(\alpha) := (\lambda_0^a(\alpha) - \lambda_1^a(\alpha))/\hbar \in \mathcal{A}_H^{/1}(\bigcirc\bigcirc) = |A| \otimes |A|.$$

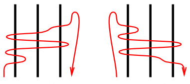
For indeed, in $\mathcal{A}_H^{/2}$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda_0^a(W(\gamma)) - \lambda_1^a(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

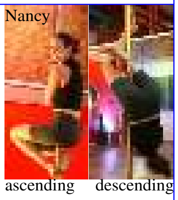


Example 1. With $\gamma_1, \gamma_2 \in |\pi|$ (or $|\bar\pi|$) set $\lambda_0(\gamma_1, \gamma_2) = \tilde\gamma_1 \cdot \tilde\gamma_2$ and $\lambda_1(\gamma_1, \gamma_2) = \tilde\gamma_2 \cdot \tilde\gamma_1$ where $\tilde\gamma_i$ are arbitrary lifts of γ_i. Then η_1 is the Goldman bracket! Note that here λ_0 and λ_1 are not well-defined, yet η_1 is.

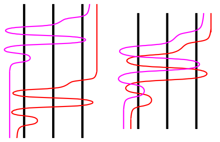


Example 2. With $\gamma_1, \gamma_2 \in \pi$ (or $\bar\pi$) and with λ_0, λ_1 as on the right, we get the "double bracket" $\eta_2\colon \pi \otimes \pi \to \pi \otimes \pi$ (or $\bar\pi \otimes \bar\pi \to \bar\pi \otimes \bar\pi$).





Example 3. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3\colon \bar\pi \to \bar\pi \otimes |\bar\pi|$. Closing the first component and anti-symmetrizing, this is the Turaev cobracket.



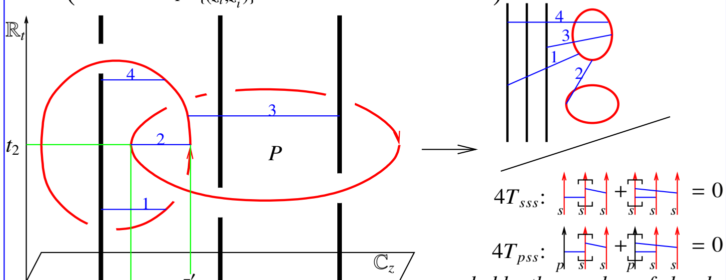
Example 4 [Ma]. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4\colon \bar\pi \to \bar\pi \otimes \bar\pi$. After some massaging, it too becomes the Turaev cobracket.

The rest is essentially **Exercises:** **1.** Lemma 1? **2.** $\mathcal{A}$? **3.** Fact 2? **4.** $\mathcal{A}^{/1}$? Especially, $\mathcal{A}^{/1}(\bigcirc) \cong |A|$! **5.** Explain why Kontsevich likes our λ's. **6.** Figure out η_i^a, $i = 1, \ldots, 4$.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-2208/`

Kontsevich in a Pole Dance Studio. (w/o poles? See [Ko, BN])

$$Z = \left(\sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \underset{\substack{t_1<\ldots<t_m \\ P=\{(z_i,z_i')\}}}{\sum\!\!\!\!\!\!\int} (-1)^{\#P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \right)^{\sim} \in \mathcal{A}$$



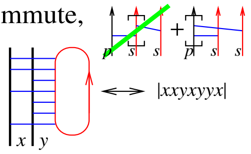
graded by the number of chords
filtered by the number of ss chords

Comments on the Kontsevich Integral.

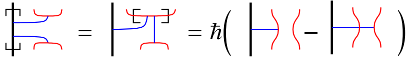
1. In the tangle case, the endpoints are fixed at top and bottom.
2. The $(\cdots)^{\sim}$ means "a correction is needed near the caps and the cups" (for the framed version, see [LM2, Da]).
3. There are never pp chords, and no $4T_{pps}$ and $4T_{ppp}$ relations.
4. Z is an "expansion".
5. Z respects the ss filtration and so descends to $Z^{/s}\colon \mathcal{K}^{/s} \to \mathcal{A}^{/s}$.

Comments on $\mathcal{A}$.

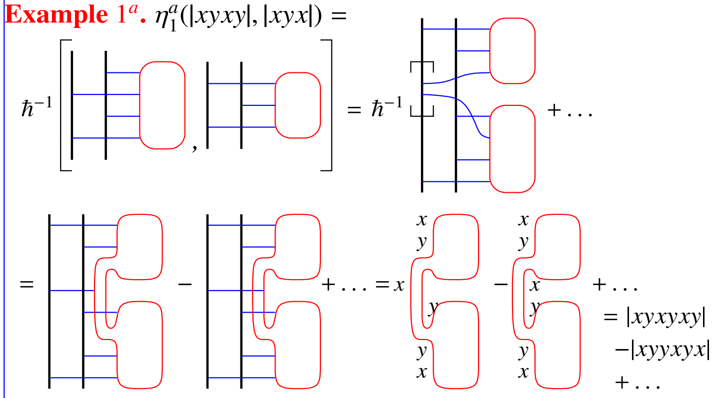
In $\mathcal{A}^{/1}$ legs on poles commute, so $\mathcal{A}^{/1}(\bigcirc) = |A|!$



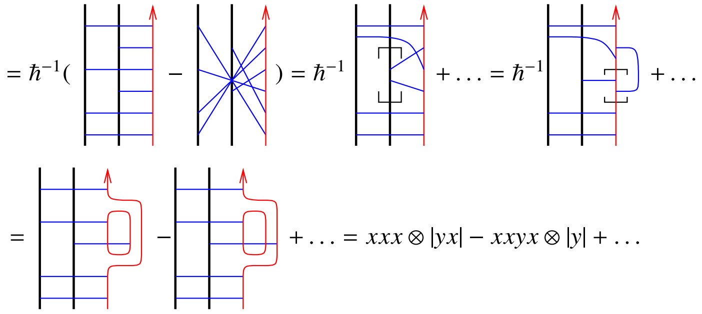
In $\mathcal{A}_H^{/2}$ we have:



Example 1^a. $\eta_1^a(|xyxy|, |xyx|) =$



Example 3^a. Ignoring complications, $\eta_3^a(xxyxyx) =$



$+\ldots = xxx \otimes |yx| - xxyx \otimes |y| + \ldots$

Proof of Lemma 1. We partially prove Theorem 2 instead:

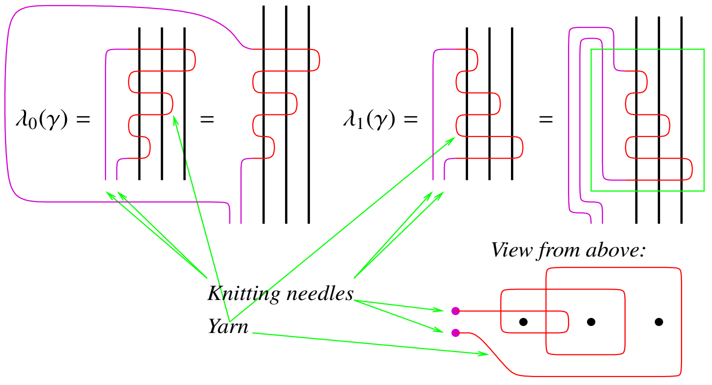
Theorem 2. $\operatorname{gr}^{\bullet} \mathcal{K}_H \cong \mathbb{F}[\![\hbar]\!] \otimes (\mathcal{K}^{/1})_0$.

Proof mod $\hbar^2$. The map ← is obvious. To go →, map $\mathcal{K}_H \to \mathbb{F}[\![\hbar]\!] \otimes \mathcal{K}^{/1}$ using $\nwarrow\!\!\!\!\nearrow \mapsto \nwarrow\!\!\!\!\nearrow + \frac{\hbar}{2})($ and $\nwarrow\!\!\!\!\nearrow \mapsto \nwarrow\!\!\!\!\nearrow - \frac{\hbar}{2})($ and apply the functor $\operatorname{gr}^{\bullet}$.

Unignoring the Complications.

We need λ_0 and λ_1 such that:

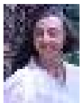
1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. Up to conjugation, $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma)) = W(\gamma)$.



Homework

1. Is there more than Examples 1–4?
2. Derive the bialgebra axioms from this perspective.
3. What more do we get if we don't mod out by HOMFLY-PT?
4. What more do we get if we allow more than one strand-strand interaction?
5. In this language, recover Kashiwara-Vergne [AKKN1, AKKN2].
6. How is all this related to w-knots?
7. Do the same with associators. Use that to derive formulas for solutions of Kashiwara-Vergne.
8. What's the relationship with the Habiro-Massuyeau invariants of links in handlebodies [HM] (different filtration!).
9. Pole dance on other surfaces!
10. Explore the action of the mapping class group.

Kashiwara



Vergne

Acknowledgement. This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC). I also wish to thanks A. Alekseev, F. Naef, and M. Ren for listening to an earlier version and catching some bugs, and Dhanya S. for the dance studio photos. And of course, thanks for listening!

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[LM1] T. Q. T. Le & J. Murakami, *Kontsevich's Integral for the HOMFLY Polynomial and Relations Between Values of Multiple Zeta Functions,* Top. and its Appl. **62-2** (1995) 193–206.

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[Ma] G. Massuyeau, *Formal Descriptions of Turaev's Loop Operations,* Quant. Topol. **9-1** (2018) 39-–117, arXiv:1511.03974.

Kashaev's Signature Conjecture

CMS Winter 2021 Meeting, December 4, 2021

Dror Bar-Natan with Sina Abbasi

Agenda. Show and tell with signatures.

Abstract. I will display side by side two nearly identical computer programs whose inputs are knots and whose outputs seem to always be the same. I'll then admit, very reluctantly, that I don't know how to prove that these outputs are always the same. One program I wrote mostly in Bedlewo, Poland, in the summer of 2003 and as of recently I understand why it computes the Levine-Tristram signature of a knot. The other is based on the 2018 preprint *On Symmetric Matrices Associated with Oriented Link Diagrams* by Rinat Kashaev (arXiv:1801.04632), where he conjectures that a certain simple algorithm also computes that same signature.

If you can, please turn your video on! (And mic, whenever needed).

These slides and all the code within are available at http://drorbn.net/cms21.

(I'll post the video there too)

```
Bed[K_, ω_] :=
  Module[{t, r, XingsByArmpits, bends, faces, p, A, is},
   t = 1 - ω; r = t + t*;
   XingsByArmpits =
    List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
      If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
   bends = Times @@ XingsByArmpits /.
     _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
   faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
   A = Table[0, Length@faces, Length@faces];
   Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
    A[[is, is]] += If[Head[x] === X+,
      ( -r   -t  2 t  t* )   (  r   -t  -2 t*  t* )
      ( -t*   0   t*   0 )   ( -t*   0   t*     0 )
      ( 2 t*  t  -r  -t* ) , ( -2 t  t   r    -t* ) ],
      ( t     0  -t    0 )   (  t    0  -t      0 )
    {x, XingsByArmpits}];
   MatrixSignature[A]];
```

```
Kas[K_, ω_] :=
  Module[{u, v, XingsByArmpits, bends, faces, p, A, is},
   u = Re[ω^{1/2}]; v = Re[ω];
   XingsByArmpits =
    List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
      If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
   bends = Times @@ XingsByArmpits /.
     _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
   faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
   A = Table[0, Length@faces, Length@faces];
   Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
    A[[is, is]] += If[Head[x] === X+,
      ( v u 1 u )     ( v u 1 u )
      ( u 1 u 1 )     ( u 1 u 1 )
      ( 1 u v u ) , - ( 1 u v u ) ],
      ( u 1 u 1 )     ( u 1 u 1 )
    {x, XingsByArmpits}];
   (MatrixSignature[A] - Writhe[K]) / 2];
```

Why am I showing you 💀code💀?

- I love code — it's fun!
- Believe it or not, it is more expressive than math-talk (though I'll do the math-talk as well, to confirm with prevailing norms).
- It is directly verifiable. Once it is up and running, you'll never ask yourself "did he misplace a sign somewhere"?

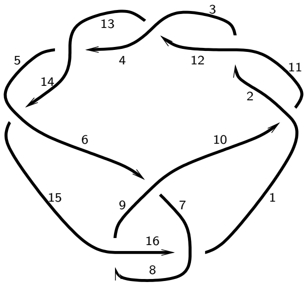
```
Bed[K_, ω_] :=
  Module[{t, r, XingsByArmpits, bends, faces, p, A, is},
   t = 1 - ω; r = t + t*;
   XingsByArmpits =
    List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
      If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
   bends = Times @@ XingsByArmpits /.
     _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
   faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
   A = Table[0, Length@faces, Length@faces];
   Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
    A[[is, is]] += If[Head[x] === X+,
      ( -r   -t  2 t  t* )   (  r   -t  -2 t*  t* )
      ( -t*   0   t*   0 )   ( -t*   0   t*     0 )
      ( 2 t*  t  -r  -t* ) , ( -2 t  t   r    -t* ) ],
      ( t     0  -t    0 )   (  t    0  -t      0 )
    {x, XingsByArmpits}];
   MatrixSignature[A]];
```

```
Kas[K_, ω_] :=
  Module[{u, v, XingsByArmpits, bends, faces, p, A, is},
   u = Re[ω^{1/2}]; v = Re[ω];
   XingsByArmpits =
    List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
      If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
   bends = Times @@ XingsByArmpits /.
     _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
   faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
   A = Table[0, Length@faces, Length@faces];
   Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
    A[[is, is]] += If[Head[x] === X+,
      ( v u 1 u )     ( v u 1 u )
      ( u 1 u 1 )     ( u 1 u 1 )
      ( 1 u v u ) , - ( 1 u v u ) ],
      ( u 1 u 1 )     ( u 1 u 1 )
    {x, XingsByArmpits}];
   (MatrixSignature[A] - Writhe[K]) / 2];
```

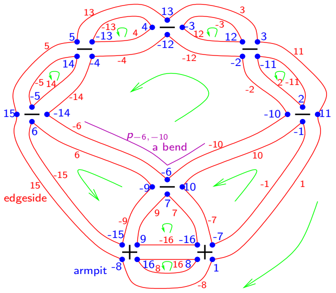
Verification.

```
Once[<< KnotTheory`]
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
MatrixSignature[A_] :=
  Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-12 &]]];
Writhe[K_] := Sum[If[PositiveQ[x], 1, -1], {x, List @@ PD@K}];
Sum[ω = e^{i RandomReal[{0,2π}]}; Bed[K, ω] == Kas[K, ω], {10},
 {K, AllKnots[{3, 10}]}]
KnotTheory: Loading precomputed data in PD4Knots`.
2490 True
```

Label everything!



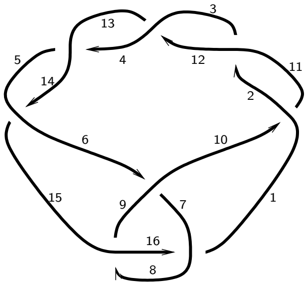
$PD[X[10,1,11,2],X[2,11,3,12],\dots]$



$\{X_-[-1,11,2,-10],X_-[-11,3,12,-2],\dots\}$

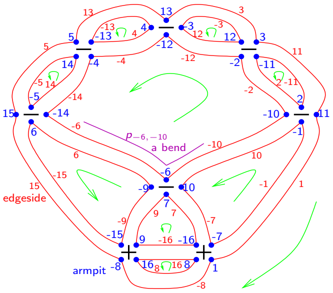
Lets run our code line by line...

```
PD[8₂] = PD[X[10, 1, 11, 2],
   X[2, 11, 3, 12], X[12, 3, 13, 4],
   X[4, 13, 5, 14], X[14, 5, 15, 6],
   X[8, 16, 9, 15], X[16, 8, 1, 7],
   X[6, 9, 7, 10]];
K = 8₂;
```



```
XingsByArmpits =
 List @@ PD[K] /.
  x : X[i_, j_, k_, l_] :>
   If[PositiveQ[x], X+[-i, j, k, -l],
    X-[-j, k, l, -i]]
```

$\{X_-[-1, 11, 2, -10], X_-[-11, 3, 12, -2],$
$X_-[-3, 13, 4, -12], X_-[-13, 5, 14, -4],$
$X_-[-5, 15, 6, -14], X_+[-8, 16, 9, -15],$
$X_+[-16, 8, 1, -7], X_-[-9, 7, 10, -6]\}$

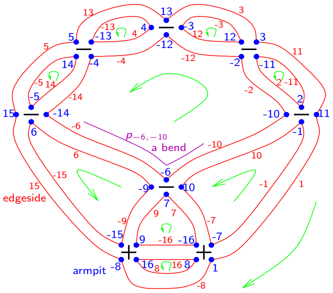


```
bends = Times @@ XingsByArmpits /.
   _[X][a_, b_, c_, d_] :>
    p[a,-d] p[b,-a] p[c,-b] p[d,-c]
```

$p_{-16,7}\, p_{-15,-9}\, p_{-14,-6}\, p_{-13,4}\, p_{-12,-4}\, p_{-11,2}$
$p_{-10,-2}\, p_{-9,6}\, p_{-8,15}\, p_{-7,-1}\, p_{-6,-10}\, p_{-5,14}$
$p_{-4,-14}\, p_{-3,12}\, p_{-2,-12}\, p_{-1,10}\, p_{1,-8}\, p_{2,-11}$
$p_{3,11}\, p_{4,-13}\, p_{5,13}\, p_{6,-15}\, p_{7,9}\, p_{8,16}\, p_{9,-16}$
$p_{10,-7}\, p_{11,1}\, p_{12,-3}\, p_{13,3}\, p_{14,-5}\, p_{15,5}\, p_{16,8}$

```
faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z]
```

$p_{-13,4,-13}\, p_{-11,2,-11}\, p_{-5,14,-5}\, p_{-3,12,-3}$
$p_{8,16,8}\, p_{6,-15,-9,6}\, p_{9,-16,7,9}\, p_{10,-7,-1,10}$
$p_{-10,-2,-12,-4,-14,-6,-10}\, p_{1,-8,15,5,13,3,11,1}$



```
A = Table[0, Length@faces, Length@faces];
A // MatrixForm
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\text{Do}\Big[is = \text{Position}[faces, \#][\![1, 1]\!] \,\&\, /@ \text{List} @@ x;$$
$$A[\![is, is]\!] \mathrel{+}= \text{If}\Big[\text{Head}[x] === X_+,$$
$$\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}, -\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}\Big],$$
$$\{x, XingsByArmpits\}\Big];$$

```
x = XingsByArmpits[[1]]
```

$X_-[-1, 11, 2, -10]$

```
faces
```

$p_{-13,4,-13}\, p_{-11,2,-11}\, p_{-5,14,-5}\, p_{-3,12,-3}\, p_{8,16,8}\, p_{6,-15,-9,6}$
$p_{9,-16,7,9}\, p_{10,-7,-1,10}\, p_{-10,-2,-12,-4,-14,-6,-10}\, p_{1,-8,15,5,13,3,11,1}$

```
is = Position[faces, #][[1, 1]] & /@ List @@ x
```

$\{8, 10, 2, 9\}$

$$A[\![is, is]\!] \mathrel{+}= \text{If}\Big[\text{Head}[x] === X_+,$$
$$\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}, -\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}\Big];$$

```
A // MatrixForm
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&-v&0&0&0&0&0&-1&-u&-u\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&-v&-u&-u\\ 0&-u&0&0&0&0&0&-u&-1&-1\\ 0&-u&0&0&0&0&0&-u&-1&-1 \end{pmatrix}$$

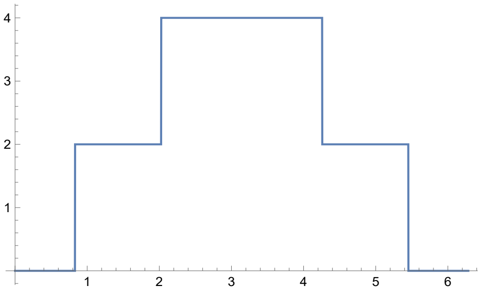
Recall, $is = \{8, 10, 2, 9\}$

$$\text{Do}\Big[is = \text{Position}[faces, \#][\![1, 1]\!] \,\&\, /@ \text{List} @@ x;$$
$$A[\![is, is]\!] \mathrel{+}= \text{If}\Big[\text{Head}[x] === X_+,$$
$$\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}, -\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}\Big],$$
$$\{x, \text{Rest}@XingsByArmpits\}\Big]$$

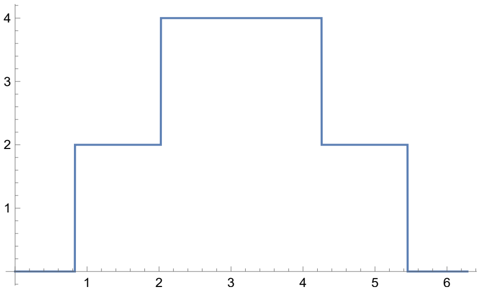
```
A // MatrixForm
```

$$\begin{pmatrix} -2v&0&-1&-1&0&0&0&0&-2u&-2u\\ 0&-2v&0&-1&0&0&0&-1&-2u&-2u\\ -1&0&-2v&0&0&-1&0&0&-2u&-2u\\ -1&-1&0&-2v&0&0&0&0&-2u&-2u\\ 0&0&0&0&2&1&2u&1&0&2u\\ 0&0&-1&0&1&1-2v&0&-1&-2u&0\\ 0&0&0&0&2u&0&-1+2v&0&-1&2\\ 0&-1&0&0&1&-1&0&1-2v&-2u&0\\ -2u&-2u&-2u&-2u&0&-2u&-1&-2u&-6&-5\\ -2u&-2u&-2u&-2u&2u&0&2&0&-5&-5+2v \end{pmatrix}$$

```
Plot[ω = ⅇ^(ⅈ t); u = Re[ω^(1/2)]; v = Re[ω];
 (MatrixSignature[A] - Writhe[K]) / 2,
 {t, 0, 2 π}]
```

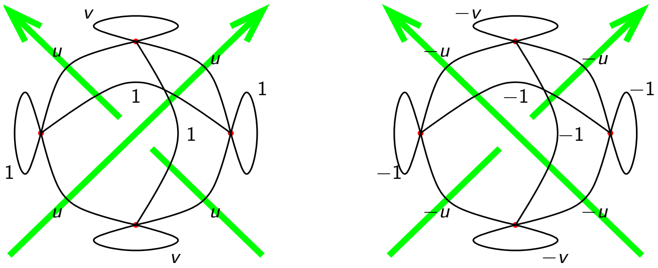


```
Plot[Bed[Knot[8, 2], ⅇ^(ⅈ t)], {t, 0, 2 π}]
```



Kashaev for Mathematicians.

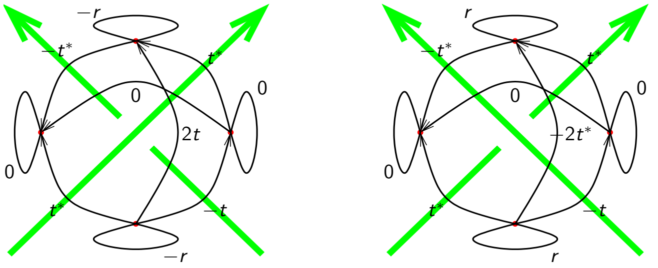
For a knot K and a complex unit ω set $u = \Re(\omega^{1/2})$, $v = \Re(\omega)$, make an $F \times F$ matrix A with contributions



and output $\frac{1}{2}(\sigma(A) - w(K))$.

Bedlewo for Mathematicians.

For a knot K and a complex unit ω set $t = 1 - \omega$, $r = 2\Re(t)$, make an $F \times F$ matrix A with contributions



(conjugate if going against the flow) and output $\sigma(A)$.

Why are they equal?

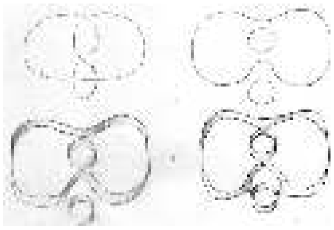
I dunno, yet note that

- Kashaev is over the $\mathbb{R}$eals, Bedlewo is over the $\mathbb{C}$omplex numbers.
- There's a factor of 2 between them, and a shift.

. . . so it's not merely a matrix manipulation.

Theorem. The Bedlewo program computes the Levine-Tristram signature of K at ω.

(Easy) **Proof.** Levine and Tristram tell us to look at $\sigma((1-\omega)L + (1-\omega^*)L^T)$, where L is the linking matrix for a Seifert surface S for K: $L_{ij} = \text{lk}(\gamma_i, \gamma_j^+)$ where γ_i run over a basis of $H_1(S)$ and γ_i^+ is the pushout of γ_i. But signatures don't change if you run over and over-determined basis, and the faces make such and over-determined basis whose linking numbers are controlled by the crossings. The rest is details.



Art by Emily Redelmeier

Thank You!

Warning. The second formula on page (−2) "**Conclusion**" is silly-wrong. A fix will be posted here soon: some of the numbers written in this handout are a bit off, yet the qualitative results remain exactly the same (namely, for finite type, 3D seems to beat 2D, with the same algorithms).

Yarn-Ball Knots

[K-OS] on October 21, 2021

Dror Bar-Natan with Itai Bar-Natan, Iva Halacheva, and Nancy Scherich

Agenda. A modest light conversation on how knots should be measured.

Abstract. Let there be scones! Our view of knot theory is biased in favour of pancakes.
Technically, if K is a 3D knot that fits in volume V (assuming fixed-width yarn), then its projection to 2D will have about $V^{4/3}$ crossings. You'd expect genuinely 3D quantities associated with K to be computable straight from a 3D presentation of K. Yet we can hardly ever circumvent this $V^{4/3} \gg V$ "projection fee". Exceptions include linking numbers (as we shall prove), the hyperbolic volume, and likely finite type invariants (as we shall discuss in detail). But knot polynomials and knot homologies seem to always pay the fee. Can we exempt them?

More at http://drorbn.net/kos21

Thanks for inviting me to speak at [K-OS]!

Most important: `http://drorbn.net/kos21`

See also arXiv:2108.10923.

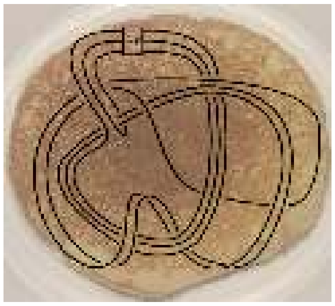
If you can, please turn your video on! (And mic, whenever needed).

A recurring question in knot theory is "do we have a 3D understanding of our invariant?"

- See Witten and the Jones polynomial.
- See Khovanov homology.

I'll talk about my perspective on the matter…

We often think of knots as planar diagrams. 3-dimensionally, they are embedded in "pancakes".



Knot by Lisa Piccirillo, pancake by DBN

But real life knots are 3D!

A Yarn Ball

'Connector' by Alexandra Griess and Jorel Heid (Hamburg, Germany). Image from www.waterfrontbia.com/ice-breakers-2019-presented-by-ports/.

The difference matters when

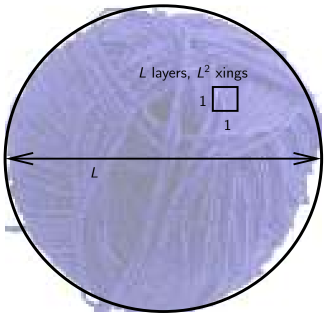
- We make statements about "random knots".
- We figure out computational complexity.

Let's try to make it quantitative…

$V \sim L^3$

$n =$ xing number $\sim L^2L^2 = L^4 = V^{4/3}$

("$\sim$" means "equal up to constant terms and log terms")



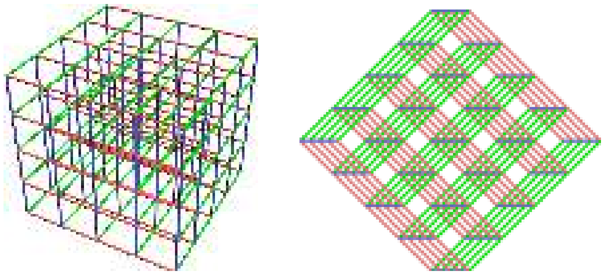
Conversation Starter 1. A knot invariant ζ is said to be Computationally 3D, or C3D, if

$$C_\zeta(3D, V) \ll C_\zeta(2D, V^{4/3}).$$

This isn't a rigorous definition! It is time- and naïveté-dependent! But there's room for less-stringent rigour in mathematics, and on a philosophical level, our definition means something.

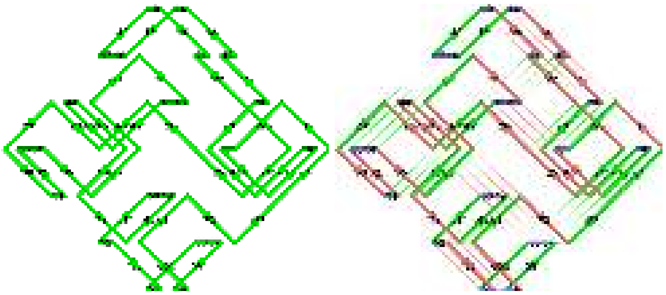
Theorem 1. Let lk denote the linking number of a 2-component link. Then $C_{lk}(2D, n) \sim n$ while $C_{lk}(3D, V) \sim V$, so lk is C3D!

Proof. WLOG, we are looking at a link in a grid, which we project as on the right:



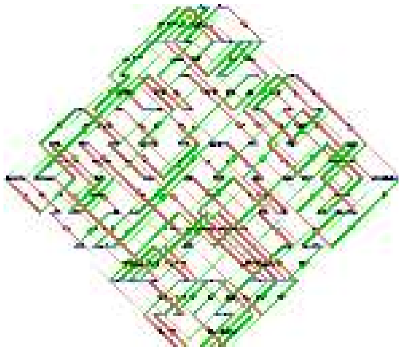
\red /green −blue

Here's what it look like, in the case of a knot:



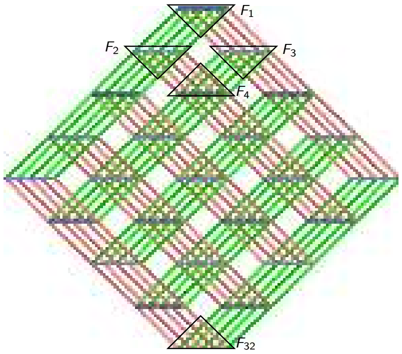
And here's a bigger knot.

This may look like a lot of information, but if V is big, it's less than the information in a planar diagram, and it is easily computable.



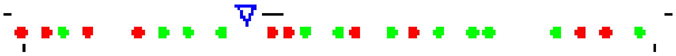
There are $2L^2$ triangular "crossings fields" F_k in such a projection.

WLOG, in each F_k all over strands and all under strands are oriented in the same way and all green edges belong to one component and all red edges to the other.

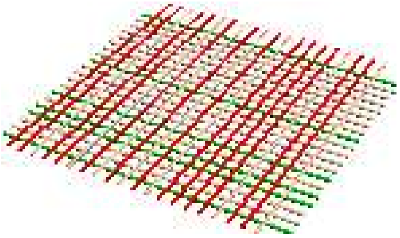


So $2L^2$ times we have to solve the problem "given two sets R and G of integers in $[0, L]$, how many pairs $\{(r, g) \in R \times G : r < g\}$ are there?". This takes time $\sim L$ (see below), so the overall computation takes time $\sim L^3$.

Below. Start with $rb = cf = 0$ ("reds before" and "cases found") and slide $\triangledown$ from left to right, incrementing rb by one each time you cross a ● and incrementing cf by rb each time you cross a ●:



In general, with our limited tools, speedup arises because appropriately projected 3D knots have many uniform "red over green" regions:



Video and more at http://www.math.toronto.edu/~drorbn/Talks/KOS-211021/

Great Embarrassment 1. I don't know if any of the Alexander, Jones, HOMFLY-PT, and Kauffman polynomials is C3D. I don't know if any Reshetikhin-Turaev invariant is C3D. I don't know if any knot homology is C3D.

Or maybe it's a cause for optimism — there's still something very basic we don't know about (say) the Jones polynomial. Can we understand it well enough 3-dimensionally to compute it well? If not, why not?

Conversation Starter 2. Similarly, if η is a stingy quantity (a quantity we expect to be small for small knots), we will say that η has Savings in 3D, or "has S3D" if $M_\eta(3D,V) \ll M_\eta(2D,V^{4/3})$.

Example (R. van der Veen, D. Thurston, private communications). The hyperbolic volume has *S3D*.

Great Embarrassment 2. I don't know if the genus of a knot has S3D! In other words, even if a knot is given in a 3-dimensional, the best way I know to find a Seifert surface for it is to first project it to 2D, at a great cost.

Next we argue that most finite type invariants are probably C3D…

(What a weak statement!)

Theorem FT2D. If ζ is a finite type invariant of type d then $C_\zeta(2D,n)$ is at most $\sim n^{\lfloor 3d/4\rfloor}$. With more effort, $C_\zeta(2D,n)\lesssim n^{(\frac{2}{3}+\epsilon)d}$.

Note that there are some exceptional finite type invariants, e.g. high coefficients of the Alexander polynomial and other poly-time knot polynomials, which can be computed much faster!

Theorem FT3D. If ζ is a finite type invariant of type d then $C_\zeta(3D,V)$ is at most $\sim V^{6d/7+1/7}$. With more effort, $C_\zeta(2D,V)\lesssim V^{(\frac{4}{5}+\epsilon)d}$.

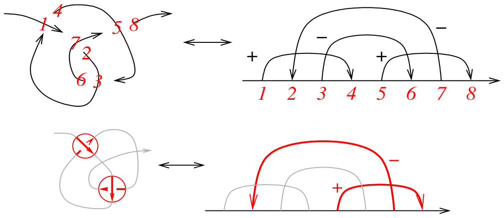
Tentative Conclusion. As

$n^{3d/4}\sim(V^{4/3})^{3d/4}=V\gg V^{6d/7+1/7}$ $\quad n^{2d/3}\sim(V^{4/3})^{2d/3}=V^{8d/9}\gg V^{4d/5}$

these theorems say "most finite type invariants are probably C3D; the ones in greater doubt are the lucky few that can be computed unusually quickly".

All pre-categorification knot polynomials are power series whose coefficients are finite type invariants. (This is sometimes helpful for the computation of finite type invariants, but rarely helpful for the computation of knot polynomials).

Gauss diagrams and sub-Gauss-diagrams:

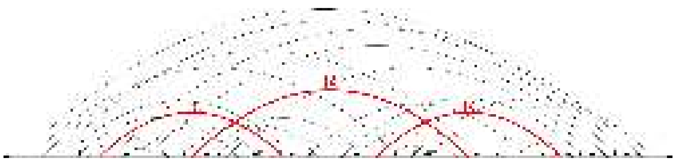


Let $\varphi_d\colon\{\text{knot diagrams}\}\to\langle\text{Gauss diagrams}\rangle$ map every knot diagram to the sum of all the sub-diagrams of its Gauss diagram which have at most d arrows.

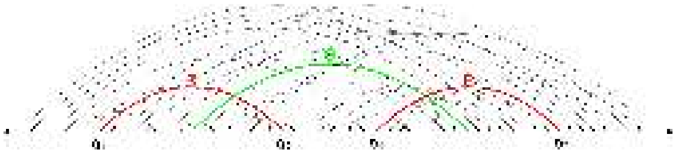
Under-Explained Theorem (Goussarov-Polyak-Viro). A knot invariant ζ is of type d iff there is a linear functional ω on $\langle\text{Gauss diagrams}\rangle$ such that $\zeta=\omega\circ\varphi_d$.

Theorem FT2D. If ζ is a finite type invariant of type d then $C_\zeta(2D,n)$ is at most $\sim n^{\lfloor 3d/4\rfloor}$. With more effort, $C_\zeta(2D,n)\lesssim n^{(\frac{2}{3}+\epsilon)d}$.

Proof of Theorem FT2D.



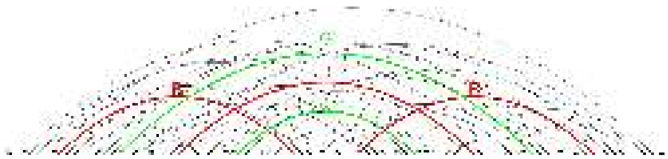
We need to count how many times a diagram such as the red appears within a bigger diagram, having n arrows. Clearly this can be done in time $\sim n^3$, and in general, in time $\sim n^d$.



With an appropriate look-up table, it can also be done in time $\sim n^2$ (in general, $\sim n^{d-1}$). That look-up table $(T^{p_1,p_2}_{q_1,q_2})$ is of size (and production cost) $\sim n^4$ if you are naive, and $\sim n^2$ if you are just a bit smarter. Indeed

$$T^{p_1,p_2}_{q_1,q_2}=T^{0,p_2}_{0,q_2}-T^{0,p_1}_{0,q_2}-T^{0,p_2}_{0,q_1}+T^{0,p_1}_{0,q_1},$$

and $(T^{0,p}_{0,q})$ is easy to compute.



With multiple uses of the same lookup table, what naively takes $\sim n^5$ can be reduced to $\sim n^3$.

In general within a big d-arrow diagram we need to find an as-large-as possible collection of arrows to delay. These must be non-adjacent to each other. As the adjacency graph for the arrows is at worst quadrivalent, we can always find $\lceil \frac{d}{4} \rceil$ non-adjacent arrows, and hence solve the counting problem in time $\sim n^{d-\lceil \frac{d}{4} \rceil} = n^{\lfloor 3d/4 \rfloor}$.

On to

Theorem FT3D. If ζ is a finite type invariant of type d then $C_\zeta(3D, V)$ is at most $\sim V^{6d/7+1/7}$. With more effort, $C_\zeta(2D, V) \lesssim V^{(\frac{4}{5}+\epsilon)d}$.

The line/feather method:

Accurate but takes forever.

In reality, you take a few shark bites and feather the rest ...

... and then there's an optimization problem to solve: when to stop biting and start feathering.

Note that this counting argument works equally well if each of the d arrows is pulled from a different set!

It follows that we can compute φ_d in time $\sim n^{\lfloor 3d/4 \rfloor}$.

$\square$

With bigger look-up tables that allow looking up "clusters" of G arrows, we can reduce this to $\sim n^{(\frac{2}{3}+\epsilon)d}$.

$\square$

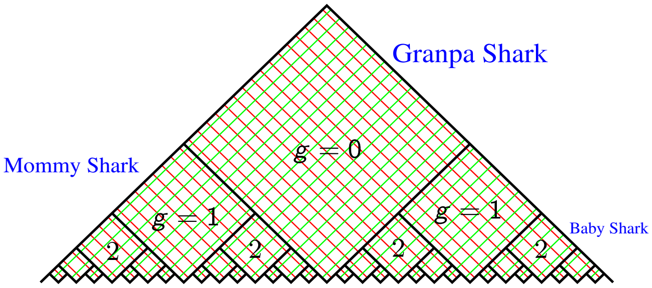
An image editing problem:

(Yarn ball and background coutesy of Heather Young)

The rectangle/shark method:

Coarse but fast.

The structure of a crossing field.



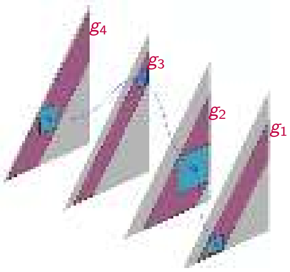
There are about $\log_2 L$ "generations". There are 2^g bites in generation g, and the total number of crossings in them is $\sim L^2/2^g$. Let's go hunt!

Multi-feathers and multi-sharks.

For a type d invariant we need to count d-tuples of crossings, and each has its own "generation" g_i. So we have the "multi-generation"

$$\bar{g} = (g_1, \ldots, g_d).$$

Let $G := \sum g_i$ be the "overall generation". We will choose between a "multi-feather" method and a "multi-shark" method based on the size of G.



The effort to take a single multi-bite is tiny. Indeed,

Lemma Given $2d$ finite sets $B_i = \{t_{i1}, t_{i2}, \ldots\} \subset [1..L^3]$ and a permutation $\pi \in S_{2n}$ the quantity

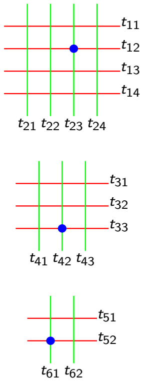
$$N = \left|\left\{(b_i) \in \prod_{i=1}^{2d} B_i \colon \text{the } b_i\text{'s are ordered as } \pi\right\}\right|$$

can be computed in time $\sim \sum |B_i| \sim \max |B_i|$.

Proof. WLOG $\pi = Id$. For $\iota \in [1..2d]$ and $\beta \in B := \cup B_i$ let

$$N_{\iota,\beta} = \left|\left\{(b_i) \in \prod_{i=1}^{\iota} B_i \colon b_1 < b_2 < \ldots < b_\iota \leq \beta\right\}\right|.$$

We need to know $N_{2d,\max B}$; compute it inductively using $N_{\iota,\beta} = N_{\iota,\beta'} + N_{\iota-1,\beta'}$, where β' is the predecessor of β in B. $\square$



Conclusion. We wish to compute the contribution to φ_d coming from d-tuples of crossings of multi-generation $\bar{g}$.

- The multi-shark method does it in time

$$\sim (\text{no. of bites}) \cdot (\text{time per bite}) = L^{2d} 2^G \cdot \frac{L}{2^{\min \bar{g}}} < L^{2d+1} 2^G$$

(increases with G).

- The multi-feather method (project and use the 2D algorithm) does it in time

$$\sim (\text{no. of crossings})^{\lfloor \frac{3}{4} d \rfloor} = \left(\prod_{i=1}^{d} L^2 \frac{L^2}{2^{g_i}}\right)^{\lfloor \frac{3}{4} d \rfloor} < \frac{L^{3d}}{(2^G)^{3/4}}$$

(decreases with G).

Of course, for any specific G we are free to choose whichever is better, shark or feather.

The two methods agree (and therefore are at their worst) if $2^G = L^{\frac{4}{7}(d-1)}$, and in that case, they both take time $\sim L^{\frac{18}{7}d+\frac{3}{7}} = V^{\frac{6}{7}d+\frac{1}{7}}$.

The same reasoning, with the $n^{(\frac{2}{3}+\epsilon)d}$ feather, gives $V^{(\frac{4}{5}+\epsilon)d}$.

$\square$

If time — a word about braids.

Thank You!

I Still Don't Understand the Alexander Polynomial

Dror Bar-Natan, http://drorbn.net/mo21

Moscow by Web, April 2021

Abstract. As an algebraic knot theorist, I still don't understand the Alexander polynomial. There are two conventions as for how to present tangle theory in algebra: one may name the strands of a tangle, or one may name their ends. The distinction might seem too minor to matter, yet it leads to a completely different view of the set of tangles as an algebraic structure. There are lovely formulas for the Alexander polynomial as viewed from either perspective, and they even agree where they meet. But the "strands" formulas know about strand doubling while the "ends" ones don't, and the "ends" formulas know about skein relations while the "strands" ones don't. There ought to be a common generalization, but I don't know what it is.

Thanks for inviting me to Moscow! As most of you have never seen it, here's a picture of the lecture room:

If you can, please turn your video on! (And mic, whenever needed).

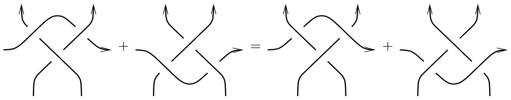
I use talks to self-motivate; so often I choose a topic and write an abstract when I know I can do it, yet when I haven't done it yet. This time it turns out my abstract was wrong — I'm still uncomfortable with the Alexander polynomial, but in slightly different ways than advertised two slides before.

My discomfort.

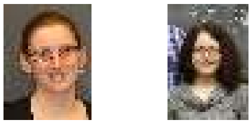
- I can compute the multivariable Alexander polynomial real fast:

$\longrightarrow (uvw)^{-1/2}(u-1)(v-1)(w-1).$

- But I can only prove "skein relations" real slow:



This talk is to a large extent an elucidation of the Ph.D. theses of my former students Jana Archibald and Iva Halacheva. See [Ar, Ha1, Ha2].



Also thanks to Roland van der Veen for comments.

A technicality. There's supposed to be fire alarm testing in my building today. Don't panic!

1. Virtual Skein Theory Heaven

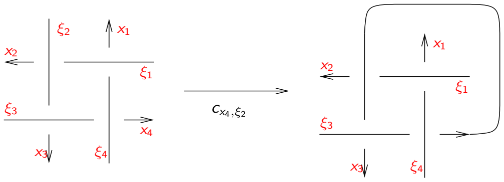
Definition. A "Contraction Algebra" assigns a set $\mathcal{T}(\mathcal{X}, X)$ to any pair of finite sets $\mathcal{X} = \{\xi \ldots\}$ and $X = \{x, \ldots\}$ provided $|\mathcal{X}| = |X|$, and has operations

- "Disjoint union" $\sqcup\colon \mathcal{T}(\mathcal{X}, X) \times \mathcal{T}(\mathcal{Y}, Y) \to \mathcal{T}(\mathcal{X} \sqcup \mathcal{Y}, X \sqcup Y)$, provided $\mathcal{X} \cap \mathcal{Y} = X \cap Y = \emptyset$.
- "Contractions" $c_{x,\xi}\colon \mathcal{T}(\mathcal{X}, X) \to \mathcal{T}(\mathcal{X} \setminus \xi, X \setminus x)$, provided $x \in X$ and $\xi \in \mathcal{X}$.
- Renaming operations $\sigma^{\xi}_{\eta}\colon \mathcal{T}(\mathcal{X} \sqcup \{\xi\}, X) \to \mathcal{T}(\mathcal{X} \sqcup \{\eta\}, X)$ and $\sigma^{x}_{y}\colon \mathcal{T}(\mathcal{X}, X \sqcup \{x\}) \to \mathcal{T}(\mathcal{X}, X \sqcup \{y\})$.

Subject to axioms that will be specified right after the two examples in the next three slides.

If R is a ring, a contraction algebra is said to be "R-linear" if all the $\mathcal{T}(\mathcal{X}, X)$'s are R-modules, if the disjoint union operations are R-bilinear, and if the contractions $c_{x,\xi}$ and the renamings $\sigma^{\cdot}_{\cdot}$ are R-linear.

(Contraction algebras with some further "unit" properties are called "wheeled props" in [MMS, DHR])

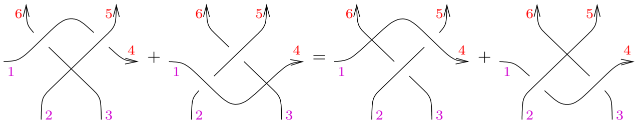


Example 1. Let $\mathcal{T}(\mathcal{X}, X)$ be the set of virtual tangles with incoming ends ("tails") labeled by $\mathcal{X}$ and outgoing ends ("heads") labeled by X, with $\sqcup$ and $\sigma^{\cdot}_{\cdot}$ the obvious disjoint union and end-renaming operations, and with $c_{x,\xi}$ the operation of attaching a head x to a tail ξ while introducing no new crossings.

Note 1. $\mathcal{T}$ can be made linear by allowing formal linear combinations.

Note 2. $\mathcal{T}$ is finitely presented, with generators the positive and negative crossings, and with relations the Reidemeister moves! (If you want, you can take this to be the definition of "virtual tangles").

Note 3. A contraction algebra morphism out of $\mathcal{T}$ is an invariant of virtual tangles (and hence of virtual knots and links) and would be an ideal tool to prove Skein Relations:



Example 2. Let V be a finite dimensional vector space and set $\mathcal{V}(\mathcal{X}, X) := (V^*)^{\otimes\mathcal{X}} \otimes V^{\otimes X}$, with $\sqcup = \otimes$, with $\sigma^{\cdot}_{\cdot}$ the operation of renaming a factor, and with $c_{x,\xi}$ the operation of contraction: the evaluation of tensor factor ξ (which is a V^*) on tensor factor x (which is a V).

Axioms. One axiom is primary and interesting,

- Contractions commute! Namely, $c_{x,\xi}/\!\!/c_{y,\eta} = c_{y,\eta}/\!\!/c_{x,\xi}$ (or in old-speak, $c_{y,\eta} \circ c_{x,\xi} = c_{x,\xi} \circ c_{y,\eta}$).

And the rest are just what you'd expect:

- $\sqcup$ is commutative and associative, and it commutes with $c_{\cdot,\cdot}$ and with $\sigma^{\cdot}_{\cdot}$ whenever that makes sense.
- $c_{\cdot,\cdot}$ is "natural" relative to renaming: $c_{x,\xi} = \sigma^x_y/\!\!/\sigma^\xi_\eta/\!\!/c_{y,\eta}$.
- $\sigma^\xi_\xi = \sigma^x_x = Id$, $\sigma^\xi_\eta/\!\!/\sigma^\eta_\zeta = \sigma^\xi_\zeta$, $\sigma^x_y/\!\!/\sigma^y_z = \sigma^x_z$, and renaming operations commute where it makes sense.

Comments.

- We can relax $|\mathcal{X}| = |X|$ at no cost.
- We can lose the distinction between $\mathcal{X}$ and X and get "circuit algebras".
- There is a "coloured version", where $\mathcal{T}(\mathcal{X}, X)$ is replaced with $\mathcal{T}(\mathcal{X}, X, \lambda, l)$ where $\lambda\colon \mathcal{X} \to C$ and $l\colon X \to C$ are "colour functions" into some set C of "colours", and contractions $c_{x,\xi}$ are allowed only if x and ξ are of the same colour, $l(x) = \lambda(\xi)$. In the world of tangles, this is "coloured tangles".

2. Heaven is a Place on Earth

(A version of the main results of Archibald's thesis, [Ar]).

Let us work over the base ring $\mathcal{R} = \mathbb{Q}[\{T^{\pm 1/2}\colon T \in C\}]$. Set

$$\mathcal{A}(\mathcal{X}, X) := \{w \in \Lambda(\mathcal{X} \sqcup X)\colon \deg_{\mathcal{X}} w = \deg_X w\}$$

(so in particular the elements of $\mathcal{A}(\mathcal{X}, X)$ are all of even degree). The union operation is the wedge product, the renaming operations are changes of variables, and $c_{x,\xi}$ is defined as follows. Write $w \in \mathcal{A}(\mathcal{X}, X)$ as a sum of terms of the form uw' where $u \in \Lambda(\xi, x)$ and $w' \in \mathcal{A}(\mathcal{X} \setminus \xi, X \setminus x)$, and map u to 1 if it is 1 or $x\xi$ and to 0 is if is ξ or x:

$$1w' \mapsto w', \qquad \xi w' \mapsto 0, \qquad xw' \mapsto 0, \qquad x\xi w' \mapsto w'.$$

Proposition. $\mathcal{A}$ is a contraction algebra.

Alternative Formulations.

- $c_{x,\xi} w = \iota_\xi \iota_x \mathbb{e}^{x\xi} w,$ where $\iota_\cdot$ denotes interior multiplication.
- Using Fermionic integration, $c_{x,\xi} w = \displaystyle\int \mathbb{e}^{x\xi} w\, d\xi dx.$
- $c_{x,\xi}$ represents composition in exterior algebras! With $X^* := \{x^*\colon x \in X\}$, we have that $\operatorname{Hom}(\Lambda X, \Lambda Y) \cong \Lambda(X^* \sqcup Y)$ and the following square commutes:

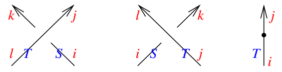
$$\begin{tikzcd}
\operatorname{Hom}(\Lambda X, \Lambda Y) \otimes \operatorname{Hom}(\Lambda Y, \Lambda Z) \arrow[r, "/\!\!/"] \arrow[d, leftrightarrow] & \operatorname{Hom}(\Lambda X, \Lambda Z) \arrow[d, leftrightarrow] \\
\Lambda(X^* \sqcup Y \sqcup Y^* \sqcup Z) \arrow[r, "\prod_{y \in Y} c_{y,y^*}"] & \Lambda(X^*, Z)
\end{tikzcd}$$

- Similarly, $\Lambda(\mathcal{X} \sqcup X) \cong (H^*)^{\otimes \mathcal{X}} \otimes H^{\otimes X}$ where H is a 2-dimensional "state space" and H^* is its dual. Under this identification, $c_{x,\xi}$ becomes the contraction of an H factor with an H^* factor.

We construct a morphism of coloured contraction algebras $\mathcal{A}\colon \mathcal{T} \to \mathcal{A}$ by declaring

$$\begin{aligned}
X_{ijkl}[S,T] &\mapsto T^{-1/2} \exp\left(\begin{pmatrix}\xi_l & \xi_i\end{pmatrix} \begin{pmatrix}1 & 1-T \\ 0 & T\end{pmatrix} \begin{pmatrix}x_j \\ x_k\end{pmatrix}\right) \\
\bar{X}_{ijkl}[S,T] &\mapsto T^{1/2} \exp\left(\begin{pmatrix}\xi_i & \xi_j\end{pmatrix} \begin{pmatrix}T^{-1} & 0 \\ 1-T^{-1} & 1\end{pmatrix} \begin{pmatrix}x_k \\ x_l\end{pmatrix}\right) \\
P_{ij}[T] &\mapsto \exp(\xi_i x_j)
\end{aligned}$$

with



$X_{ijkl}[S,T]$ $\qquad$ $\bar{X}_{ijkl}[S,T]$ $\qquad$ $P_{ij}[T]$

(Note that the matrices appearing in these formulas are the Burau matrices).

Theorem.

If D is a classical link diagram with k components coloured $T_1, \ldots, T_k$ whose first component is open and the rest are closed, if MVA is the multivariable Alexander polynomial of the closure of D (with these colours), and if ρ_j is the counterclockwise rotation number of the jth component of D, then

$$\mathcal{A}(D) = T_1^{-1/2}(T_1 - 1)\left(\prod_j T_j^{\rho_j/2}\right) \cdot MVA \cdot (1 + \xi_{\text{in}} \wedge x_{\text{out}}).$$

($\mathcal{A}$ vanishes on closed links).

3. An Implementation of $\mathcal{A}$

If I didn't implement I wouldn't believe myself.

Written in Mathematica [Wo], available as the notebook `Alpha.nb` at `http://drorbn.net/mo21/ap`. Code lines are highlighted in grey, demo lines are plain. We start with an implementation of elements ("`Wedge`") of exterior algebras, and of the wedge product ("`WP`"):

```
WP[Wedge[u___], Wedge[v___]] := Signature[{u, v}] * Wedge @@ Sort[{u, v}];
WP[0, _] = WP[_, 0] = 0;
WP[A_, B_] :=
  Expand[Distribute[A ** B] /.
    (a_. * u_Wedge) ** (b_. * v_Wedge) :→ a b WP[u, v]];
```

```
WP[Wedge[‸] + Wedge[a] - 2 b ∧ a, Wedge[‸] - 3 Wedge[b] + 7 c ∧ d]
```

```
Wedge[] + Wedge[a] - 3 Wedge[b] - a ∧ b + 7 c ∧ d + 7 a ∧ c ∧ d + 14 a ∧ b ∧ c ∧ d
```

We then define the exponentiation map in exterior algebras ("`WExp`") by summing the series and stopping the sum once the current term ("`t`") vanishes:

```
WExp[A_] := Module[{s = Wedge[‸], t = Wedge[‸], k = 0},
  While[t =!= 0, s += (t = Expand[WP[t, A] / (++k)])]; s]
```

```
WExp[a ∧ b + c ∧ d + e ∧ f]
```

```
Wedge[] + a ∧ b + c ∧ d + e ∧ f + a ∧ b ∧ c ∧ d + a ∧ b ∧ e ∧ f + c ∧ d ∧ e ∧ f + a ∧ b ∧ c ∧ d ∧ e ∧ f
```

Contractions!

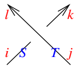
```
c_{x_,y_}[w_Wedge] := Module[{i, j},
   {i} = FirstPosition[w, x, {0}]; {j} = FirstPosition[w, y, {0}];
   { w                                              (i == 0) ∧ (j == 0)
   { (-1)^(i+j+If[i>j,0,1]) Delete[w, {{i}, {j}}]   (i > 0) ∧ (j > 0)
  ];
c_{x_,y_}[ℰ_] := ℰ /. w_Wedge :→ c_{x,y}[w]
```

```
WExp[a∧b + 2 c∧d]
c_{d,c}@WExp[a∧b + 2 c∧d]
```

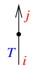
$$\text{Wedge}[\,] + a\wedge b + 2\, c\wedge d + 2\, a\wedge b\wedge c\wedge d$$

$$-\text{Wedge}[\,] - a\wedge b$$

The negative crossing and the "point":



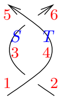
$\bar{X}_{ijkl}[S,T]$



$P_{ij}[T]$

```
𝒜[X̄_{i_,j_,k_,l_}[S_, T_]] := 𝒜[{i, j}, {k, l}, <|ξ_i → S, ξ_j → T, x_k → S, x_l → T|>,
   Expand[T^(1/2) WExp[Expand[{ξ_i, ξ_j}.( T^-1     0 ).{x_k, x_l}] /. ξ_a_ x_b_ :→ ξ_a ∧ x_b]]];
                                         ( 1 - T^-1 1 )
𝒜[X̄_{i_,j_,k_,l_}] := 𝒜[X̄_{i,j,k,l}[τ_i, τ_j]];
```

```
𝒜[P_{i_,j_}[T_]] := 𝒜[{i}, {j}, <|ξ_i → T, x_j → T|>, WExp[ξ_i ∧ x_j]];
𝒜[P_{i_,j_}] := 𝒜[P_{i,j}[τ_i]]
```



The union operation on $\mathcal{A}$'s (implemented as "multiplication"):

```
𝒜 /: 𝒜[is1_, os1_, cs1_, w1_] × 𝒜[is2_, os2_, cs2_, w2_] :=
 𝒜[is1 ⋃ is2, os1 ⋃ os2, Join[cs1, cs2], WP[w1, w2]]
```

```
Short[𝒜[X_{2,4,3,1}[S, T]] × 𝒜[X̄_{3,4,6,5}], 5]
```

$$\mathcal{A}\Big[\{1,2,3,4\},\{3,4,5,6\},\langle|\xi_2\to S,x_4\to T,x_3\to S,\xi_1\to T,\xi_3\to\tau_3,\xi_4\to\tau_4,x_6\to\tau_3,x_5\to\tau_4|\rangle,\frac{\sqrt{\tau_4}\,\text{Wedge}[\,]}{\sqrt{T}}-\frac{\sqrt{\tau_4}\,x_3\wedge\xi_1}{\sqrt{T}}+\sqrt{T}\sqrt{\tau_4}\,x_3\wedge\xi_1-\sqrt{T}\sqrt{\tau_4}\,x_3\wedge\xi_2-\frac{\sqrt{\tau_4}\,x_4\wedge\xi_1}{\sqrt{T}}-\frac{\sqrt{\tau_4}\,x_5\wedge\xi_4}{\sqrt{T}}-\frac{x_6\wedge\xi_3}{\sqrt{T}\sqrt{\tau_4}}+\ll 40\gg+\frac{\sqrt{T}\,x_3\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_3\wedge\xi_4}{\sqrt{\tau_4}}-\frac{\sqrt{T}\,x_3\wedge x_5\wedge x_6\wedge\xi_2\wedge\xi_3\wedge\xi_4}{\sqrt{\tau_4}}-\frac{x_4\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_3\wedge\xi_4}{\sqrt{T}\sqrt{\tau_4}}+\frac{\sqrt{T}\,x_3\wedge x_4\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_2\wedge\xi_3\wedge\xi_4}{\sqrt{\tau_4}}\Big]$$

Automatic and intelligent multiple contractions:

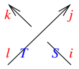
```
c@𝒜[is_, os_, cs_, w_] := Fold[c_{#2,#2}[#1] &, 𝒜[is, os, cs, w], is ⋂ os]
𝒜[{A_𝒜}] := c[A];
𝒜[{A1_𝒜, As__𝒜}] := Module[{A2},
  A2 = First@MaximalBy[{As}, Length[A1⟦1⟧ ⋂ #⟦2⟧] + Length[A1⟦2⟧ ⋂ #⟦1⟧] &];
  𝒜[Join[{c[A1 A2]}, DeleteCases[{As}, A2]]] ]
𝒜[𝒪s_List] := 𝒜[𝒜 /@ 𝒪s]
```

```
c[𝒜[X_{2,4,3,1}[S, T]] × 𝒜[X̄_{3,4,6,5}]]
```

$$\mathcal{A}[\{1,2\},\{5,6\},\langle|\xi_2\to S,\xi_1\to T,x_6\to S,x_5\to T|\rangle,\text{Wedge}[\,]-x_5\wedge\xi_1-x_6\wedge\xi_2-x_5\wedge x_6\wedge\xi_1\wedge\xi_2]$$

```
𝒜@{𝒜[X_{2,4,3,1}[S, T]], 𝒜[X̄_{3,4,6,5}]}
```

$$\mathcal{A}[\{1,2\},\{5,6\},\langle|\xi_2\to S,\xi_1\to T,x_6\to S,x_5\to T|\rangle,\text{Wedge}[\,]-x_5\wedge\xi_1-x_6\wedge\xi_2-x_5\wedge x_6\wedge\xi_1\wedge\xi_2]$$



$X_{ijkl}[S,T]$

$\mathcal{A}$`[is,os,cs,w]` is also a container for the values of the $\mathcal{A}$-invariant of a tangle. In it, `is` are the labels of the input strands, `os` are the labels of the output strands, `cs` is an assignment of colours (namely, variables) to all the ends $\{\xi_i\}_{i\in\texttt{is}}\sqcup\{x_j\}_{j\in\texttt{os}}$, and `w` is the "payload": an element of $\Lambda(\{\xi_i\}_{i\in\texttt{is}}\sqcup\{x_j\}_{j\in\texttt{os}})$.

```
𝒜[X_{i_,j_,k_,l_}[S_, T_]] := 𝒜[{l, i}, {j, k}, <|ξ_i → S, x_j → T, x_k → S, ξ_l → T|>,
   Expand[T^(-1/2) WExp[Expand[{ξ_l, ξ_i}.( 1 1 - T ).{x_j, x_k}] /. ξ_a_ x_b_ :→ ξ_a ∧ x_b]]];
                                          ( 0   T   )
```

```
𝒜[X_{1,2,3,4}[u, v]]
```

$$\mathcal{A}\Big[\{4,1\},\{2,3\},\langle|\xi_1\to u,x_2\to v,x_3\to u,\xi_4\to v|\rangle,\frac{\text{Wedge}[\,]}{\sqrt{v}}-\frac{x_2\wedge\xi_4}{\sqrt{v}}-\sqrt{v}\,x_3\wedge\xi_1-\frac{x_3\wedge\xi_4}{\sqrt{v}}+\sqrt{v}\,x_3\wedge\xi_4+\sqrt{v}\,x_2\wedge x_3\wedge\xi_1\wedge\xi_4\Big]$$

```
𝒜[X_{i_,j_,k_,l_}] := 𝒜[X_{i,j,k,l}[τ_i, τ_l]]
```

The linear structure on $\mathcal{A}$'s:

```
𝒜 /: α_ × 𝒜[is_, os_, cs_, w_] := 𝒜[is, os, cs, Expand[α w]]
𝒜 /: 𝒜[is1_, os1_, cs1_, w1_] + 𝒜[is2_, os2_, cs2_, w2_] /;
  (Sort@is1 == Sort@is2) ∧ (Sort@os1 == Sort@os2) ∧
   (Sort@Normal@cs1 == Sort@Normal@cs2) := 𝒜[is1, os1, cs1, w1 + w2]
```

Deciding if two $\mathcal{A}$'s are equal:

```
𝒜 /: 𝒜[is1_, os1_, _, w1_] ≡ 𝒜[is2_, os2_, _, w2_] :=
 TrueQ[(Sort@is1 === Sort@is2) ∧ (Sort@os1 === Sort@os2) ∧
   PowerExpand[w1 == w2]]
```

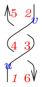
Contractions of $\mathcal{A}$-objects:

```
c_{h_,t_}@𝒜[is_, os_, cs_, w_] := 𝒜[
    DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {x_h, ξ_t}], c_{x_h,ξ_t}[w]
   ] /. If[MatchQ[cs[ξ_t], τ_], cs[ξ_t] → cs[x_h], cs[x_h] → cs[ξ_t]];
```

```
c_{4,4}[𝒜[X_{2,4,3,1}[S, T]] × 𝒜[X̄_{3,4,6,5}]]
```

$$\mathcal{A}\Big[\{1,2,3\},\{3,5,6\},\langle|\xi_2\to S,x_3\to S,\xi_1\to T,\xi_3\to\tau_3,x_6\to\tau_3,x_5\to T|\rangle,\text{Wedge}[\,]-x_3\wedge\xi_1+T\,x_3\wedge\xi_1-T\,x_3\wedge\xi_2-x_5\wedge\xi_1-x_6\wedge\xi_1+\frac{x_6\wedge\xi_1}{T}-\frac{x_6\wedge\xi_3}{T}+T\,x_3\wedge x_5\wedge\xi_1\wedge\xi_2-x_3\wedge x_6\wedge\xi_1\wedge\xi_2+T\,x_3\wedge x_6\wedge\xi_1\wedge\xi_2+x_3\wedge x_6\wedge\xi_1\wedge\xi_3-\frac{x_3\wedge x_6\wedge\xi_1\wedge\xi_3}{T}-x_3\wedge x_6\wedge\xi_2\wedge\xi_3-\frac{x_5\wedge x_6\wedge\xi_1\wedge\xi_3}{T}-x_3\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_2\wedge\xi_3\Big]$$

4. Skein relations and evaluations for $\mathcal{A}$

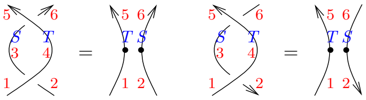


```
𝒜@{X̄_{4,1,6,3}[v, u], X̄_{3,2,5,4}}
```

$$\mathcal{A}\Big[\{1,2\},\{5,6\},\langle|\xi_2\to v,x_5\to u,\xi_1\to u,x_6\to v|\rangle,\sqrt{u}\sqrt{v}\,\text{Wedge}[\,]-\frac{\sqrt{u}\,x_5\wedge\xi_1}{\sqrt{v}}+\frac{\sqrt{u}\,x_5\wedge\xi_2}{\sqrt{v}}-\sqrt{u}\sqrt{v}\,x_5\wedge\xi_2+\frac{\sqrt{v}\,x_6\wedge\xi_1}{\sqrt{u}}-\sqrt{u}\sqrt{v}\,x_6\wedge\xi_1-\frac{\sqrt{v}\,x_6\wedge\xi_2}{\sqrt{u}}-\frac{\sqrt{u}\,x_5\wedge x_6\wedge\xi_1\wedge\xi_2}{\sqrt{v}}-\frac{\sqrt{v}\,x_5\wedge x_6\wedge\xi_1\wedge\xi_2}{\sqrt{u}}+\sqrt{u}\sqrt{v}\,x_5\wedge x_6\wedge\xi_1\wedge\xi_2\Big]$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/

Reidemeister 2



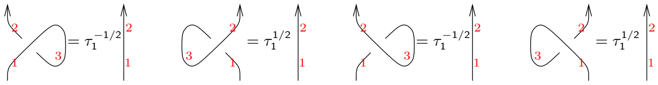
$$\mathcal{A}@\left\{X_{2,4,3,1}[S,\ T],\ \overline{X}_{3,4,6,5}\right\} \equiv \mathcal{A}@\{P_{1,5}[T],\ P_{2,6}[S]\}$$

True

$$\mathcal{A}@\left\{\overline{X}_{3,1,2,4}[S,\ T],\ X_{6,5,3,4}\right\} \equiv \mathcal{A}@\{P_{1,5}[T],\ P_{6,2}[S]\}$$

True

Reidemeister 1

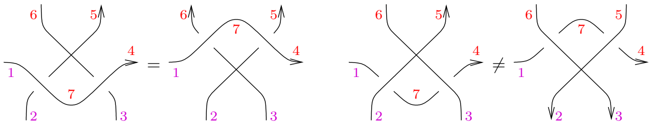


$$\left\{\mathcal{A}@\{X_{3,3,2,1}\} \equiv \tau_1^{-1/2}\,\mathcal{A}@\{P_{1,2}\},\ \mathcal{A}@\{X_{1,2,3,3}\} \equiv \tau_1^{1/2}\,\mathcal{A}@\{P_{1,2}\},\ \mathcal{A}@\{\overline{X}_{1,3,3,2}\} \equiv \tau_1^{-1/2}\,\mathcal{A}@\{P_{1,2}\},\ \mathcal{A}@\{\overline{X}_{3,1,2,3}\} \equiv \tau_1^{1/2}\,\mathcal{A}@\{P_{1,2}\}\right\}$$

{True, True, True, True}

(So we have an invariant, up to rotation numbers).

Overcrossings Commute but Undercrossings don't



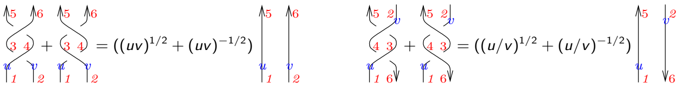
$$\mathcal{A}@\{X_{2,7,5,1},\ X_{3,4,6,7}\} \equiv \mathcal{A}@\{X_{3,7,6,1},\ X_{2,4,5,7}\}$$

True

$$\mathcal{A}@\left\{\overline{X}_{1,2,7,5},\ \overline{X}_{7,3,4,6}\right\} \equiv \mathcal{A}@\left\{\overline{X}_{1,3,7,6},\ \overline{X}_{7,2,4,5}\right\}$$

False

Conway's Second Set of Identities (see [Co])



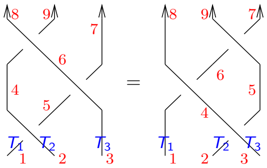
$$\mathcal{A}@\{X_{2,4,3,1}[v,\ u],\ X_{4,6,5,3}\} + \mathcal{A}@\left\{\overline{X}_{1,2,4,3}[u,\ v],\ \overline{X}_{3,4,6,5}\right\} \equiv \left(u^{1/2}\,v^{1/2} + u^{-1/2}\,v^{-1/2}\right)\mathcal{A}@\{P_{1,5}[u],\ P_{2,6}[v]\}$$

True

$$\mathcal{A}@\left\{\overline{X}_{4,1,6,3}[v,\ u],\ \overline{X}_{3,2,5,4}\right\} + \mathcal{A}@\{X_{1,6,3,4}[u,\ v],\ X_{2,5,4,3}\} \equiv \left(u^{1/2}\,v^{-1/2} + u^{-1/2}\,v^{1/2}\right)\mathcal{A}@\{P_{1,5}[u],\ P_{2,6}[v]\}$$

True

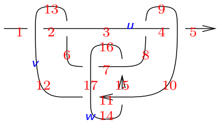
Reidemeister 3



$$\mathcal{A}@\{X_{2,5,4,1}[T_2,\ T_1],\ X_{3,7,6,5}[T_3,\ T_1],\ X_{6,9,8,4}\} \equiv \mathcal{A}@\{X_{3,5,4,2}[T_3,\ T_2],\ X_{4,6,8,1}[T_3,\ T_1],\ X_{5,7,9,6}\}$$

True

The Relation with the Multivariable Alexander Polynomial



$$\mathtt{MVA} = u^{-1/2}\,v^{-1/2}\,w^{-1/2}\,(u-1)\,(v-1)\,(w-1);$$

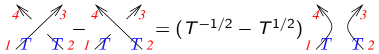
$$\mathtt{A} = \left\{\overline{X}_{1,12,2,13}[u,\ v],\ \overline{X}_{13,2,6,3},\ X_{8,4,9,3},\ X_{4,10,5,9},\ X_{6,17,7,16}[v,\ w],\ X_{15,8,16,7},\ \overline{X}_{14,10,15,11},\ \overline{X}_{11,17,12,14}\right\}\ //\ \mathcal{A}\ //\ \mathtt{Last}\ //\ \mathtt{Factor}$$

$$\frac{(-1+u)^2\,(-1+v)\,(-1+w)\,(\mathtt{Wedge}[\,] - x_5 \wedge \xi_1)}{u\,v}$$

$$\mathtt{A} == u^{-1/2}\,(u-1)\,u^0\,v^{-1/2}\,w^{1/2}\,\mathtt{MVA}\,(\mathtt{Wedge}[{}_\wedge] - x_5 \wedge \xi_1)$$

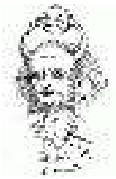
True

The Conway Relation (see [Co])

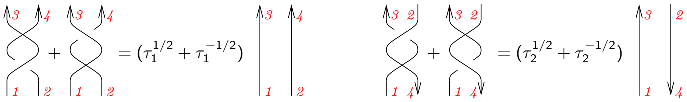


$$\mathcal{A}@\{X_{2,3,4,1}[T,\ T]\} - \mathcal{A}@\left\{\overline{X}_{1,2,3,4}[T,\ T]\right\} \equiv \left(T^{-1/2} - T^{1/2}\right)\mathcal{A}@\{P_{1,4}[T],\ P_{2,3}[T]\}$$

True



Virtual versions (Archibald, [Ar])



$$\mathcal{A}@\{X_{2,3,4,1}\} + \mathcal{A}@\{\overline{X}_{2,1,4,3}\} \equiv \left(\tau_1^{1/2} + \tau_1^{-1/2}\right)\mathcal{A}@\{P_{1,3},\ P_{2,4}\}$$

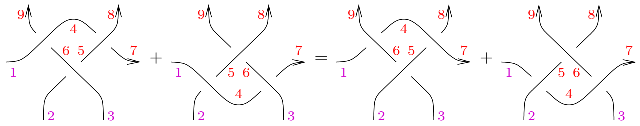
True

$$\mathcal{A}@\{\overline{X}_{1,2,3,4}\} + \mathcal{A}@\{X_{1,4,3,2}\} \equiv \left(\tau_2^{1/2} + \tau_2^{-1/2}\right)\mathcal{A}@\{P_{1,3},\ P_{2,4}\}$$

True

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/

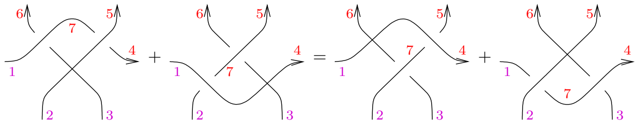
Conway's Third Identity (see [Co])



$$\mathcal{A}@\{X_{6,4,9,1}, \overline{X}_{4,5,7,8}, \overline{X}_{2,3,5,6}\} + \mathcal{A}@\{X_{2,4,5,1}, \overline{X}_{4,3,7,6}, X_{6,8,9,5}\} \equiv$$
$$\mathcal{A}@\{\overline{X}_{1,6,4,9}, X_{5,7,8,4}, X_{3,5,6,2}\} + \mathcal{A}@\{\overline{X}_{1,2,4,5}, X_{3,7,6,4}, \overline{X}_{5,6,8,9}\}$$

True

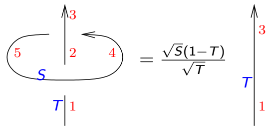
Virtual version (Archibald, [Ar])



$$\mathcal{A}@\{X_{3,7,6,1}, \overline{X}_{7,2,4,5}\} + \mathcal{A}@\{X_{2,4,7,1}, X_{3,5,6,7}\} \equiv$$
$$\mathcal{A}@\{X_{3,7,6,2}, X_{7,4,5,1}\} + \mathcal{A}@\{\overline{X}_{1,2,7,5}, X_{3,4,6,7}\}$$

True

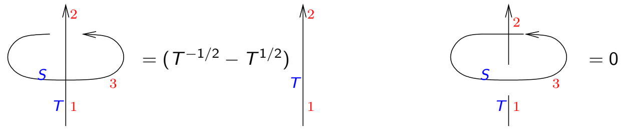
Jun Murakami's Fifth Axiom (see [Mu])



$$\mathcal{A}@\{X_{1,4,2,5}[T, S], X_{4,3,5,2}\} \equiv \frac{\sqrt{S}\ (1 - T)}{\sqrt{T}} \mathcal{A}@\{P_{1,3}[T]\}$$

True

Virtual versions (Archibald, [Ar])



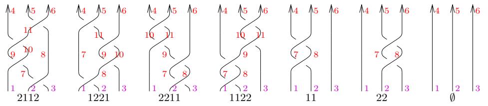
$$\mathcal{A}@\{X_{3,2,3,1}[S, T]\} \equiv \left(T^{-1/2} - T^{1/2}\right) \mathcal{A}@\{P_{1,2}[T]\}$$

True

$$\mathcal{A}@\{X_{1,3,2,3}\}$$

$$\mathcal{A}[\{1\}, \{2\}, \langle| \xi_1 \to \tau_1, x_2 \to \tau_1 |\rangle, 0]$$

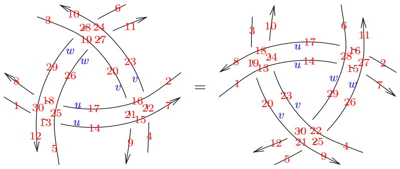
Jun Murakami's Third Axiom (see [Mu])



$$\mathcal{A}_{2112} = \mathcal{A}@\{X_{3,8,7,2}, X_{7,10,9,1}, X_{10,11,4,9}, X_{8,6,5,11}\};$$
$$\mathcal{A}_{1221} = \mathcal{A}@\{X_{2,8,7,1}, X_{3,10,9,8}, X_{10,6,11,9}, X_{11,5,4,7}\};$$
$$\mathcal{A}_{2211} = \mathcal{A}@\{X_{3,8,7,2}, X_{8,6,9,7}, X_{9,11,10,1}, X_{11,5,4,10}\};$$
$$\mathcal{A}_{1122} = \mathcal{A}@\{X_{2,8,7,1}, X_{8,9,4,7}, X_{3,11,10,9}, X_{11,6,5,10}\};$$
$$\mathcal{A}_{11} = \mathcal{A}@\{X_{2,8,7,1}, X_{8,5,4,7}, P_{3,6}\}; \quad \mathcal{A}_{22} = \mathcal{A}@\{X_{3,8,7,2}, X_{8,6,5,7}, P_{1,4}\};$$
$$\mathcal{A}_{\phi} = \mathcal{A}@\{P_{1,4}, P_{2,5}, P_{3,6}\};$$
$$g_+[z\_] := z^{1/2} + z^{-1/2}; \quad g_-[z\_] := z^{1/2} - z^{-1/2};$$
$$g_+[\tau_1]\, g_-[\tau_2]\, \mathcal{A}_{2112} - g_-[\tau_2]\, g_+[\tau_3]\, \mathcal{A}_{1221} - g_-[\tau_3 / \tau_1]\, (\mathcal{A}_{2211} + \mathcal{A}_{1122}) +$$
$$g_-[\tau_2\, \tau_3 / \tau_1]\, g_+[\tau_3]\, \mathcal{A}_{11} - g_+[\tau_1]\, g_-[\tau_1\, \tau_2 / \tau_3]\, \mathcal{A}_{22} \equiv g_-\left[\tau_3^2 / \tau_1^2\right] \mathcal{A}_{\phi}$$

True

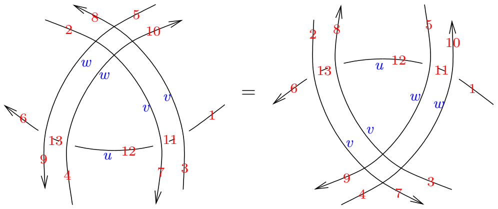
The Naik-Stanford Double Delta Move (see [NS])



$$\text{Timing}\left[\mathcal{A}@\{X_{6,10,28,24}[w, v], \overline{X}_{28,3,29,19}[w, v], X_{26,20,27,19}[w, v], \overline{X}_{27,23,11,24}[w, v],\right.$$
$$X_{1,12,13,30}[u, w], \overline{X}_{13,5,14,25}[u, w], X_{17,26,18,25}[u, w], \overline{X}_{18,29,8,30}[u, w],$$
$$X_{4,7,22,15}[v, u], \overline{X}_{22,2,23,16}[v, u], X_{20,17,21,16}[v, u], \overline{X}_{21,14,9,15}[v, u]\} \equiv$$
$$\mathcal{A}@\{X_{5,9,25,21}[w, v], \overline{X}_{25,4,26,22}[w, v], X_{29,23,30,22}[w, v], \overline{X}_{30,20,12,21}[w, v],$$
$$X_{2,11,16,27}[u, w], \overline{X}_{16,6,17,28}[u, w], X_{14,29,15,28}[u, w], \overline{X}_{15,26,7,27}[u, w],$$
$$\left.X_{3,8,19,18}[v, u], \overline{X}_{19,1,20,13}[v, u], X_{23,14,24,13}[v, u], \overline{X}_{24,17,10,18}[v, u]\}\right]$$

{190.422, True}

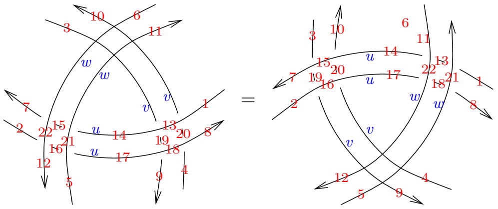
Virtual Version 1 (Archibald, [Ar])



$$\mathcal{A}@\{X_{1,8,11,3}[u, v], \overline{X}_{11,2,12,7}[u, v], X_{12,10,13,4}[u, w], \overline{X}_{13,5,6,9}[u, w]\} \equiv$$
$$\mathcal{A}@\{X_{1,10,11,4}[u, w], \overline{X}_{11,5,12,9}[u, w], X_{12,8,13,3}[u, v], \overline{X}_{13,2,6,7}[u, v]\}$$

True

Virtual Version 2 (Archibald, [Ar])



$$\mathcal{A}@\{\overline{X}_{20,1,10,13}[v, u], X_{3,14,19,13}[v, u], X_{14,11,15,21}[u, w], \overline{X}_{15,6,7,22}[u, w],$$
$$X_{2,12,16,22}[u, w], \overline{X}_{16,5,17,21}[u, w], \overline{X}_{19,17,9,18}[v, u], X_{4,8,20,18}[v, u]\} \equiv$$
$$\mathcal{A}@\{X_{1,11,13,21}[u, w], \overline{X}_{13,6,14,22}[u, w], \overline{X}_{20,14,10,15}[v, u], X_{3,7,19,15}[v, u],$$
$$\overline{X}_{19,2,9,16}[v, u], X_{4,17,20,16}[v, u], X_{17,12,18,22}[u, w], \overline{X}_{18,5,8,21}[u, w]\}$$

True

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/`

5. Some Problems in Heaven

Unfortunately, $\dim\mathcal{A}(\mathcal{X},X)=\dim\Lambda(\mathcal{X},X)=4^{|X|}$ is big. Fortunately, we have the following theorem, a version of one of the main results in Halacheva's thesis, [Ha1, Ha2]:

Theorem. Working in $\Lambda(\mathcal{X}\cup X)$, if $w=\omega e^{\lambda}$ is a balanced Gaussian (namely, a scalar ω times the exponential of a quadratic $\lambda=\sum_{\zeta\in\mathcal{X},z\in X}\alpha_{\zeta,z}\zeta z$), then generically so is $c_{x,\xi}e^{\lambda}$.

(This is great news! The space of balanced quadratics is only $|\mathcal{X}||X|$-dimensional!)

Γ-calculus.

Thus we have an almost-always-defined "Γ-calculus": a contraction algebra morphism $\mathcal{T}(\mathcal{X},X)\to R\times(\mathcal{X}\otimes_{R/R}X)$ whose behaviour under contractions is given by

$$c_{x,\xi}(\omega,\lambda=\mu+\eta x+\xi y+\alpha\xi x)=((1-\alpha)\omega,\mu+\eta y/(1-\alpha)).$$

(Γ is fully defined on pure tangles – tangles without closed components – and hence on long knots).

Multiplying and comparing Γ objects:

```
Γ /: Γ[is1_, os1_, cs1_, ω1_, λ1_] × Γ[is2_, os2_, cs2_, ω2_, λ2_] :=
 Γ[is1 ⋃ is2, os1 ⋃ os2, Join[cs1, cs2], ω1 ω2, λ1 + λ2]
Γ /: Γ[is1_, os1_, _, ω1_, λ1_] ≡ Γ[is2_, os2_, _, ω2_, λ2_] :=
 TrueQ[(Sort@is1 === Sort@is2) ∧ (Sort@os1 === Sort@os2) ∧
   Simplify[ω1 == ω2] ∧ CF@λ1 == CF@λ2]
```

No rules for linear operations!

The crossings and the point:

```
Γ[X_{i_,j_,k_,l_}[S_, T_]] := Γ[{l, i}, {j, k}, <|ξ_i → S, x_j → T, x_k → S, ξ_l → T|>,
    T^(-1/2), CF[{ξ_l, ξ_i}.({{1, 1 - T}, {0, T}}).{x_j, x_k}]];
Γ[X̄_{i_,j_,k_,l_}[S_, T_]] := Γ[{i, j}, {k, l}, <|ξ_i → S, ξ_j → T, x_k → S, x_l → T|>,
    T^(1/2), CF[{ξ_i, ξ_j}.({{T^-1, 0}, {1 - T^-1, 1}}).{x_k, x_l}]];
Γ[X_{i_,j_,k_,l_}] := Γ[X_{i,j,k,l}[τ_i, τ_l]];
Γ[X̄_{i_,j_,k_,l_}] := Γ[X̄_{i,j,k,l}[τ_i, τ_j]];
```

```
Γ[P_{i_,j_}[T_]] := Γ[{i}, {j}, <|ξ_i → T, x_j → T|>, 1, ξ_i x_j];
Γ[P_{i_,j_}] := Γ[P_{i,j}[τ_i]];
```

Proof. Recall that $c_{x,\xi}\colon(1,\xi,x,x\xi)w'\mapsto(1,0,0,1)w'$, write $\lambda=\mu+\eta x+\xi y+\alpha\xi x$, and ponder $e^{\lambda}=$

$$\dots+\frac{1}{k!}\underbrace{(\mu+\eta x+\xi y+\alpha\xi x)(\mu+\eta x+\xi y+\alpha\xi x)\cdots(\mu+\eta x+\xi y+\alpha\xi x)}_{k\text{ factors}}+\dots.$$

Then $c_{x,\xi}e^{\lambda}$ has three contributions:

- e^{μ}, from the term proportional to 1 (namely, independent of ξ and x) in e^{λ}
- $-\alpha e^{\mu}$, from the term proportional to $x\xi$, where the x and the ξ come from the same factor above.
- $\eta y e^{\mu}$, from the term proportional to $x\xi$, where the x and the ξ come from different factors above.

So $c_{x,\xi}e^{\lambda}=e^{\mu}(1-\alpha+\eta y)=(1-\alpha)e^{\mu}(1+\eta y/(1-\alpha))=(1-\alpha)e^{\mu}e^{\eta y/(1-\alpha)}=(1-\alpha)e^{\mu+\eta y/(1-\alpha)}$.

☐

6. An Implementation of Γ.

If I didn't implement I wouldn't believe myself.

Written in Mathematica [Wo], available as the notebook `Gamma.nb` at `http://drorbn.net/mo21/ap`. Code lines are highlighted in grey, demo lines are plain. We start with canonical forms for quadratics with rational function coefficients:

```
CCF[ℰ_] := Factor[ℰ];
CF[ℰ_] := Module[{vs = Union@Cases[ℰ, (ξ | x)_, ∞]},
   Total[(CCF[#⟦2⟧] (Times @@ vs^#⟦1⟧)) & /@ CoefficientRules[ℰ, vs]]];
```

Contractions:

```
c_{h_,t_}@Γ[is_, os_, cs_, ω_, λ_] := Module[{α, η, y, μ},
    α = ∂_{ξ_t,x_h} λ;  μ = λ /. ξ_t | x_h → 0;
    η = ∂_{x_h} λ /. ξ_t → 0;  y = ∂_{ξ_t} λ /. x_h → 0;
     Γ[
      DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {x_h, ξ_t}],
      CCF[(1 - α) ω], CF[μ + η y / (1 - α)]
     ] /. If[MatchQ[cs[ξ_t], τ_], cs[ξ_t] → cs[x_h], cs[x_h] → cs[ξ_t]]];
c@Γ[is_, os_, cs_, ω_, λ_] := Fold[c_{#2,#2}[#1] &, Γ[is, os, cs, ω, λ], is ⋂ os]
```

Automatic intelligent contractions:

```
Γ[{γ_Γ}] := c[γ];
Γ[{γ1_Γ, γs__Γ}] := Module[{γ2},
  γ2 = First@MaximalBy[{γs}, Length[γ1⟦1⟧ ⋂ #⟦2⟧] + Length[γ1⟦2⟧ ⋂ #⟦1⟧] &];
  Γ[Join[{c[γ1 γ2]}, DeleteCases[{γs}, γ2]]] ]
Γ[𝒪s_List] := Γ[Γ /@ 𝒪s]
```

Conversions $\mathcal{A} \leftrightarrow \Gamma$:

```
Γ@𝒜[is_, os_, cs_, w_] := Module[{i, j, ω = Coefficient[w, Wedge[‸]]},
   Γ[is, os, cs, ω, Sum[Cancel[-Coefficient[w, x_j ∧ ξ_i] ξ_i x_j / ω],
     {i, is}, {j, os}]]
  ];
𝒜@Γ[is_, os_, cs_, ω_, λ_] :=
  𝒜[is, os, cs, Expand[ω WExp[Expand[λ] /. ξ_a_ x_b_ :→ ξ_a ∧ x_b]]];
```

The conversions are inverses of each other:

```
γ = Γ[{1, 2, 3}, {1, 2, 3}, {x_1 → τ_1, x_2 → τ_2, x_3 → τ_3, ξ_1 → τ_1, ξ_2 → τ_2, ξ_3 → τ_3},
    ω, a_11 x_1 ξ_1 + a_12 x_2 ξ_1 + a_13 x_3 ξ_1 + a_21 x_1 ξ_2 + a_22 x_2 ξ_2 + a_23 x_3 ξ_2 + a_31 x_1 ξ_3 +
     a_32 x_2 ξ_3 + a_33 x_3 ξ_3];
Γ@𝒜@γ == γ
```

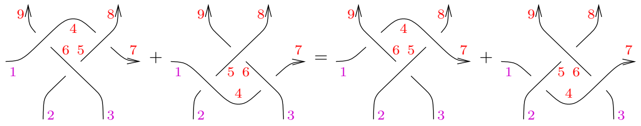
```
True
```

The conversions commute with contractions:

```
Γ@c_3,3@𝒜@γ ≡ c_3,3@γ
```

```
True
```

Conway's Third Identity



Sorry, Γ has nothing to say about that...'

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- I. Halacheva, *Alexander Type Invariants of Tangles, Skew Howe Duality for Crystals and The Cactus Group,* University of Toronto Ph.D. thesis, 2016, `http://drorbn.net/mo21/HT`.
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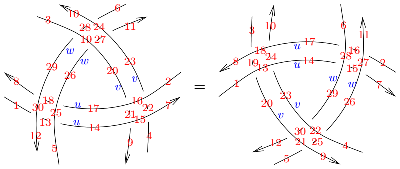
Thank You!

The Naik-Stanford Double Delta Move (again)



```
Timing[Γ@{X_6,10,28,24[w, v], X̄_28,3,29,19[w, v], X_26,20,27,19[w, v], X̄_27,23,11,24[w, v],
    X_1,12,13,30[u, w], X̄_13,5,14,25[u, w], X_17,26,18,25[u, w], X̄_18,29,8,30[u, w],
    X_4,7,22,15[v, u], X̄_22,2,23,16[v, u], X_20,17,21,16[v, u], X̄_21,14,9,15[v, u]} ≡
  Γ@{X_5,9,25,21[w, v], X̄_25,4,26,22[w, v], X_29,23,30,22[w, v], X̄_30,20,12,21[w, v],
    X_2,11,16,27[u, w], X̄_16,6,17,28[u, w], X_14,29,15,28[u, w], X̄_15,26,7,27[u, w],
    X_3,8,19,18[v, u], X̄_19,1,20,13[v, u], X_23,14,24,13[v, u], X̄_24,17,10,18[v, u]}]
```

```
{0.703125, True}
```

What I still don't understand.

- What becomes of $c_{x,\xi}e^{\lambda}$ if we have to divide by 0 in order to write it again as an exponentiated quadratic? Does it still live within a very small subset of $\Lambda(\mathcal{X} \sqcup X)$?
- How do cablings and strand reversals fit within $\mathcal{A}$?
- Are there "classicality conditions" satisfied by the invariants of classical tangles (as opposed to virtual ones)?

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- J. Murakami, *A State Model for the Multivariable Alexander Polynomial,* Pacific J. Math. **157-1** (1993) 109-–135.
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Video and more at `http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/`

Thanks for inviting me to speak in my basement!

The Alexander Polynomial is a Quantum Invariant in a Different Way

ωεβ:=http://drorbn.net/cat20/

▶ On a chat window here I saw a comment "Alexander is the quantum $gl(1|1)$ invariant". I have an opinion about this, and I'd like to share it. First, some stories.

I left the wonderful subject of Categorification nearly 15 years ago. It got crowded, lots of very smart people had things to say, and I feared I will have nothing to add. Also, clearly the next step was to categorify all other "quantum invariants". Except it was not clear what "categorify" means. Worse, I felt that I (perhaps "we all") didn't understand "quantum invariants" well enough to try to categorify them, whatever that might mean.

I still feel that way! I learned a lot since 2006, yet I'm still not comfortable with quantum algebra, quantum groups, and quantum invariants. I still don't feel that I know what God had in mind when She created this topic.

Yet I'm not here to rant about my philosophical quandaries, but only about things that I learned about the Alexander polynomial after 2006.

Yes, the Alexander polynomial fits within the Dogma, "one invariant for every Lie algebra and representation" (it's $gl(1|1)$, I hear). But it's better to think of it as a quantum invariant arising by other means, outside the Dogma.

Alexander comes from (or in) practically any non-Abelian Lie algebra. Foremost from the not-even-semi-simple 2D "$ax+b$" algebra. You get a polynomially-sized extension to tangles using some lovely formulas (can you categorify them?). It generalizes to higher dimensions and it has an organized family of siblings. (There are some questions too, beyond categorification).

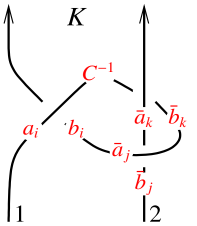
I note the spectacular existing categorification of Alexander by Ozsváth and Szabó. The theorems are proven and a lot they say, the programs run and fast they run. Yet if that's where the story ends, She has abandoned us. Or at least abandoned me: a simpleton will never be able to catch up.

If you care only about categorification, the take-home from my talk will be a challenge: Categorify what I believe is the best Alexander invariant for tangles.

The Yang-Baxter Technique. Given an algebra U (typically some $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and suitable elements R, C,

$$R=\sum a_i\otimes b_i\in U\otimes U \quad\text{with}\quad R^{-1}=\sum \bar{a}_i\otimes\bar{b}_i \quad\text{and}\quad C,C^{-1}\in U,$$

form
$$Z(K)=\sum_{i,j,k} a_iC^{-1}\bar{b}_k\bar{a}_jb_i\otimes\bar{b}_j\bar{a}_k.$$



Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

Example 1. Let $\mathfrak{a}:=L\langle a,x\rangle/([a,x]=x)$, $\mathfrak{b}:=\mathfrak{a}^\star=\langle b,y\rangle$, and $\mathfrak{g}:=\mathfrak{b}\rtimes\mathfrak{a}=\mathfrak{b}\oplus\mathfrak{a}$ with $[a,x]=x$, $[a,y]=-y$, $[b,\cdot]=0$, and $[x,y]=b$ and with $\deg(y,b,a,x)=(1,1,0,0)$. Let $U=\hat{\mathcal{U}}(\mathfrak{g})$ and

$$R:=\mathbb{e}^{b\otimes a+y\otimes x}\in U\otimes U \quad\text{or better}\quad R_{ij}:=\mathbb{e}^{b_ia_j+y_ix_j}\in U_i\otimes U_j,\quad\text{and}\quad C_i=\mathbb{e}^{-b_i/2}.$$

Gentle's Agreement. Everything converges!

Theorem 1. With "scalars":=power series in $\{b_i\}$ which are rational functions in $\{b_i\}$ and $\{B_i:=\mathbb{e}^{b_i}\}$,

$$Z(K)=\mathbb{O}_{ybax}\left(\omega^{-1}\mathbb{e}^{\underbrace{l^{ij}b_ia_j}_{L}+\underbrace{q^{ij}y_ix_j}_{Q}}(1+\epsilon P_1+\epsilon^2P_2+\dots)\right)$$

a tangle w/o closed components

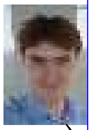
the "i over j" linking numbers (integers)

categorify us! scalars

a docile perturbation for other Lie algebras; semisimple algebras have a hidden parameter ϵ!

"normal ordering" at *ybax* order

a scalar; if K is a long knot, the Alexander poly $\Delta(T)$ categorify me!


With Roland van der Veen


Continues Lev Rozansky

Example 2. Let $\mathfrak{h}:=A\langle p,x\rangle/([p,x]=1)$ be the Heisenberg algebra, with $C_i=\mathbb{e}^{t/2}$ and $R_{ij}=\mathbb{e}^{t/2}\mathbb{e}^{t(p_i-p_j)x_j}$.

I just told you the whole Alexander story! Everything else is details.

Claim. $R_{ij}=\mathbb{O}_{px}\left(\mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j}\right)$.

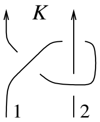
Theorem 2. $Z(K)=\mathbb{O}_{px}\left(\omega^{-1}\mathbb{e}^{q^{ij}p_ix_j}\right)$ where ω and the q^{ij} are rational functions in $T=\mathbb{e}^t$. In fact ω and ωq^{ij} are Laurent polynomials (categorify us!). When K is a long knot, ω is the Alexander polynomial.

Packaging. Write $\mathbb{O}_{px}\left(\omega^{-1}\mathbb{e}^{q^{ij}p_ix_j}\right)$ as

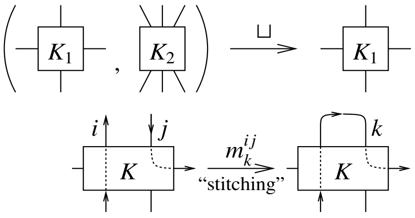
$$\mathbb{E}_{p_1,\dots,x_1,\dots}[\omega,Q]\leftrightarrow\begin{array}{c|ccc}\omega & x_1 & x_2 & \cdots\\\hline p_1 & q^{11} & q^{12} & \cdots\\ p_2 & q^{21} & q^{22} & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{array}$$

The "First Tangle". $Z(K)=$

$$\mathbb{E}_{12}\left[\frac{2T-1}{T},\frac{(T-1)(p_1-p_2)(Tx_1-x_2)}{2T-1}\right]=\begin{array}{c|cc}2-T^{-1} & x_1 & x_2\\\hline p_1 & \frac{T(T-1)}{2T-1} & \frac{1-T}{2T-1}\\ p_2 & \frac{T(1-T)}{2T-1} & \frac{T-1}{2T-1}\end{array}$$



(v-)Tangles. Generated by $\{\nearrow,\nwarrow\}$!



There's also strand doubling and reversal…

Theorem 3. Full evaluation via

$$\left(\,_i\nearrow_j,\ _j\nwarrow_i\right)\to\begin{array}{c|cc}1 & x_i & x_j\\\hline p_i & 0 & T^{\pm1}-1\\ p_j & 0 & 1-T^{\pm1}\end{array}\qquad(1)\square$$

$$K_1\sqcup K_2\to\begin{array}{c|cc}\omega_1\omega_2 & X_1 & X_2\\\hline P_1 & A_1 & 0\\ P_2 & 0 & A_2\end{array}\qquad(2)\square$$

$$\begin{array}{c|ccc}\omega & x_i & x_j & \cdots\\\hline p_i & \alpha & \beta & \theta\\ p_j & \gamma & \delta & \epsilon\\ \vdots & \phi & \psi & \Xi\end{array}\xrightarrow{hm^{ij}_k}\qquad(3)$$

$$\begin{array}{c|cc}(1+\gamma)\omega & x_k & \cdots\\\hline p_k & 1+\beta-\frac{(1-\alpha)(1-\delta)}{1+\gamma} & \theta+\frac{(1-\alpha)\epsilon}{1+\gamma}\\ \vdots & \psi+\frac{(1-\delta)\phi}{1+\gamma} & \Xi-\frac{\phi\epsilon}{1+\gamma}\end{array}$$

"Γ-calculus" relates via $A\leftrightarrow I-A^T$ and has slightly simpler formulas: $\omega\to(1-\beta)\omega$,

$$\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}\to\begin{pmatrix}\gamma+\frac{\alpha\delta}{1-\beta}&\epsilon+\frac{\delta\theta}{1-\beta}\\\phi+\frac{\alpha\psi}{1-\beta}&\Xi+\frac{\psi\theta}{1-\beta}\end{pmatrix}$$

Why Should You Categorify This? The simplest and fastest Alexander for tangles, easily generalizes to the multi-variable case, generalizes to v-tangles and w-tangles, generalizes to other Lie algebras. In fact, it's in almost any Lie algebra, and you don't even need to know what is $gl(1|1)$! But you'll have to deal with denominators and/or divisions!

Note. Example 1 ⇝ Example 2 via $\mathfrak{g}\hookrightarrow\mathfrak{h}(t)$ via $(y,b,a,x)\mapsto(-tp,t,px,x)$.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/LearningSeminarOnCategorification-2006/`

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $(p,x)^* = (\pi,\xi)$

The Generating Series $\mathcal{G}$: $\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \to \mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \underset{\mathcal{G}}{\xrightarrow{\sim}} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}L(z_A^n) = L\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p\in\mathbb{Q}[z_A].$$

Claim. If $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $M\in\operatorname{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Examples. • $\mathcal{G}(id\colon \mathbb{Q}[p,x]\to\mathbb{Q}[p,x]) = \mathbb{e}^{\pi p+\xi x}$.

• Consider $R_{ij}\in(\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!] \cong \operatorname{Hom}\left(\mathbb{Q}[]\to\mathbb{Q}[p_i,x_i,p_j,x_j]\right)[\![t]\!]$. Then $\mathcal{G}(R_{ij}) = \mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j} = \mathbb{e}^{(T-1)(p_i-p_j)x_j}$.

Heisenberg Algebras. Let $\mathfrak{h} = A\langle p,x\rangle/([p,x]=1)$, let $\mathbb{O}_i\colon\mathbb{Q}[p_i,x_i]\to\mathfrak{h}_i$ is the "p before x" PBW normal ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[p_i,x_i,p_j,x_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathfrak{h}_i\otimes\mathfrak{h}_j \xrightarrow{m_k^{ij}} \mathfrak{h}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[p_k,x_k].$$

Then $\mathcal{G}(hm_k^{ij}) = \mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}$.

Proof. Recall the "Weyl CCR" $\mathbb{e}^{\xi x}\mathbb{e}^{\pi p} = \mathbb{e}^{-\xi\pi}\mathbb{e}^{\pi p}\mathbb{e}^{\xi x}$, and find

$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= \mathbb{e}^{\pi_ip_i+\xi_ix_i+\pi_jp_j+\xi_jx_j}/\!\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1}\\ &= \mathbb{e}^{\pi_ip_i}\mathbb{e}^{\xi_ix_i}\mathbb{e}^{\pi_jp_j}\mathbb{e}^{\xi_jx_j}/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{\pi_ip_k}\mathbb{e}^{\xi_ix_k}\mathbb{e}^{\pi_jp_k}\mathbb{e}^{\xi_jx_k}/\!\!/\mathbb{O}_k^{-1}\\ &= \mathbb{e}^{-\xi_i\pi_j}\mathbb{e}^{(\pi_i+\pi_j)p_k}\mathbb{e}^{(\xi_i+\xi_j)x_k}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}.\end{aligned}$$

GDO := The category with objects finite sets and

$$\operatorname{mor}(A\to B) = \left\{\mathcal{L} = \omega\mathbb{e}^Q\right\}\subset\mathbb{Q}[\![\zeta_A,z_B]\!],$$

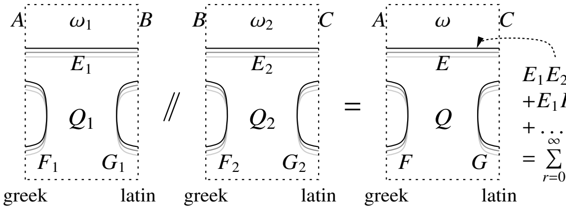
where: • ω is a scalar. • Q is a "small" quadratic in $\zeta_A\cup z_B$. • Compositions: $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

Compositions. In $\operatorname{mor}(A\to B)$,


R. Feynman

$$Q = \sum_{i\in A, j\in B}E_{ij}\zeta_iz_j + \frac12\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j + \frac12\sum_{i,j\in B}G_{ij}z_iz_j,$$

and so (remember, $e^x = 1+x+xx/2+xxx/6+\dots$)



$$E_1E_2+E_1F_2G_1E_2 +E_1F_2G_1F_2G_1E_2 +\dots = \sum_{r=0}^\infty E_1(F_2G_1)^rE_2$$

where • $E = E_1(I-F_2G_1)^{-1}E_2$ • $F = F_1+E_1F_2(I-G_1F_2)^{-1}E_1^T$ • $G = G_2+E_2^TG_1(I-F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2\det(I-F_2G_1)^{-1/2}$

Proof of Claim in Example 2. Let $\Phi_1 := \mathbb{e}^{t(p_i-p_j)x_j}$ and $\Phi_2 := \mathbb{O}_{p_jx_j}\left(\mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j}\right) =: \mathbb{O}(\Psi)$. We show that $\Phi_1=\Phi_2$ in $(\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!]$ by showing that both solve the ODE $\partial_t\Phi = (p_i-p_j)x_j\Phi$ with $\Phi|_{t=0}=1$. For Φ_1 this is trivial. $\Phi_2|_{t=0}=1$ is trivial, and

$$\partial_t\Phi_2 = \mathbb{O}(\partial_t\Psi) = \mathbb{O}(\mathbb{e}^t(p_i-p_j)x_j\Psi)$$

$$\begin{aligned}(p_i-p_j)x_j\Phi_2 &= (p_i-p_j)x_j\mathbb{O}(\Psi) = (p_i-p_j)\mathbb{O}(x_j\Psi-\partial_{p_j}\Psi)\\ &= \mathbb{O}\left((p_i-p_j)(x_j\Psi+(\mathbb{e}^t-1)x_j\Psi)\right) = \mathbb{O}(\mathbb{e}^t(p_i-p_j)x_j\Psi)\end{aligned}$$

□

Implementation.

Without, don't trust!

```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
```

```
𝔼_{A1_→B1_}[ω1_, Q1_] 𝔼_{A2_→B2_}[ω2_, Q2_] ^:= 𝔼_{A1⋃A2→B1⋃B2}[ω1 ω2, Q1 + Q2]
```

```
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂_{i,j}Q1, {i, A1}, {j, B1}]; E2 = M[∂_{i,j}Q2, {i, A2}, {j, B2}];
  F1 = M[∂_{i,j}Q1, {i, A1}, {j, A1}]; F2 = M[∂_{i,j}Q2, {i, A2}, {j, A2}];
  G1 = M[∂_{i,j}Q1, {i, B1}, {j, B1}]; G2 = M[∂_{i,j}Q2, {i, B2}, {j, B2}];
  𝔼_{A1→B2}[CF[ω1 ω2 Det[I - F2.G1]^{1/2}], CF@Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1^T).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2^T.G1.Inverse[I - F2.G1].E2).B2]]]]
```

```
A_ \ B_ := Complement[A, B];
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* =!= A2) :=
 𝔼_{A1⋃(A2\B1*)→B1⋃A2*}[ω1, Q1 + Sum[ζ* ζ, {ζ, A2 \ B1*}]] //
  𝔼_{B1*⋃A2→B2⋃(B1\A2*)}[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

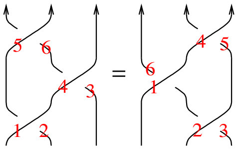
```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)_i;
l_List* := #* & /@ l;
```

```
R_{i_,j_} := 𝔼_{{}→{p_i,x_i,p_j,x_j}}[T^{-1/2}, (1 - T) p_j x_j + (T - 1) p_i x_j];
R̄_{i_,j_} := 𝔼_{{}→{p_i,x_i,p_j,x_j}}[T^{1/2}, (1 - T^{-1}) p_j x_j + (T^{-1} - 1) p_i x_j];
C_{i_} := 𝔼_{{}→{p_i,x_i}}[T^{-1/2}, 0];
C̄_{i_} := 𝔼_{{}→{p_i,x_i}}[T^{1/2}, 0];
```

```
hm_{i_,j_→k_} := 𝔼_{{π_i,ξ_i,π_j,ξ_j}→{p_k,x_k}}[1, -ξ_i π_j + (π_i + π_j) p_k + (ξ_i + ξ_j) x_k]
```

```
𝔼_{{}→vs_}[ωi_, Q_]_h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[ωi, 1 + Length@ps, 1 + Length@xs];
  M〚2 ;;, 2 ;;〛 = Table[CF[∂_{i,j}Q], {i, ps}, {j, xs}];
  M〚2 ;;, 1〛 = ps; M〚1, 2 ;;〛 = xs;
  MatrixForm[M]_h]
```

Proof of Reidemeister 3.



```
(R_{1,2} R_{4,3} R_{5,6} // hm_{1,4→1} hm_{2,5→2} hm_{3,6→3}) ==
 (R_{2,3} R_{1,6} R_{4,5} // hm_{1,4→1} hm_{2,5→2} hm_{3,6→3})

True
```

□

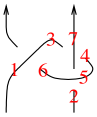
The "First Tangle".

```
Factor /@
 (z = R_{1,6} C̄_3 R̄_{7,4} R̄_{5,2} // hm_{1,3→1} // hm_{1,4→1} // hm_{1,5→1} // hm_{1,6→1} // hm_{2,7→2})
```

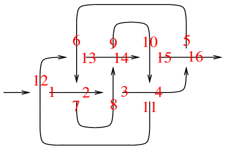
$$\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{-1+2T}{T}, \frac{(-1+T)(p_1-p_2)(Tx_1-x_2)}{-1+2T}\right]$$

```
z_h
```

$$\begin{pmatrix}\frac{-1+2T}{T} & x_1 & x_2\\ p_1 & \frac{-T+T^2}{-1+2T} & \frac{1-T}{-1+2T}\\ p_2 & \frac{T-T^2}{-1+2T} & \frac{-1+T}{-1+2T}\end{pmatrix}_h$$



The knot 8_{17}.



```
z = R̄_{12,1} R̄_{27} R̄_{83} R̄_{4,11} R_{16,5} R_{6,13} R_{14,9} R_{10,15};
Table[z = z // hm_{1k→1}, {k, 2, 16}] // Last
```

$$\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3}, 0\right]$$

Proof of Theorem 3, (3).

```
{(γ1 = 𝔼_{{}→{p_1,x_1,p_2,x_2,p_3,x_3}}[ω, {p_1, p_2, p_3}.(α β θ; γ δ ϵ; ϕ ψ Ξ).{x_1, x_2, x_3}])_h,
 (γ1 // hm_{1,2→0})_h}
```

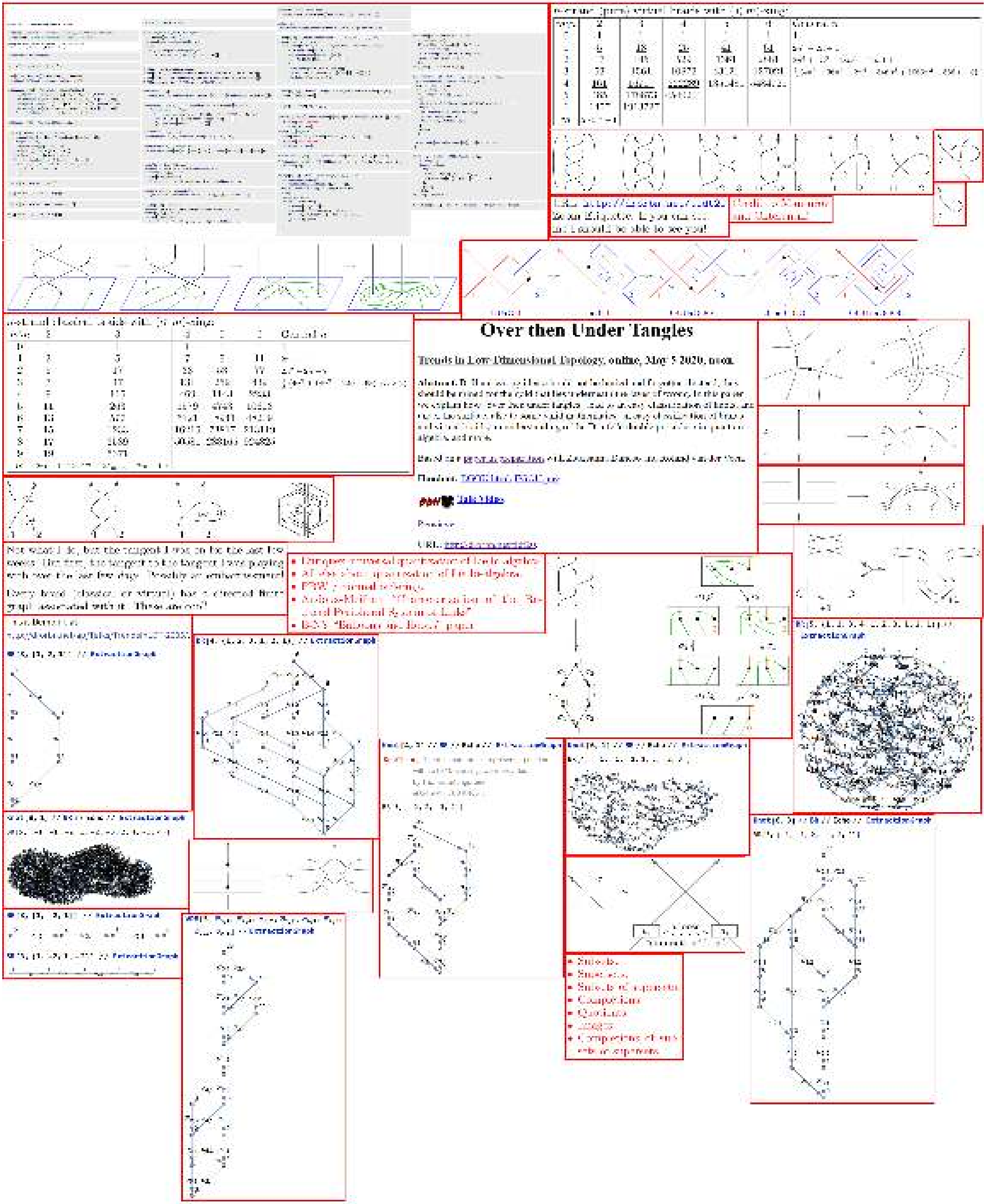
$$\left\{\begin{pmatrix}\omega & x_1 & x_2 & x_3\\ p_1 & \alpha & \beta & \theta\\ p_2 & \gamma & \delta & \epsilon\\ p_3 & \phi & \psi & \Xi\end{pmatrix}_h, \begin{pmatrix}\omega+\gamma\,\omega & x_0 & x_3\\ p_0 & \frac{\alpha+\beta+\gamma+\beta\,\gamma+\delta-\alpha\,\delta}{1+\gamma} & \frac{\epsilon-\alpha\,\epsilon+\theta+\gamma\,\theta}{1+\gamma}\\ p_3 & \frac{\phi-\delta\,\phi+\psi+\gamma\,\psi}{1+\gamma} & \frac{\Xi+\gamma\,\Xi-\epsilon\,\phi}{1+\gamma}\end{pmatrix}_h\right\}$$

□

References.

On ωεβ=http://drorbn.net/cat20

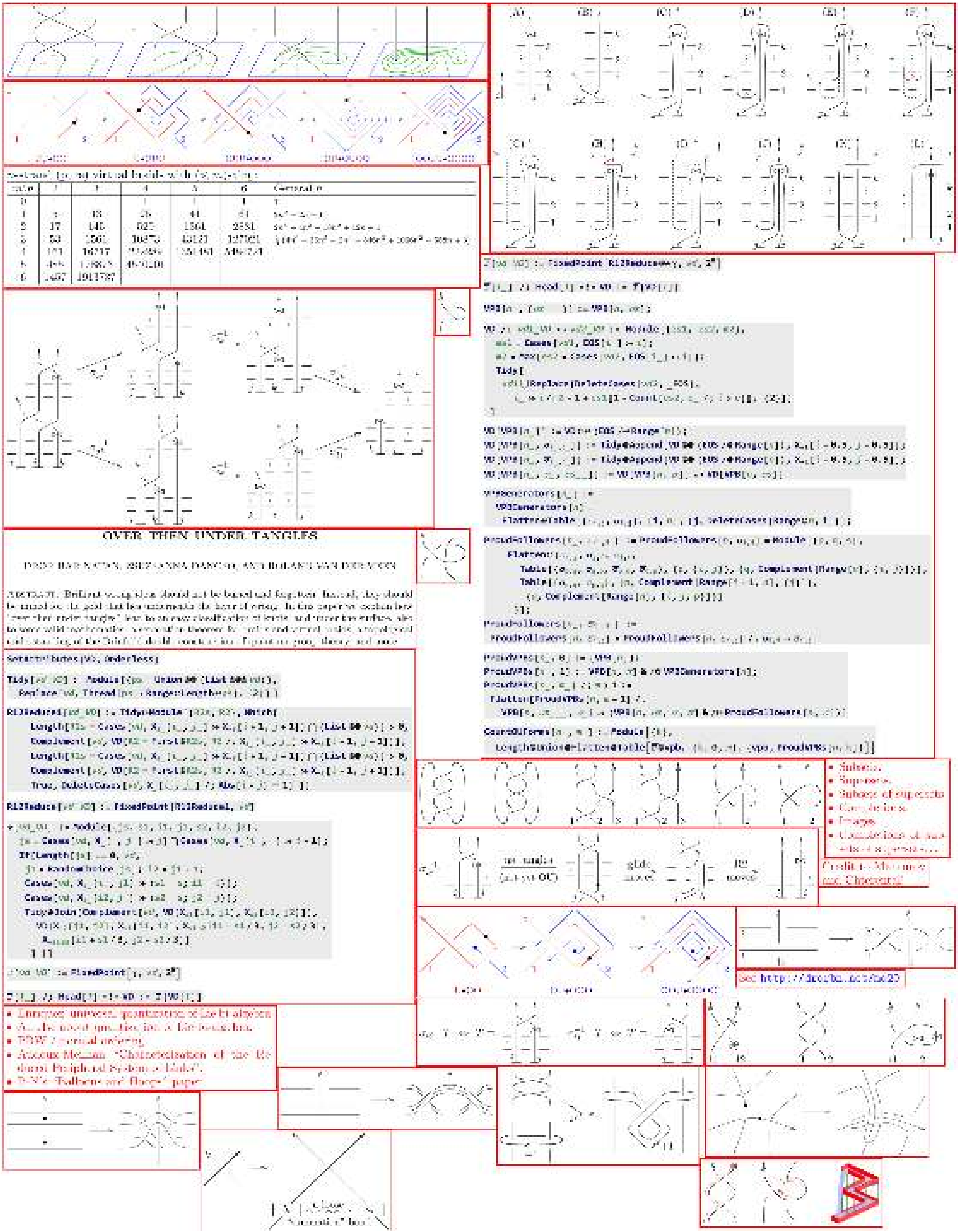
Video and more at http://www.math.toronto.edu/~drorbn/Talks/LearningSeminarOnCategorification-2006/



Over then Under Tangles

Trends in Low Dimensional Topology, online, May 5 2020, noon.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/TrendsInLDT-2005//`



OVER THEN UNDER TANGLES

Audio and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2004//

Dror Bar-Natan: Talks: Toronto-1912:

Chord Diagrams, Knots, and Lie Algebras

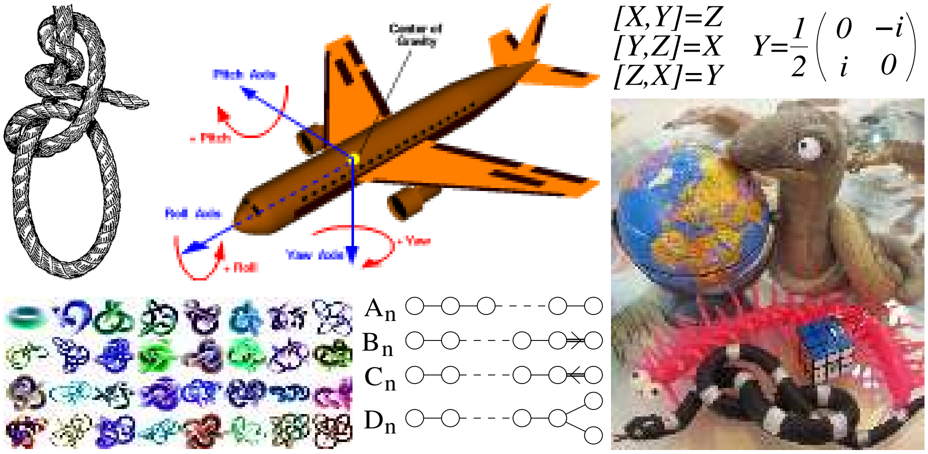
Thanks for inviting me to the *Chord Diagrams Everywhere* session / Winter 2019 CMS meeting!
ωεβ:=`http://drorbn.net/to19`

Abstract. This will be a service talk on ancient material — I will briefly describe how the exact same type of chord diagrams (and relations between them) occur in a natural way in both knot theory and in the theory of Lie algebras.

While preparing for this talk I realized that I've done it before, much better, within a book review. So here's that review! It has been modified from its original version: it had been formatted to fit this page, parts were highlighted, and commentary had been added in green italics.

[Book] *Introduction to Vassiliev Knot Invariants,* by S. Chmutov, S. Duzhin, and J. Mostovoy, Cambridge University Press, Cambridge UK, 2012, xvi+504 pp., hardback, $70.00, ISBN 978-1-10702-083-2.

Merely ~~30~~ *36* years ago, if you had asked even the best informed mathematician about the relationship between knots and Lie algebras, she would have laughed, for there isn't and there can't be. Knots are flexible; Lie algebras are rigid. Knots are irregular; Lie algebras are symmetric. The list of knots is a lengthy mess; the collection of Lie algebras is well-organized. Knots are useful for sailors, scouts, and hangmen; Lie algebras for navigators, engineers, and high energy physicists. Knots are blue collar; Lie algebras are white. They are as similar as worms and crystals: both well-studied, but hardly ever together.



$$[X,Y]=Z \quad [Y,Z]=X \quad [Z,X]=Y \qquad Y=\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

A knot and a Lie algebra, a list of knots and a list of Lie algebras, and an unusual conference of the symmetric and the knotted.

Then in the 1980s came Jones, and Witten, and Reshetikhin and Turaev [Jo, Wi, RT] and showed that if you really are the best informed, and you know your quantum field theory and conformal field theory and quantum groups, then you know that the two disjoint fields are in fact intricately related. This "quantum" approach remains the most powerful way to get computable knot invariants out of (certain) Lie algebras (and representations thereof). Yet shortly later, in the late 80s and early 90s, an alternative perspective arose, that of "finite-type" or "Vassiliev-Goussarov" invariants [Va1, Va2, Go1, Go2, BL, Ko1, Ko2, BN1], which made the surprising relationship between knots and Lie algebras appear simple and almost inevitable.

The reviewed [Book] is about that alternative perspective, the one reasonable sounding but not entirely trivial theorem that is crucially needed within it (the "Fundamental Theorem" or the "Kontsevich integral"), and the many threads that begin with that perspective. Let me start with a brief summary of the mathematics, and even before, an even briefer summary.

In briefest, a certain space $\mathcal{A}$ of chord diagrams is the dual to the dual of the space of knots, and at the same time, it is dual to Lie algebras.

The briefer summary is that in some combinatorial sense it is possible to "differentiate" knot invariants, and hence it makes sense to talk about "polynomials" on the space of knots — these are functions on the set of knots (namely, these are knot invariants) whose sufficiently high derivatives vanish. Such polynomials can be fairly conjectured to separate knots — elsewhere in math in lucky cases polynomials separate points, and in our case, specific computations are encouraging. Also, such polynomials are determined by their "coefficients", and each of these, by the one-side-easy "Fundamental Theorem", is a linear functional on some finite space of

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2010 *Mathematics Subject Classification.* Primary 57M25.

Published Bull. Amer. Math. Soc. **50** (2013) 685–690. TEX at `http://drorbn.net/AcademicPensieve/2013-01/CDMReview/`, copyleft at `http://www.math.toronto.edu/~drorbn/Copyleft/`. This review was written while I was a guest at the Newton Institute, in Cambridge, UK. I wish to thank N. Bar-Natan, I. Halacheva, and P. Lee for comments and suggestions.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/`

graphs modulo relations. These same graphs turn out to parameterize formulas that make sense in a wide class of Lie algebras, and the said relations match exactly with the relations in the definition of a Lie algebra — anti-symmetry and the Jacobi identity. Hence what is more or less dual to knots (invariants), is also, after passing to the coefficients, dual to certain graphs which are more or less dual to Lie algebras. QED, and on to the less brief summary[1].

Let V be an arbitrary invariant of oriented knots in oriented space with values in (say) $\mathbb{Q}$. Extend V to be an invariant of 1-singular knots, knots that have a single singularity that locally looks like a double point $\doublepoint$, using the formula

$$(1)\qquad V(\doublepoint) = V(\overcrossing) - V(\undercrossing).$$

Further extend V to the set $\mathcal{K}^m$ of m-singular knots (knots with m such double points) by repeatedly using (1).

Definition 1. We say that V is of type m (or "Vassiliev of type m") if its extension $V|_{\mathcal{K}^{m+1}}$ to $(m+1)$-singular knots vanishes identically. We say that V is of finite type (or "Vassiliev") if it is of type m for some m.

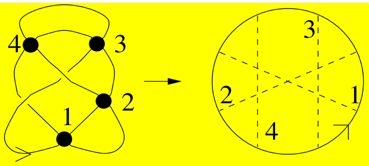
Repeated differences are similar to repeated derivatives and hence it is fair to think of the definition of $V|_{\mathcal{K}^m}$ as repeated differentiation. With this in mind, the above definition imitates the definition of polynomials of degree m. Hence finite type invariants can be thought of as "polynomials" on the space of knots[2]. It is known (see e.g. [Book]) that the class of finite type invariants is large and powerful. Yet the first question on finite type invariants remains unanswered:

Problem 2. *Honest polynomials are dense in the space of functions. Are finite type invariants dense within the space of all knot invariants? Do they separate knots?*

The top derivatives of a multi-variable polynomial form a system of constants that determine that polynomial up to polynomials of lower degree. Likewise the mth derivative[3] $V^{(m)} = V|_{\mathcal{K}^m} = V\left(\doublepoint\overset{m}{\cdots}\doublepoint\right)$ of a type m invariant V is a constant in the sense that it does not see the difference between overcrossings and undercrossings and so it is blind to 3D topology. Indeed

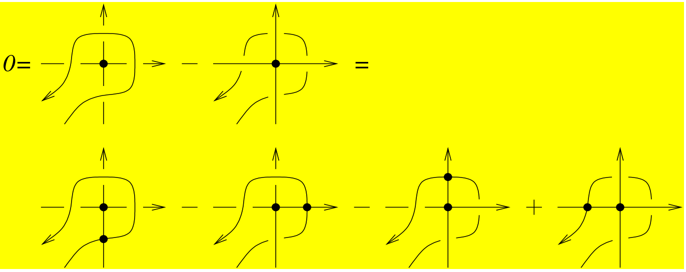
$$V\left(\doublepoint\overset{m}{\cdots}\doublepoint\overcrossing\right) - V\left(\doublepoint\overset{m}{\cdots}\doublepoint\undercrossing\right) = V\left(\doublepoint\overset{m+1}{\cdots}\doublepoint\right) = 0.$$

Also, clearly $V^{(m)}$ determines V up to invariants of lower type. Hence a primary tool in the study of finite type invariants is the study of the "top derivative" $V^{(m)}$, also known as "the weight system of V".

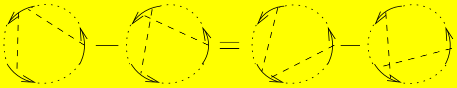


Blind to 3D topology, $V^{(m)}$ only sees the combinatorics of the circle that parameterizes an m-singular knot. On this circle there are m pairs of points that are pairwise identified in the image; standardly one indicates those by drawing a circle with m chords marked (an "m-chord diagram") as above. Let $\mathcal{D}_m$ denote the space of all formal linear combinations with rational coefficients of m-chord diagrams. Thus $V^{(m)}$ is a linear functional on $\mathcal{D}_m$.

I leave it for the reader to figure out or read in [Book, pp. 88] how the following figure easily implies the "4T" relations of the "easy side" of the theorem that follows:



Theorem 3. *(The Fundamental Theorem, details in [Book]).*



- *(Easy side) If V is a rational valued type m invariant then $V^{(m)}$ satisfies the "4T" relations shown above, and hence it descends to a linear functional on $\mathcal{A}_m := \mathcal{D}_m/4T$. If in addition $V^{(m)} \equiv 0$, then V is of type $m-1$.*
- *(Hard side, slightly misstated by avoiding "framings") For any linear functional W on $\mathcal{A}_m$ there is a rational valued type m invariant V so that $V^{(m)} = W$.*

Thus to a large extent the study of finite type invariants is reduced to the finite (though super-exponential in m) algebraic study of $\mathcal{A}_m$.

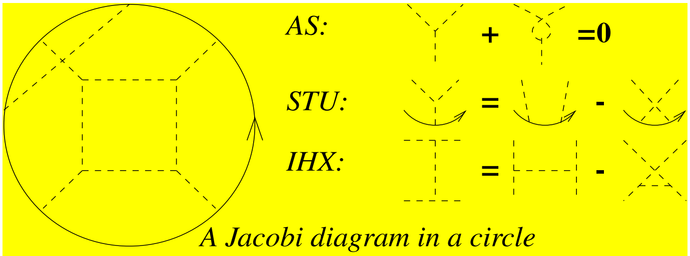
Much of the richness of finite type invariants stems from their relationship with Lie algebras. Theorem 4 below suggests this relationship on an abstract level and Theorem 5 makes that relationship concrete.

---

[1] Partially self-plagiarized from [BN2].

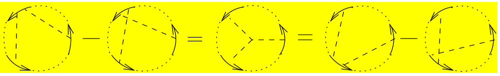
[2] Keep this apart from invariants of knots whose values are polynomials, such as the Alexander or the Jones polynomial. A posteriori related, these are a priori entirely different.

[3] As common in the knot theory literature, in the formulas that follow a picture such as $\doublepoint\overset{m}{\cdots}\doublepoint\overcrossing$ indicates "some knot having m double points and a further (right-handed) crossing". Furthermore, when two such pictures appear within the same formula, it is to be understood that the parts of the knots (or diagrams) involved *outside* of the displayed pictures are to be taken as the same.



A Jacobi diagram in a circle

Theorem 4. [BN1] *The space $\mathcal{A}_m$ is isomorphic to the space $\mathcal{A}^t_m$ generated by "Jacobi diagrams in a circle" (chord diagrams that are also allowed to have oriented internal trivalent vertices) that have exactly 2m vertices, modulo the AS, STU and IHX relations. See the figure above.*

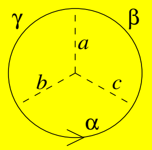


The key to the proof of Theorem 4 is the figure above, which shows that the $4T$ relation is a consequence of two STU relations. The rest is more or less an exercise in induction.

Thinking of internal trivalent vertices as graphical analogs of the Lie bracket, the AS relation becomes the anti-commutativity of the bracket, STU becomes the equation $[x, y] = xy - yx$ and IHX becomes the Jacobi identity. This analogy is made concrete within the following construction, originally due to Penrose [Pe] and to Cvitanović [Cv]. Given a finite dimensional metrized Lie algebra $\mathfrak{g}$ (e.g., any semi-simple Lie algebra) and a finite-dimensional representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of $\mathfrak{g}$, choose an orthonormal basis[4] $\{X_a\}_{a=1}^{\dim\mathfrak{g}}$ of $\mathfrak{g}$ and some basis $\{v_\alpha\}_{\alpha=1}^{\dim V}$ of V, let f_{abc} and $r^\gamma_{\alpha\beta}$ be the "structure constants" defined by

$$f_{abc} := \langle [X_a, X_b], X_c\rangle \qquad \text{and} \qquad \rho(X_a)(v_\beta) = \sum_\gamma r^\gamma_{a\beta} v_\gamma.$$

Now given a Jacobi diagram D label its circle-arcs with Greek letters $\alpha, \beta, \ldots$, and its chords with Latin letters a, $b, \ldots$, and map it to a sum as suggested by the following example:



$$\longrightarrow \sum_{a,b,c,\alpha,\beta,\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

$\left(\begin{array}{l}\text{internal vertices go to } f\text{'s,}\\ \text{circle-vertices to } r\text{'s}\end{array}\right)$

Theorem 5. *This construction is well defined, and the basic properties of Lie algebras imply that it respects the AS, STU, and IHX relations. Therefore it defines a linear functional $W_{\mathfrak{g},\rho} : \mathcal{A}_m \to \mathbb{Q}$, for any m.*

The last assertion along with Theorem 3 show that associated with any $\mathfrak{g}$, ρ and m there is a weight system and hence a knot invariant. Thus knots are indeed linked with Lie algebras.

The above is of course merely a sketch of the beginning of a long story. You can read the details, and some of the rest, in [Book].

What I like about [Book]. Detailed, well thought out, and carefully written. Lots of pictures! Many excellent exercises! A complete discussion of "the algebra of chord diagrams". A nice discussion of the pairing of diagrams with Lie algebras, including examples aplenty. The discussion of the Kontsevich integral (meaning, the proof of the hard side of Theorem 3) is terrific — detailed and complete and full of pictures and examples, adding a great deal to the original sources. The subject of "associators" is huge and worthy of its own book(s); yet in as much as they are related to Vassiliev invariants, the discussion in [Book] is excellent. A great many further topics are touched — multiple ζ-values, the relationship of the Hopf link with the Duflo isomorphism, intersection graphs and other combinatorial aspects of chord diagrams, Rozansky's rationality conjecture, the Melvin-Morton conjecture, braids, n-equivalence, etc.

For all these, I'd certainly recommend [Book] to any newcomer to the subject of knot theory, starting with my own students.

However, some proofs other than that of Theorem 3 are repeated as they appear in original articles with only a superficial touch-up, or are omitted altogether, thus missing an opportunity to clarify some mysterious points. This includes Vogel's construction of a non-Lie-algebra weight system and the Goussarov-Polyak-Viro proof of the existence of "Gauss diagram formulas".

What I wish there was in the book, but there isn't. The relationship with Chern-Simons theory, Feynman diagrams, and configuration space integrals, culminating in an alternative (and more "3D") proof of the Fundamental Theorem. This is a major omission.

Why I hope there will be a continuation book, one day. There's much more to the story! There are finite type invariants of 3-manifolds, and of certain classes of 2-dimensional knots in $\mathbb{R}^4$, and of "virtual knots", and they each have their lovely yet non-obvious theories, and these theories link with each other and with other branches of Lie theory, algebra, topology, and quantum field theory. Volume 2 is sorely needed.

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---

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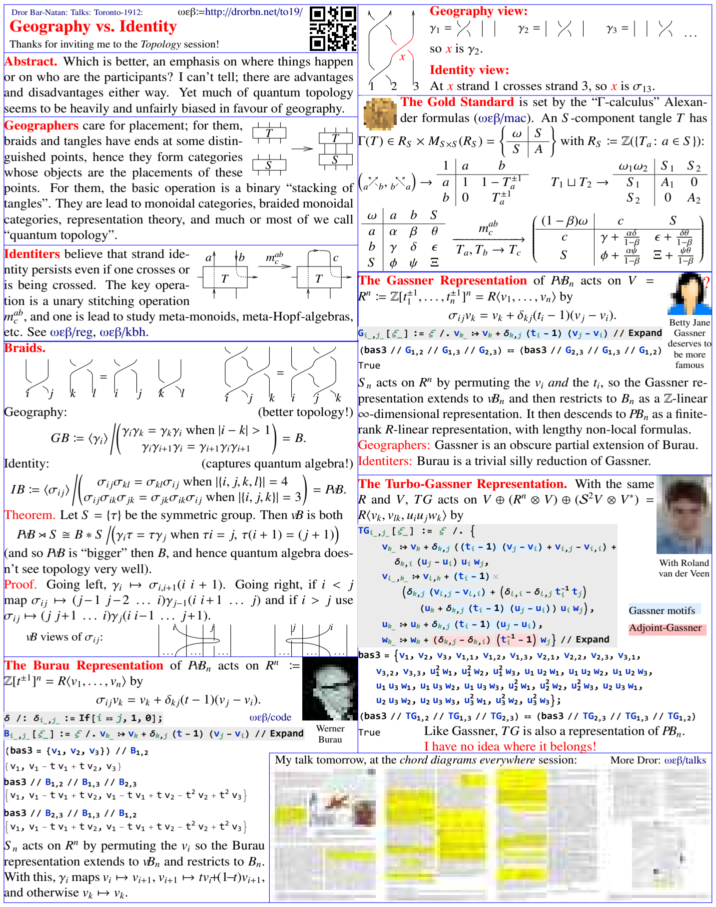
Dror Bar-Natan
University of Toronto, Canada
December 6, 2019 (first edition February 7, 2013)

My talk yesterday:

More Dror: ωεβ/talks



Dror Bar-Natan: Talks: Toronto-1912: ωεβ:=http://drorbn.net/to19/

Geography vs. Identity

Thanks for inviting me to the *Topology* session!

Abstract. Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

Geographers care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".

Identiters believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation m^{ab}_c, and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See ωεβ/reg, ωεβ/kbh.

Braids.

Geography: (better topology!)

$$GB := \langle \gamma_i\rangle \Big/\!\!\Big/ \begin{pmatrix} \gamma_i\gamma_k = \gamma_k\gamma_i \text{ when } |i-k|>1 \\ \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1}\end{pmatrix} = B.$$

Identity: (captures quantum algebra!)

$$IB := \langle \sigma_{ij}\rangle \Big/\!\!\Big/ \begin{pmatrix} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \text{ when } |\{i,j,k,l\}|=4 \\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \text{ when } |\{i,j,k\}|=3\end{pmatrix} = PvB.$$

Theorem. Let $S=\{\tau\}$ be the symmetric group. Then vB is both

$$PvB \rtimes S \cong B * S \Big/\!\!\Big/ \big(\gamma_i\tau = \tau\gamma_j \text{ when } \tau i = j,\ \tau(i+1)=(j+1)\big)$$

(and so PvB is "bigger" then B, and hence quantum algebra doesn't see topology very well).

Proof. Going left, $\gamma_i \mapsto \sigma_{i,i+1}(i\ i+1)$. Going right, if $i<j$ map $\sigma_{ij} \mapsto (j{-}1\ j{-}2\ \dots\ i)\gamma_{j-1}(i\ i{+}1\ \dots\ j)$ and if $i>j$ use $\sigma_{ij}\mapsto (j\ j{+}1\ \dots\ i)\gamma_j(i\ i{-}1\ \dots\ j{+}1)$.

vB views of σ_{ij}:

Geography view:

$\gamma_1 =$ … $\gamma_2 =$ … $\gamma_3 =$ … …

so x is γ_2.

Identity view:

At x strand 1 crosses strand 3, so x is σ_{13}.

The Gold Standard is set by the "Γ-calculus" Alexander formulas (ωεβ/mac). An S-component tangle T has $\Gamma(T)\in R_S\times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{T_a\colon a\in S\})$:

$$\left(\ {}_a\!\nearrow_b,\ {}_b\!\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a,T_b\to T_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

The Gassner Representation of PvB_n acts on $V = R^n := \mathbb{Z}[t_1^{\pm1},\dots,t_n^{\pm1}]^n = R\langle v_1,\dots,v_n\rangle$ by

$$\sigma_{ij}v_k = v_k + \delta_{kj}(t_i-1)(v_j-v_i).$$

```
Gi_,j_[ξ_] := ξ /. vk_ ⧴ vk + δk,j (ti - 1) (vj - vi) // Expand
(bas3 // G1,2 // G1,3 // G2,3) == (bas3 // G2,3 // G1,3 // G1,2)
True
```

Betty Jane Gassner deserves to be more famous ?

S_n acts on R^n by permuting the v_i *and* the t_i, so the Gassner representation extends to vB_n and then restricts to B_n as a $\mathbb{Z}$-linear ∞-dimensional representation. It then descends to PB_n as a finite-rank R-linear representation, with lengthy non-local formulas.

Geographers: Gassner is an obscure partial extension of Burau.

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The Turbo-Gassner Representation. With the same R and V, TG acts on $V\oplus(R^n\otimes V)\oplus(\mathcal{S}^2V\otimes V^*) = R\langle v_k, v_{lk}, u_iu_jw_k\rangle$ by

```
TGi_,j_[ξ_] := ξ /. {
    vk_ ⧴ vk + δk,j ((ti - 1) (vj - vi) + vi,j - vi,i) +
      δk,i (uj - ui) ui wj,
    vl_,k_ ⧴ vl,k + (ti - 1) ×
       (δk,j (vl,j - vl,i) + (δl,i - δl,j ti^-1 tj)
          (uk + δk,j (ti - 1) (uj - ui)) ui wj),
    uk_ ⧴ uk + δk,j (ti - 1) (uj - ui),
    wk_ ⧴ wk + (δk,j - δk,i) (ti^-1 - 1) wj} // Expand
bas3 = {v1, v2, v3, v1,1, v1,2, v1,3, v2,1, v2,2, v2,3, v3,1,
   v3,2, v3,3, u1^2 w1, u1^2 w2, u1^2 w3, u1 u2 w1, u1 u2 w2, u1 u2 w3,
   u1 u3 w1, u1 u3 w2, u1 u3 w3, u2^2 w1, u2^2 w2, u2^2 w3, u2 u3 w1,
   u2 u3 w2, u2 u3 w3, u3^2 w1, u3^2 w2, u3^2 w3};
(bas3 // TG1,2 // TG1,3 // TG2,3) == (bas3 // TG2,3 // TG1,3 // TG1,2)
True
```

With Roland van der Veen

Gassner motifs

Adjoint-Gassner

Like Gassner, TG is also a representation of PB_n.

I have no idea where it belongs!

The Burau Representation of PvB_n acts on $R^n := \mathbb{Z}[t^{\pm1}]^n = R\langle v_1,\dots,v_n\rangle$ by

$$\sigma_{ij}v_k = v_k + \delta_{kj}(t-1)(v_j-v_i).$$

ωεβ/code

Werner Burau

```
δ /: δi_,j_ := If[i == j, 1, 0];
Bi_,j_[ξ_] := ξ /. vk_ ⧴ vk + δk,j (t - 1) (vj - vi) // Expand
(bas3 = {v1, v2, v3}) // B1,2
{v1, v1 - t v1 + t v2, v3}
bas3 // B1,2 // B1,3 // B2,3
{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}
bas3 // B2,3 // B1,3 // B1,2
{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}
```

S_n acts on R^n by permuting the v_i so the Burau representation extends to vB_n and restricts to B_n. With this, γ_i maps $v_i\mapsto v_{i+1}$, $v_{i+1}\mapsto tv_i{+}(1{-}t)v_{i+1}$, and otherwise $v_k\mapsto v_k$.

My talk tomorrow, at the *chord diagrams everywhere* session: More Dror: ωεβ/talks

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Picture credits: Rope from "The Project Gutenberg eBook, Knots, Splices and Rope Work, by A. Hyatt Verrill", `http://www.gutenberg.org/files/13510/13510-h/13510-h.htm`. Plane from NASA, `http://www.grc.nasa.gov/WWW/k-12/airplane/rotations.html`.

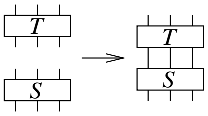
Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/`

Dror Bar-Natan: Talks: Toronto-1912: ωεβ:=http://drorbn.net/to19/

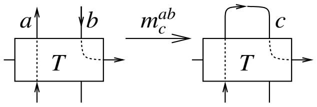
Geography vs. Identity

Thanks for inviting me to the *Topology* session!

Abstract. Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

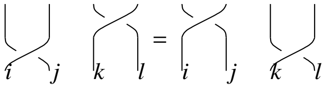


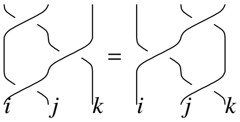
Geographers care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".



Identiters believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation m^{ab}_c, and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See ωεβ/reg, ωεβ/kbh.

Braids.





Geography: (better topology!)

$$GB := \langle \gamma_i \rangle \Big/ \left(\begin{array}{c} \gamma_i\gamma_k = \gamma_k\gamma_i \text{ when } |i-k|>1 \\ \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1} \end{array} \right) = B.$$

Identity: (captures quantum algebra!)

$$IB := \langle \sigma_{ij} \rangle \Big/ \left(\begin{array}{c} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \text{ when } |\{i,j,k,l\}| = 4 \\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \text{ when } |\{i,j,k\}| = 3 \end{array} \right) = PvB.$$

Theorem. Let $S = \{\tau\}$ be the symmetric group. Then vB is both

$$PvB \rtimes S \cong B * S \Big/ \big(\gamma_i\tau = \tau\gamma_j \text{ when } \tau i = j,\ \tau(i+1) = (j+1) \big)$$

(and so PvB is "bigger" then B, and hence quantum algebra doesn't see topology very well).

Proof. Going left, $\gamma_i \mapsto \sigma_{i,i+1}(i\ i+1)$. Going right, if $i<j$ map $\sigma_{ij} \mapsto (j{-}1\ j{-}2\ \dots\ i)\gamma_{j-1}(i\ i{+}1\ \dots\ j)$ and if $i>j$ use $\sigma_{ij} \mapsto (j\ j{+}1\ \dots\ i)\gamma_j(i\ i{-}1\ \dots\ j{+}1)$.

vB views of σ_{ij}:

The Burau Representation


Werner Burau

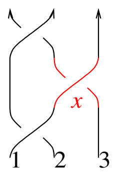
The Burau Representation of PvB_n acts on $R^n := \mathbb{Z}[t^{\pm 1}]^n = R\langle v_1,\dots,v_n\rangle$ by

$$\sigma_{ij}v_k = v_k + \delta_{kj}(t-1)(v_j - v_i).$$

ωεβ/code

```
δ /: δ_{i_,j_} := If[i == j, 1, 0];
B_{i_,j_}[ℰ_] := ℰ /. v_{k_} :→ v_k + δ_{k,j} (t - 1) (v_j - v_i) // Expand
(bas3 = {v_1, v_2, v_3}) // B_{1,2}
{v_1, v_1 - t v_1 + t v_2, v_3}
bas3 // B_{1,2} // B_{1,3} // B_{2,3}
{v_1, v_1 - t v_1 + t v_2, v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3}
bas3 // B_{2,3} // B_{1,3} // B_{1,2}
{v_1, v_1 - t v_1 + t v_2, v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3}
```

S_n acts on R^n by permuting the v_i so the Burau representation extends to vB_n and restricts to B_n. With this, γ_i maps $v_i \mapsto v_{i+1}$, $v_{i+1} \mapsto tv_i{+}(1{-}t)v_{i+1}$, and otherwise $v_k \mapsto v_k$.



Geography view:

$\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  …

so x is γ_2.

Identity view:

At x strand 1 crosses strand 3, so x is σ_{13}.

The Gold Standard is set by the "Γ-calculus" Alexander formulas (ωεβ/mac). An S-component tangle T has

$$\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\} \text{ with } R_S := \mathbb{Z}(\{T_a : a \in S\}):$$

$$\left({}_a\nearrow_b,\ {}_b\nwarrow_a \right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-T_a^{\pm1} \\ b & 0 & T_a^{\pm1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a, T_b \to T_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array} \right)$$

The Gassner Representation

Betty Jane Gassner deserves to be more famous

The Gassner Representation of PvB_n acts on $V = R^n := \mathbb{Z}[t_1^{\pm1},\dots,t_n^{\pm1}]^n = R\langle v_1,\dots,v_n\rangle$ by

$$\sigma_{ij}v_k = v_k + \delta_{kj}(t_i - 1)(v_j - v_i).$$

```
G_{i_,j_}[ℰ_] := ℰ /. v_{k_} :→ v_k + δ_{k,j} (t_i - 1) (v_j - v_i) // Expand
(bas3 // G_{1,2} // G_{1,3} // G_{2,3}) == (bas3 // G_{2,3} // G_{1,3} // G_{1,2})
True
```

S_n acts on R^n by permuting the v_i *and* the t_i, so the Gassner representation extends to vB_n and then restricts to B_n as a $\mathbb{Z}$-linear ∞-dimensional representation. It then descends to PB_n as a finite-rank R-linear representation, with lengthy non-local formulas.

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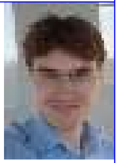
The Turbo-Gassner Representation.


With Roland van der Veen

The Turbo-Gassner Representation. With the same R and V, TG acts on $V \oplus (R^n \otimes V) \oplus (\mathcal{S}^2 V \otimes V^*) = R\langle v_k, v_{lk}, u_iu_jw_k\rangle$ by

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    v_{l_,k_} :→ v_{l,k} + (t_i - 1) ×
        (δ_{k,j} (v_{l,j} - v_{l,i}) + (δ_{l,i} - δ_{l,j} t_i^-1 t_j)
           (u_k + δ_{k,j} (t_i - 1) (u_j - u_i)) u_i w_j),
    u_{k_} :→ u_k + δ_{k,j} (t_i - 1) (u_j - u_i),
    w_{k_} :→ w_k + (δ_{k,j} - δ_{k,i}) (t_i^-1 - 1) w_j} // Expand
bas3 = {v_1, v_2, v_3, v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,1},
   v_{3,2}, v_{3,3}, u_1^2 w_1, u_1^2 w_2, u_1^2 w_3, u_1 u_2 w_1, u_1 u_2 w_2, u_1 u_2 w_3,
   u_1 u_3 w_1, u_1 u_3 w_2, u_1 u_3 w_3, u_2^2 w_1, u_2^2 w_2, u_2^2 w_3, u_2 u_3 w_1,
   u_2 u_3 w_2, u_2 u_3 w_3, u_3^2 w_1, u_3^2 w_2, u_3^2 w_3};
(bas3 // TG_{1,2} // TG_{1,3} // TG_{2,3}) == (bas3 // TG_{2,3} // TG_{1,3} // TG_{1,2})
True
```

Gassner motifs

Adjoint-Gassner

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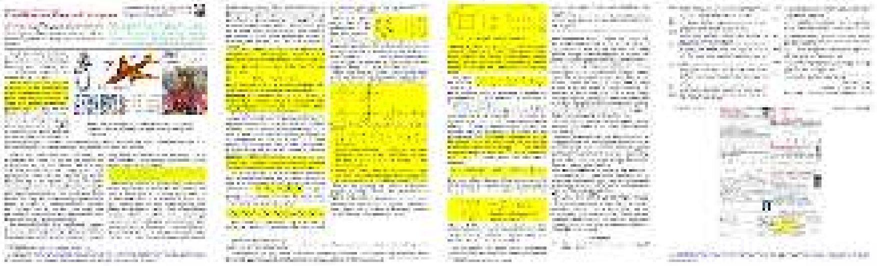
I have no idea where it belongs!

My talk tomorrow, at the *chord diagrams everywhere* session: More Dror: ωεβ/talks



Dror Bar-Natan: Talks: Columbia-191125:

Some Feynman Diagrams in Pure Algebra

With Roland van der Veen 

Thanks for allowing me in Columbia U!
ωεβ:=http://drorbn.net/co19/
Slides w/ no handout/URL should be banned!

Abstract. I will explain how the computation of compositions of maps of a certain natural class, from one polynomial ring into another, naturally leads to a certain composition operation of quadratics and to Feynman diagrams. I will also explain, with very little detail, how this is used in the construction of some very well-behaved poly-time computable knot polynomials.

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Gentle Agreement. Everything converges!

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $(y,b,a,x)^* = (\eta,\beta,\alpha,\xi)$

The Generating Series $\mathcal{G}$: $\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \to \mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \xrightarrow[\mathcal{G}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}L(z_A^n) = L\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p\in\mathbb{Q}[z_A].$$

Claim. If $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $M\in\operatorname{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Basic Examples. 1. $\mathcal{G}(id\colon \mathbb{Q}[y,a,x]\to\mathbb{Q}[y,a,x]) = \mathbb{e}^{\eta y+\alpha a+\xi x}$.

2. The standard commutative product m_k^{ij} of polynomials is given by $z_i, z_j \to z_k$. Hence $\mathcal{G}(m_k^{ij}) = m_k^{ij}(\mathbb{e}^{\zeta_i z_i+\zeta_j z_j}) = \mathbb{e}^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{tikzcd}
\mathbb{Q}[z]_i\otimes\mathbb{Q}[z]_j \arrow[r, "m_k^{ij}"] \arrow[d, equal] & \mathbb{Q}[z]_k \arrow[d, equal] \\
\mathbb{Q}[z_i,z_j] \arrow[r, "m_k^{ij}"] & \mathbb{Q}[z_k]
\end{tikzcd}$$

3. The standard co-commutative coproduct Δ^i_{jk} of polynomials is given by $z_i \to z_j + z_k$. Hence $\mathcal{G}(\Delta^i_{jk}) = \Delta^i_{jk}(\mathbb{e}^{\zeta_i z_i}) = \mathbb{e}^{\zeta_i(z_j+z_k)}$.

$$\begin{tikzcd}
\mathbb{Q}[z]_i \arrow[r, "\Delta^i_{jk}"] \arrow[d, equal] & \mathbb{Q}[z]_j\otimes\mathbb{Q}[z]_k \arrow[d, equal] \\
\mathbb{Q}[z_i] \arrow[r, "\Delta^i_{jk}"] & \mathbb{Q}[z_j,z_k]
\end{tikzcd}$$

Heisenberg Algebras. Let $\mathbb{H} = \langle x,y\rangle/[x,y] = \hbar$ (with $\hbar$ a scalar), let $\mathbb{O}_i\colon \mathbb{Q}[x_i,y_i]\to\mathbb{H}_i$ is the "x before y" PBW ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[x_i,y_i,x_j,y_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathbb{H}_i\otimes\mathbb{H}_j \xrightarrow{m_k^{ij}} \mathbb{H}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[x_k,y_k].$$

Then $\mathcal{G}(hm_k^{ij}) = \mathbb{e}^{\Lambda_\hbar}$, where $\Lambda_\hbar = -\hbar\eta_i\xi_j + (\xi_i+\xi_j)x_k + (\eta_i+\eta_j)y_k$.

Proof 1. Recall the "Weyl form of the CCR" $\mathbb{e}^{\eta y}\mathbb{e}^{\xi x} = \mathbb{e}^{-\hbar\eta\xi}\mathbb{e}^{\xi x}\mathbb{e}^{\eta y}$, and compute

$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= \mathbb{e}^{\xi_i x_i+\eta_i y_i+\xi_j x_j+\eta_j y_j}/\!\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} \\ &= \mathbb{e}^{\xi_i x_i}\mathbb{e}^{\eta_i y_i}\mathbb{e}^{\xi_j x_j}\mathbb{e}^{\eta_j y_j}/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{\xi_i x_k}\mathbb{e}^{\eta_i y_k}\mathbb{e}^{\xi_j x_k}\mathbb{e}^{\eta_j y_k}/\!\!/\mathbb{O}_k^{-1} \\ &= \mathbb{e}^{-\hbar\eta_i\xi_j}\mathbb{e}^{(\xi_i+\xi_j)x_k}\mathbb{e}^{(\eta_i+\eta_j)y_k}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{\Lambda_\hbar}.\end{aligned}$$

Proof 2. We compute in a faithful 3D representation ρ of $\mathbb{H}$: (ωεβ/hm)

```
{x̂ = (0 1 0; 0 0 0; 0 0 0), ŷ = (0 0 0; 0 0 ħ; 0 0 0), ĉ = (0 0 1; 0 0 0; 0 0 0)};
{x̂.ŷ - ŷ.x̂ == ħ ĉ, x̂.ĉ == ĉ.x̂, ŷ.ĉ == ĉ.ŷ}
{True, True, True}
Λ = -ħ ηi ξj ck + (ξi + ξj) xk + (ηi + ηj) yk;
Simplify@With[{𝔼 = MatrixExp},
  𝔼[x̂ ξi].𝔼[ŷ ηi].𝔼[x̂ ξj].𝔼[ŷ ηj] ==
   𝔼[x̂ ∂xk Λ].𝔼[ŷ ∂yk Λ].𝔼[ĉ ∂ck Λ]]
True
```

A Real DoPeGDO Example (DoPeGDO:=Docile Perturbed Gaussian Differential Operators). Let $sl^\epsilon_{2+} := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t:=\epsilon a-b$ is central and if $\exists\epsilon^{-1}$, $sl^\epsilon_{2+}\cong sl_2\oplus\langle t\rangle$. Let $CU := \mathcal{U}(sl^\epsilon_{2+})$, and let cm_k^{ij} be the composition below, where $\mathbb{O}_i\colon \mathbb{Q}[y_i,b_i,a_i,x_i]\to CU_i$ be the PBW ordering map in the order $ybax$:

$$\begin{tikzcd}
CU_i\otimes CU_j \arrow[r, "m_k^{ij}"] & CU_k \\
\mathbb{Q}[y_i,b_i,a_i,x_i,y_j,b_j,a_j,x_j] \arrow[u, "\mathbb{O}_{i,j}"'] \arrow[r, "cm_k^{ij}"] & \mathbb{Q}[y_k,b_k,a_k,x_k] \arrow[u, "\mathbb{O}_k"']
\end{tikzcd}$$

Claim. Let (all brawn and no brains)

$$\Lambda = \left(\eta_i + \frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k + \left(\beta_i+\beta_j+\frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k + \left(\alpha_i+\alpha_j+\log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k + \left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i}+\xi_j\right)x_k$$

Then $\mathbb{e}^{\eta_i y_i+\beta_i b_i+\alpha_i a_i+\xi_i x_i+\eta_j y_j+\beta_j b_j+\alpha_j a_j+\xi_j x_j}/\!\!/\mathbb{O}_{i,j}/\!\!/cm_k^{ij} = \mathbb{e}^\Lambda/\!\!/\mathbb{O}_k$, and hence $\mathcal{G}(cm_k^{ij}) = \mathbb{e}^\Lambda$.

Proof. We compute in a faithful 2D representation ρ of CU: (ωεβ/sl2)

```
{ŷ = (0 0; ϵ 0), b̂ = (0 0; 0 -ϵ), â = (1 0; 0 0), x̂ = (0 1; 0 0)};
{â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -ϵ ŷ,
 b̂.x̂ - x̂.b̂ == ϵ x̂, x̂.ŷ - ŷ.x̂ == b̂ + ϵ â}
{True, True, True, True, True}
Simplify@With[{𝔼 = MatrixExp},
  𝔼[ηi ŷ].𝔼[βi b̂].𝔼[αi â].𝔼[ξi x̂].𝔼[ηj ŷ].𝔼[βj b̂].
   𝔼[αj â].𝔼[ξj x̂] == 𝔼[ŷ ∂yk Λ].𝔼[b̂ ∂bk Λ].𝔼[â ∂ak Λ].
   𝔼[x̂ ∂xk Λ]]
True
Series[Λ, {ϵ, 0, 2}]
(ak (αi + αj) + yk (ηi + 𝕖^(-αi) ηj) +
   bk (βi + βj + ηj ξi) + xk (𝕖^(-αj) ξi + ξj)) +
 (ak ηj ξi - 1/2 bk ηj^2 ξi^2 - 𝕖^(-αi) yk ηj (βi + ηj ξi) -
   𝕖^(-αj) xk ξi (βj + ηj ξi)) ϵ +
 (-1/2 ak ηj^2 ξi^2 + 1/3 bk ηj^3 ξi^3 + 1/2 𝕖^(-αi) yk ηj (βi^2 + 2 βi ηj ξi + 2 ηj^2 ξi^2) +
   1/2 𝕖^(-αj) xk ξi (βj^2 + 2 βj ηj ξi + 2 ηj^2 ξi^2)) ϵ^2 + O[ϵ]^3
```

Note 1. If the lower half of the alphabet (a,b,α,β) is regarded as constants, then $\Lambda = C + Q + \sum_{k\geq1}\epsilon^k P^{(k)}$ is a docile perturbed Gaussian relative to the upper half of the alphabet (x,y,ξ,η): C is a scalar, Q is a quadratic, and $\deg P^{(k)}\leq 2k+2$.

Note 2. $\operatorname{wt}(x,y,\xi,\eta;a,b,\alpha,\beta;\epsilon) = (1,1,1,1;2,0,0,2;-2)$.

Quadratic Casimirs. If $t\in\mathfrak{g}\otimes\mathfrak{g}$ is the quadratic Casimir of a semi-simple Lie algebra $\mathfrak{g}$, then $\mathbb{e}^t$, regarded by PBW as an element of $\mathcal{S}^{\otimes2} = \operatorname{Hom}\left(\mathcal{S}(\mathfrak{g})^{\otimes0}\to\mathcal{S}(\mathfrak{g})^{\otimes2}\right)$, has a latin-latin dominant Gaussian factor. Likewise for R-matrices.

(Baby) **DoPeGDO** := The category with objects finite sets^{†1} and

$$\operatorname{mor}(A\to B) = \{\mathcal{L} = \omega\exp(Q+P)\}\subset\mathbb{Q}[\![\zeta_A,z_B,\epsilon]\!],$$

where: • ω is a scalar.^{†2} • Q is a "small" ϵ-free quadratic in $\zeta_A\cup z_B$.^{†3} • P is a "docile perturbation": $P=\sum_{k\geq1}\epsilon^kP^{(k)}$, where $\deg P^{(k)}\leq 2k+2$.^{†4} • Compositions:^{†6} $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

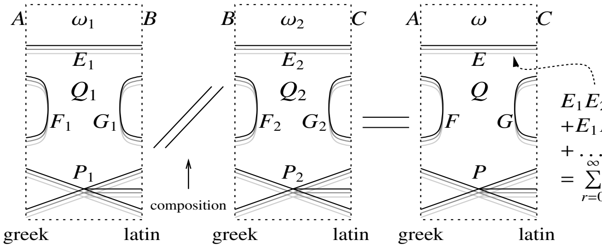
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Columbia-191125/

So What? If V is a representation, then $V^{\otimes n}$ explodes as a function of n, while in **DoPeGDO** up to a fixed power of ϵ, the ranks of mor$(A \to B)$ grow polynomially as a function of $|A|$ and $|B|$.

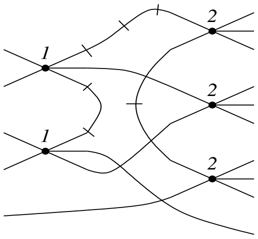
Compositions. In mor$(A \to B)$,

$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j,$$

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



$$E_1E_2 + E_1F_2G_1E_2 + E_1F_2G_1F_2G_1E_2 + \dots = \sum_{r=0}^{\infty} E_1(F_2G_1)^r E_2$$



where • $E = E_1(I - F_2G_1)^{-1}E_2$.
- $F = F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T$.
- $G = G_2 + E_2^T G_1(I - F_2G_1)^{-1}E_2$.
- $\omega = \omega_1\omega_2 \det(I - F_2G_1)^{-1}$.
- P is computed as the solution of a messy PDE or using "connected Feynman diagrams" (yet we're still in pure algebra!). Docility is preserved.

DoPeGDO Footnotes. Each variable has a "weight"$\in \{0, 1, 2\}$, and always wt z_i + wt $\zeta_i = 2$.

†1. Really, "weight-graded finite sets" $A = A_0 \sqcup A_1 \sqcup A_2$.

†2. Really, a power series in the weight-0 variables$^{\dagger 5}$.

†3. The weight of Q must be 2, so it decomposes as $Q = Q_{20}+Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series$^{\dagger 5}$.

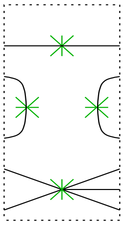
†4. Setting wt $\epsilon = -2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained)$^{\dagger 5}$.

†5. In the knot-theoretic case, all weight-0 power series are rational functions of bounded degree in the exponentials of the weight-0 variables.

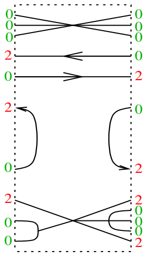
†6. There's also an obvious product

$$\text{mor}(A_1 \to B_1)\times\text{mor}(A_2 \to B_2) \to \text{mor}(A_1\sqcup A_2 \to B_1\sqcup B_2).$$

Full DoPeGDO. Compute compositions in two phases:
- A 1-1 phase over the ring of power series in the weight-0 variables, in which the weight-2 variables are spectators.
- A (slightly modified) 2-0 phase over $\mathbb{Q}$, in which the weight-1 variables are spectators.



Analog. Solve $Ax = a,\ B(x)y = b$

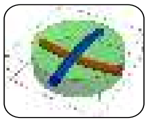


Questions. • Are there QFT precedents for "two-step Gaussian integration"?
- In QFT, one saves even more by considering "one-particle-irreducible" diagrams and "effective actions". Does this mean anything here?
- Understanding Hom$(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$ seems like a good cause.

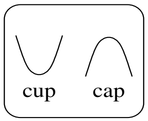
Can you find other applications for the technology here?

$QU = \mathcal{U}_\hbar(sl^\epsilon_{2+}) = A\langle y, b, a, x\rangle[\![\hbar]\!]$ with $[a, x] = x$, $[b, y] = -\epsilon y$, $[a, b] = 0$, $[a, y] = -y$, $[b, x] = \epsilon x$, and $xy - qyx = (1-AB)/\hbar$, where $q = \mathbb{e}^{\hbar\epsilon}$, $A = \mathbb{e}^{-\hbar\epsilon a}$, and $B = \mathbb{e}^{-\hbar b}$. Also $\Delta(y, b, a, x) = (y_1 + B_1y_2, b_1 + b_2, a_1 + a_2, x_1 + A_1x_2)$, $S(y, b, a, x) = (-B^{-1}y, -b, -a, -A^{-1}x)$, and $R = \sum \hbar^{j+k}y^kb^j \otimes a^jx^k/j![k]_q!$.

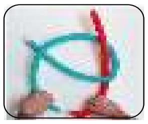
Theorem. Everything of value regrading $U = CU$ and/or its quantization $U = QU$ is **DoPeGDO**:



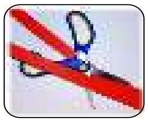
$R\in QU \otimes QU$



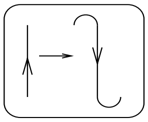
$C^{\pm1}\in QU$



$m\colon U \otimes U\to U$

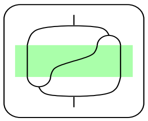


$\Delta\colon U\to U \otimes U$

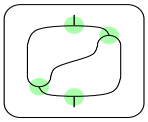


$S : U\to U$

tr: $U\to U/wx{=}xw$



$\Phi\in CU^{\otimes 3}$

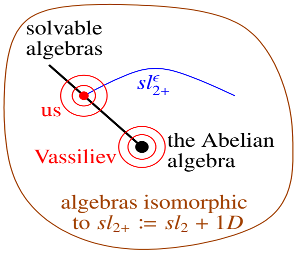


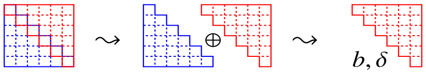
$J\in CU \otimes CU$

also Cartan's θ, the $\mathbb{D}$equantizator, and more, and all of their compositions.

Solvable Approximation. In sl_n, half is enough! Indeed $sl_n \oplus \mathfrak{a}_{n-1} = \mathcal{D}(◹, b, \delta)$. Now define $sl^\epsilon_{n+} := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is [◹, ◹] = ◹, [◺, ◺] = ϵ◺, and [◹, ◺] = ◺ + ϵ◹. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1} = 0$ always yields a solvable Lie algebra.

4D Metrized Lie Algebras



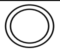
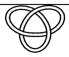
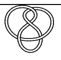
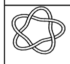
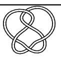
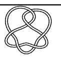
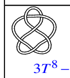
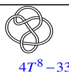
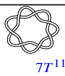


b(◹) $= b\colon$ ◹ $\otimes$ ◹ $\to$ ◹

b(◺) $\rightsquigarrow \delta\colon$ ◹ $\to$ ◹ $\otimes$ ◹

Conclusion. There are lots of poly-time-computable well-behaved near-Alexander knot invariants: • They extend to tangles with appropriate multiplicative behaviour. • They have cabling and strand reversal formulas. ωεβ/akt

The invariant for $sl^\epsilon_{2+}/(\epsilon^2 = 0)$ (prior art: ωεβ/Ov) attains 2,883 distinct values on the 2,978 prime knots with $\leq$ 12 crossings. HOMFLY-PT and Khovanov homology together attain only 2,786 distinct values.

knot diag	n^t_k $(\rho'_1)^+$	Alexander's ω^+ $(\rho'_2)^+$	genus / ribbon unknotting # / amphi?	knot diag	n^t_k $(\rho'_1)^+$	Alexander's ω^+ $(\rho'_2)^+$	genus / ribbon unknotting # / amphi?	knot diag	n^t_k $(\rho'_1)^+$	Alexander's ω^+ $(\rho'_2)^+$	genus / ribbon unknotting # / amphi?
	0^a_1 0	1 0	0 / ✔ 0 / ✔		3^a_1 T	$T-1$ $3T^3-12T^2+26T-38$	1 / ✘ 1 / ✘		4^a_1 0	$3-T$ $T^4-3T^3-15T^2+74T-110$	1 / ✘ 1 / ✔
	5^a_1 $2T^3+3T$	T^2-T+1 $5T^7-20T^6+55T^5-120T^4+217T^3-338T^2+450T-510$	2 / ✘ 2 / ✘		5^a_2 $5T-4$	$2T-3$ $-10T^4+120T^3-487T^2+1054T-1362$	1 / ✘ 1 / ✘		6^a_1 $T-4$	$5-2T$ $14T^4-16T^3-293T^2+1098T-1598$	1 / ✔ 1 / ✘
	6^a_2 T^3-4T^2+4T-4	$-T^2+3T-3$ $3T^8-21T^7+49T^6+15T^5-433T^4+1543T^3-3431T^2+5482T-6410$	2 / ✘ 1 / ✘		6^a_3 0	T^2-3T+5 $4T^8-33T^7+121T^6-203T^5-111T^4+1499T^3-4210T^2+7186T-8510$	2 / ✘ 1 / ✔		7^a_1 $3T^5+5T^3+6T$	T^3-T^2+T-1 $7T^{11}-28T^{10}+77T^9-168T^8+322T^7-560T^6+891T^5-1310T^4+1777T^3-2238T^2+2604T-2772$	3 / ✘ 3 / ✘

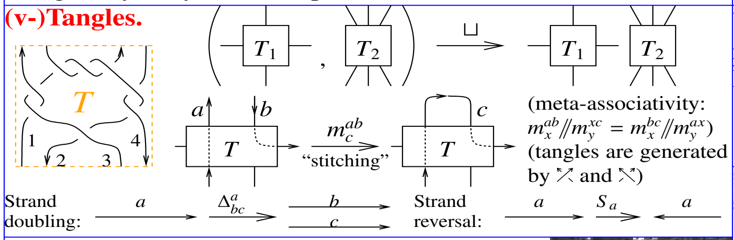
Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Columbia-191125/`

Dror Bar-Natan: Talks: Macquarie-191016: ωεβ:=http://drorbn.net/mac19/

Algebraic Knot Theory

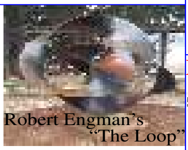
Abstract. This will be a very "light" talk: I will explain why about 13 years ago, in order to have a say on some problems in knot theory, I've set out to find tangle invariants with some nice compositional properties. In other talks in Sydney (ωεβ/talks) I have explained / will explain how such invariants were found - though they are yet to be explored and utilized.

(v-)Tangles.



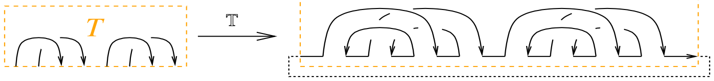
(meta-associativity: $m^{ab}_c /\!/ m^{xc}_y = m^{bc}_x /\!/ m^{ax}_y$) (tangles are generated by ⤱ and ⤲)

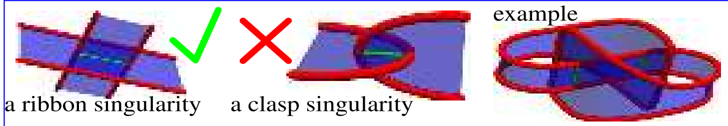
Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.



Robert Engman's "The Loop"

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via





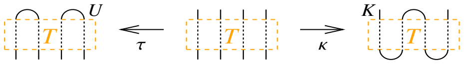
a ribbon singularity a clasp singularity

A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

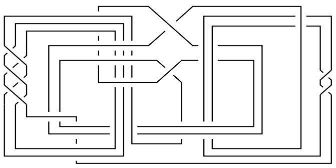
Theorem. K is ribbon iff it is κT for a tangle T for which τT is the untangle U.



Gompf, Scharlemann, Thompson [GST]

$$\begin{tikzcd} & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n \\ \mathcal{T}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] \arrow[rr, "z"] & & \mathcal{A}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] & \\ & \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1 \end{tikzcd}$$

with $\mathcal{R} := \kappa(\tau^{-1}(1))$



Faster is better, leaner is meaner!

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN]. An S-component tangle T has

$$\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A \end{array}\right\} \text{ with } R_S := \mathbb{Z}(\{T_a\colon a\in S\}):$$

$$\left({}_a\!\nearrow\!\!\!\nwarrow_b,\ {}_b\!\nearrow\!\!\!\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-T_a^{\pm1} \\ b & 0 & T_a^{\pm1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a, T_b \to T_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}\right)$$

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

Strand Doubling and Reversal.

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu:=T_a-1 \\ \nu:=\alpha-\sigma_a \\ T_a\mapsto T_bT_c}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a-\alpha T_a-\nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu \\ c & (T_c-1)\nu/\mu & (\alpha-\sigma_aT_a-\nu T_c)/\mu & (T_c-1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array}\right)$$

$$dS^a \Big\downarrow T_a\to T_a^{-1}$$

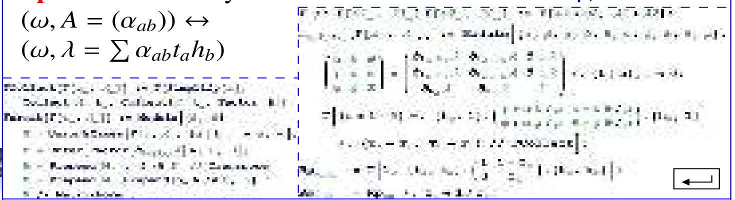
$$\left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi-\phi\theta)/\alpha \end{array}\right)$$

Where σ assigns to every $a\in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left({}_a\!\nearrow\!\!\!\nwarrow_b,\ {}_b\!\nearrow\!\!\!\nwarrow_a\right) = (a\to 1, b\to t_a^{\pm1})$, $\sigma(T_1\sqcup T_2) = \sigma(T_1)\sqcup\sigma(T_2)$, and $\sigma /\!/ m^{ab}_c = (\sigma\setminus\{a,b\})\cup(c\to\sigma_a\sigma_b)|_{t_a,t_b\to t_c}$.

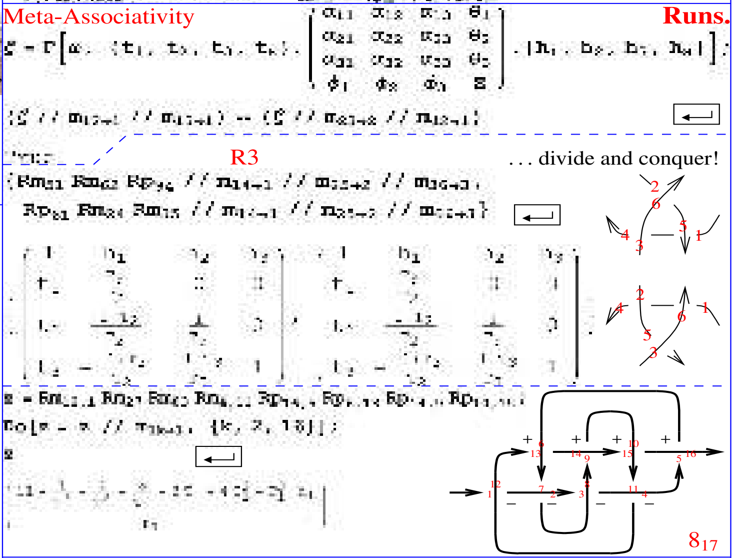
Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Implementation key idea: ωεβ/AlexDemo

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab}t_ah_b)$



Meta-Associativity **Runs.**



Fact. Γ is better viewed as an invariant of a certain class of 2D knotted objects in $\mathbb{R}^4$ [BND, BN].

Fact. Γ is the "0-loop" part of an invariant that generalizes to "n-loops" (1D tangles only, see further talks and future publications with van der Veen).

Speculation. Stepping stones to categorification?

M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

Ask me about geography vs. identity!

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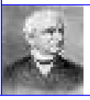
[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[Vo] H. Vo, *Alexander Invariants of Tangles via Expansions,* University of Toronto Ph.D. thesis, ωεβ/Vo.



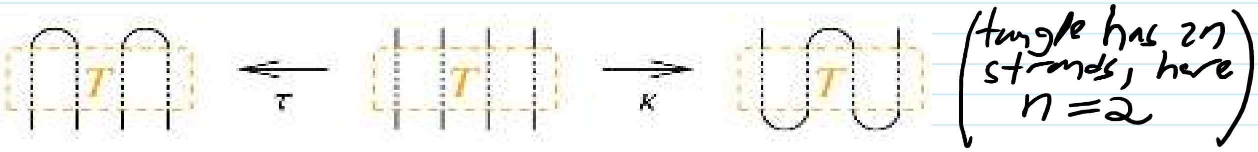
"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Macquarie-191016/

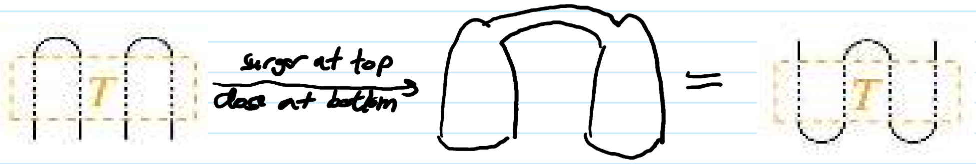
Proof of the Tangle Characterization of Ribbon Knots



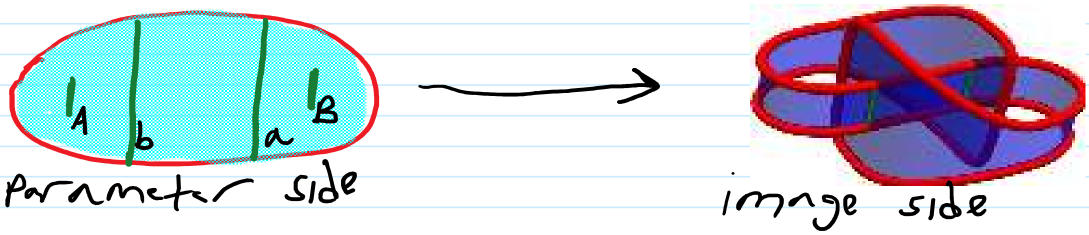
(tangle has 2n strands, here n=2)

Theorem. A knot K is ribbon iff there exists a tangle T whose τ closure is the untangle and whose κ closure is K.

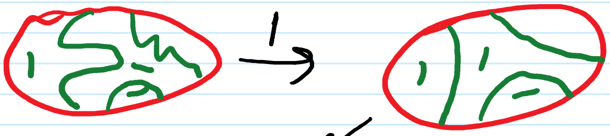
Proof. The backward $\Leftarrow$ implication is easy:



For the forward implication, follow the following 5 steps:

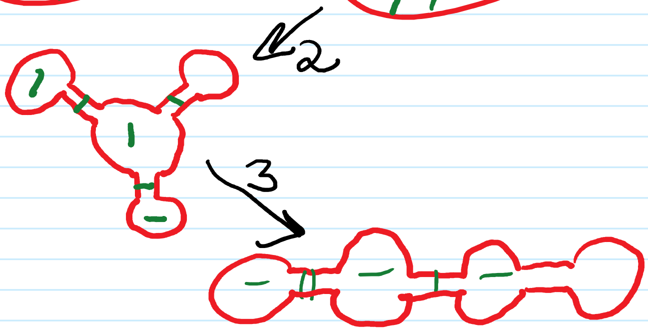


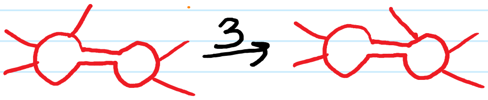
Step I: In-situ cosmetics.
At end: D is a tree of chord-and-arc polygons.

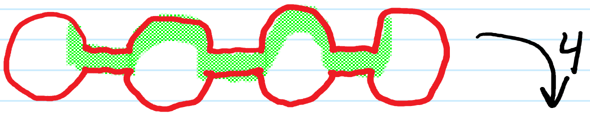


Step 2: Near-situ cosmetics.
At end: D is tree-band-sum of n unknotted disks.

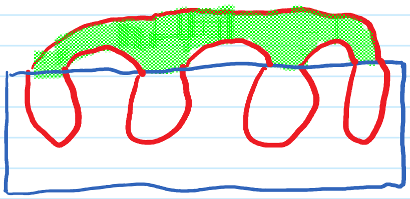
Step 3: Slides.
At end: D is a linear-band-sum of n unknotted disks.



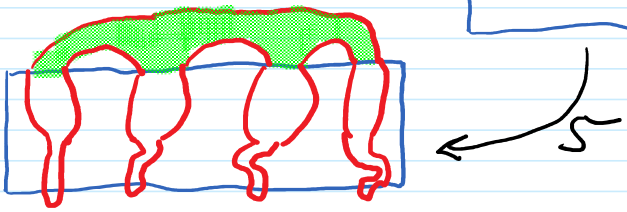




Step 4: Exposure!
The green domain is contractible - so it can be shrank, moved at will (with the blue membrane following along), and expanded back again.
At end: D has (n-1) exposed bridges which when turned, make D a union of n unknotted disks.



Step 5: Pulling bottom handles avoiding the obstacles.
At end: Theorem is proven.



Dror Bar-Natan: Talks: UCLA-191101:

Everything around sl_{2+}^ϵ is **DoPeGDO**. So what?

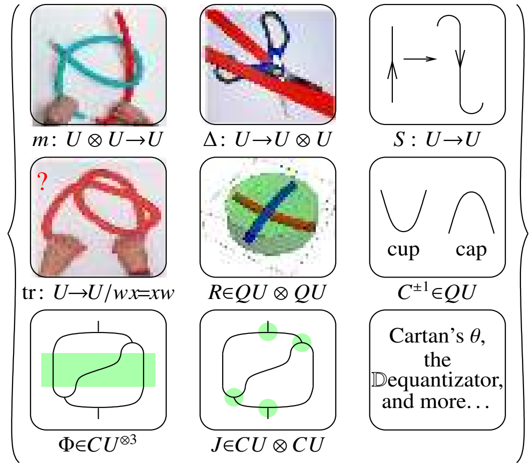
Thanks for inviting me to UCLA!
ωεβ:=http://drorbn.net/la19/
More at ωεβ/talks

Continues Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen [BV].

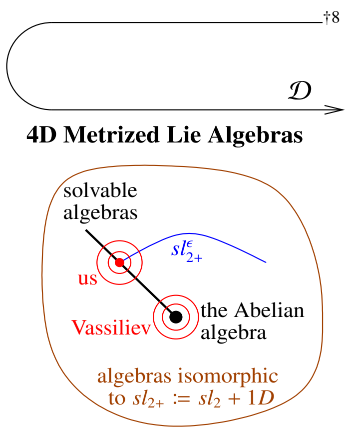
Abstract. I'll explain what "everything around" means: classical and quantum m, Δ, S, tr, R, C, and θ, as well as P, Φ, J, $\mathbb{D}$, and more, and all of their compositions. What **DoPeGDO** means: the category of Docile Perturbed Gaussian Differential Operators. And what sl_{2+}^ϵ means: a solvable approximation of the semi-simple Lie algebra sl_2.

Knot theorists should rejoice because all this leads to very powerful and well-behaved poly-time-computable knot invariants. Quantum algebraists should rejoice because it's a realistic playground for testing complicated equations and theories.

Conventions. 1. For a set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$.[†1] 2. Everything converges!



Less Abstract



DoPeGDO := The category with objects finite sets[†2] and mor($A \to B$):

$$\{\mathcal{F} = \omega \exp(Q+P)\} \subset \mathbb{Q}[\![\zeta_A, z_B, \epsilon]\!]$$

Where: • ω is a scalar.[†3] • Q is a "small" ϵ-free quadratic in $\zeta_A \cup z_B$.[†4] • P is a "docile perturbation": $P = \sum_{k\geq1}\epsilon^k P^{(k)}$, where $\deg P^{(k)} \leq 2k+2$.[†5] • Compositions:[†6]

$$\mathcal{F}/\!/\mathcal{G} = \mathcal{G}\circ\mathcal{F} := \left(\mathcal{G}|_{\zeta_i\to\partial_{z_i}}\mathcal{F}\right)_{z_i=0} = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

Cool! $(V^*)^{\otimes\Sigma}\otimes V^{\otimes S}$ explodes; the ranks of quadratics and bounded-degree polynomials grow slowly![†7] Representation theory is over-rated!

Cool! How often do you see a computational toolbox so successful?

Our Algebras. Let $sl_{2+}^\epsilon := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t := \epsilon a - b$ is central and if $\exists\epsilon^{-1}$, $sl_{2+}^\epsilon/\langle t\rangle \cong sl_2$. ωεβ/oa
U is either $CU = \mathcal{U}(sl_{2+}^\epsilon)[\![\hbar]\!]$ or $QU = \mathcal{U}_\hbar(sl_{2+}^\epsilon) = A\langle y,b,a,x\rangle[\![\hbar]\!]$ with $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $xy-qyx=(1-AB)/\hbar$, where $q=\mathbb{e}^{\hbar\epsilon}$, $A=\mathbb{e}^{-\hbar\epsilon a}$, and $B=\mathbb{e}^{-\hbar b}$. Set also $T=A^{-1}B=\mathbb{e}^{\hbar t}$.

The Quantum Leap. Also decree that in QU,

$$\Delta(y,b,a,x) = (y_1+B_1y_2, b_1+b_2, a_1+a_2, x_1+A_1x_2),$$
$$S(y,b,a,x) = (-B^{-1}y, -b, -a, -A^{-1}x),$$

and $R = \sum \hbar^{j+k}y^kb^j\otimes a^jx^k/j![k]_q!$.

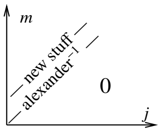
Mid-Talk Debts. • What is this good for in quantum algebra?
- In knot theory?
- How does the "inclusion" $\mathcal{D}\colon \mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S}) \rightsquigarrow$ **DoPeGDO** work?
- Proofs that everything around sl_{2+}^ϵ really is **DoPeGDO**.
- Relations with prior art.
- The rest of the "compositions" story.

Theorem ([BG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing


Melvin,
Morton,
Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq0} a_{jm}(K)d^j\hbar^m,$$

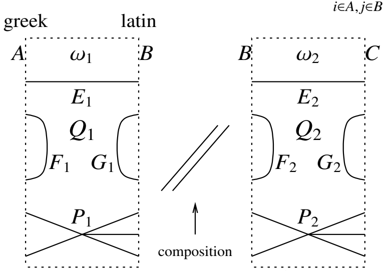
"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

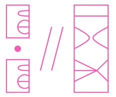
Compositions (1). In mor($A\to B$), $Q = \sum_{i\in A, j\in B}E_{ij}\zeta_iz_j + \frac12\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j + \frac12\sum_{i,j\in B}G_{ij}z_iz_j$



Where • $E = E_1(I-F_2G_1)^{-1}E_2$.
• $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$.
• $G = G_2 + E_2^TG_1(I-F_2G_1)^{-1}E_2$.
• $\omega = \omega_1\omega_2\det(I-F_2G_1)^{-1}$.
• P is computed using "connected Feynman diagrams" or as the solution of a messy PDE (yet we're still in algebra!).

Alexander lives here

One abstraction level up from tangles!
{tangles} → {▭}
with compositions:



DoPeGDO Footnotes. †1. Each variable has a "weight"$\in\{0,1,2\}$, and always $\mathrm{wt}\,z_i + \mathrm{wt}\,\zeta_i = 2$.

†2. Really, "weight-graded finite sets" $A = A_0\sqcup A_1\sqcup A_2$.

†3. Really, a power series in the weight-0 variables[†9].

†4. The weight of Q must be 2, so it decomposes as $Q = Q_{20}+Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series[†9].

†5. Setting $\mathrm{wt}\,\epsilon = -2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained[†9]).

†6. There's also an obvious product
$$\mathrm{mor}(A_1\to B_1)\times\mathrm{mor}(A_2\to B_2)\to\mathrm{mor}(A_1\sqcup A_2\to B_1\sqcup B_2).$$

†7. That is, if the weight-0 variables are ignored. Otherwise more care is needed yet the conclusion remains.

†8. $\mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S}) \rightsquigarrow \mathrm{mor}(\{\eta_i,\beta_i,\tau_i,\alpha_i,\xi_i\}_{i\in\Sigma}\to\{y_i,b_i,t_i,a_i,x_i\}_{i\in S})$, where $\mathrm{wt}(\eta_i,\xi_i,y_i,x_i)=1$ and $\mathrm{wt}(\beta_i,\tau_i,\alpha_i;b_i,t_i,a_i)=(2,2,0;0,0,2)$.

†9. For tangle invariants the wt-0 power series are always rational functions in the exponentials of the wt-0 variables (for knots: just one variable), with degrees bounded linearly by the crossing number.

Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907,
http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.

Dror Bar-Natan: Talks: Ohio-1901:

Computation without Representation

Thanks for inviting me to Ohio!

ωεβ:=http://drorbn.net/o19/

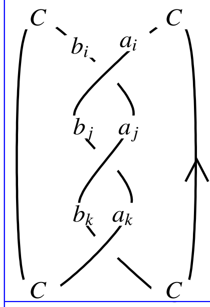
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at [BV] and at ωεβ/talks.

Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast.

In my talk, I will describe a direct method for carrying out such computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know. (experimental analysis @ωεβ/kiw)

Knotted Candies ωεβ/kc



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

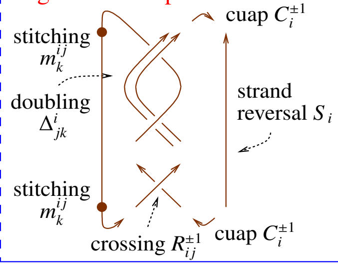
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

A Knot Theory Portfolio.

- Has operations $\sqcup$, m^{ij}_k, Δ^i_{jk}, S_i.
- All tangloids are generated by $R^{\pm 1}$ and $C^{\pm 1}$ (so "easy" to produce invariants).
- Makes some knot properties ("genus", "ribbon") become "definable".

Tangloids and Operations



A "Quantum Group" Portfolio consists of a vector space U along with maps (and some axioms…)

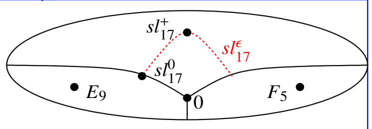
$$\begin{tikzcd}
\mathbb{Q} = \hat{U}^{\otimes\emptyset} \arrow[r, "C_i"] & \hat{U}^{\otimes\{i\}} \arrow[loop above, "S_i"] \arrow[r, bend left, "\Delta^i_{jk}"] & \hat{U}^{\otimes\{j,k\}} \arrow[l, bend left, "m^{jk}_i"] & \mathbb{Q} = \hat{U}^{\otimes\emptyset} \arrow[l, "R_{jk}"'] \\
\hat{\mathcal{S}}(\emptyset) \arrow[u, "\mathbb{O}_{()}"] \arrow[r, "C_i"] & \hat{\mathcal{S}}(B_i) \arrow[u, "\mathbb{O}_{y_i x_i \ldots}"] \arrow[loop below, "S_i"] \arrow[r, bend left, "\Delta^i_{jk}"] & \hat{\mathcal{S}}\left(B_j, B_k\right) \arrow[u, "\mathbb{O}_{y_j x_j \ldots \otimes y_k x_k \ldots}"] \arrow[l, bend left, "m^{jk}_i"] & \hat{\mathcal{S}}(\emptyset) \arrow[u, "\mathbb{O}_{()}"] \arrow[l, "R_{jk}"']
\end{tikzcd}$$

$\sqcup \to \otimes$

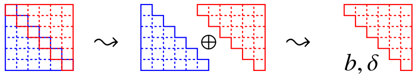
$\sqcup \to \cdot$

Our Way. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl^k_{17} := sl^\epsilon_{17}/(\epsilon^{k+1} = 0)$.



Solvable Approximation. In gl_n, half is enough! Indeed $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b\colon$ ◹ $\otimes$ ◹ $\to$ ◹

$b(◺) \rightsquigarrow \delta\colon$ ◹ $\to$ ◹ $\otimes$ ◹

Now define $gl^\epsilon_n := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is [◹, ◹] = ◹, [◺, ◺] = ϵ◺, and [◹, ◺] = ◺ + ϵ◹. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1} = 0$ always yields a solvable Lie algebra.

***CU* and *QU*.** Starting from sl_2, get $CU_\epsilon = \langle y, a, x, t\rangle / ([t, -] = 0, [a, y] = -y, [a, x] = x, [x, y] = 2\epsilon a - t)$. Quantize using standard tools (I'm sorry) and get $QU_\epsilon = \langle y, a, x, t\rangle / ([t, -] = 0, [a, y] = -y, [a, x] = x, xy - \mathbb{e}^{\hbar\epsilon} yx = (1 - T\mathbb{e}^{-2\hbar\epsilon a})/\hbar)$.

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set $B = \{y, x, \ldots\}$ of "generators" and isomorphisms $\mathbb{O}_{y,x,\ldots}\colon \hat{\mathcal{S}}(B) \to U$ defined by "ordering monomials" to some fixed $y, x, \ldots$ order. The quantum group portfolio now becomes a "symmetric algebra" portfolio, or a "power series" portfolio.

Operations are Objects.

★ $$B^* := \{z_i^* = \zeta_i\colon\ z_i \in B\},$$

$$\langle z_i^m, \zeta_i^n \rangle = \delta_{mn} n!,$$

$$\left\langle \prod z_i^{m_i}, \prod \zeta_i^{n_i} \right\rangle = \prod \delta_{m_i n_i} n_i!,$$

in general, for $f \in \mathcal{S}(z_i)$ and $g \in \mathcal{S}(\zeta_i)$,

$$\langle f, g \rangle = f(\partial_{\zeta_i}) g\big|_{\zeta_i = 0} = g(\partial_{z_i}) f\big|_{z_i = 0}.$$

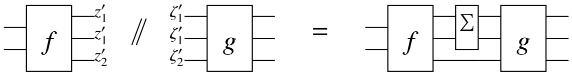
$$\begin{tikzcd}
f \in \mathrm{Hom}_{\mathbb{Q}}(S(B) \to S(B')) \arrow[d, phantom, "\parallel"] \arrow[dddd, bend right] \\
S(B)^* \otimes S(B') \arrow[d, phantom, "\parallel\bigstar"] \\
S(B^*) \otimes S(B') \arrow[d, phantom, "\parallel"] \\
S(B^* \sqcup B') \arrow[d, phantom, "\parallel"] \\
\tilde{f} \in \mathbb{Q}[\zeta_i, z'_i]
\end{tikzcd}$$

The Composition Law. If

$$\mathcal{S}(B) \xrightarrow[\tilde{f} \in \mathbb{Q}[\![\zeta_i, z'_j]\!]]{f} \mathcal{S}(B') \xrightarrow[\tilde{g} \in \mathbb{Q}[\![\zeta'_j, z''_k]\!]]{g} \mathcal{S}(B'')$$

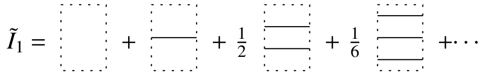
then

$$\widetilde{(f /\!\!/ g)} = \widetilde{(g \circ f)} = \left(\tilde{g}|_{\zeta'_j \to \partial_{z'_j}} \tilde{f} \right)_{z'_j = 0} = \left(\tilde{f}|_{z'_j \to \partial_{\zeta'_j}} \tilde{g} \right)_{\zeta'_j = 0}:$$



Examples

1. The 1-variable identity map $I\colon \mathcal{S}(z) \to \mathcal{S}(z)$ is given by $\tilde{I}_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by $\tilde{I}_n = \mathbb{e}^{z_1\zeta_1 + \cdots + z_n\zeta_n}$:



2. The "archetypal multiplication map $m^{ij}_k\colon \mathcal{S}(z_i, z_j) \to \mathcal{S}(z_k)$" has $\tilde{m} = \mathbb{e}^{z_k(\zeta_i + \zeta_j)}$.
3. The "archetypal coproduct $\Delta^i_{jk}\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j, z_k)$", given by $z_i \to z_j + z_k$ or $\Delta z = z \otimes 1 + 1 \otimes z$, has $\tilde{\Delta} = \mathbb{e}^{(z_j + z_k)\zeta_i}$.
4. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1 x_2} \in \mathcal{U}_q \otimes \mathcal{U}_q$. The "baby R-matrix" is $\tilde{R} = \mathbb{e}^{\hbar yx} \in \mathcal{S}(y, x)$.
5. The "Weyl form of the canonical commutation relations" states that if $[y, x] = tI$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. So with

$$\begin{tikzcd}
\mathcal{S}(y, x) \arrow[loop left, "SW_{xy}"] \arrow[r, shift left, "\mathbb{O}_{xy}"] \arrow[r, shift right, "\mathbb{O}_{yx}"'] & \mathcal{U}(y, x)
\end{tikzcd}$$

we have $\widetilde{SW}_{xy} = \mathbb{e}^{\eta y + \xi x - \eta\xi t}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901

Do Not Turn Over Until Instructed

Thanks for inviting me to the fall 2018 MAA Seaway Section meeting!

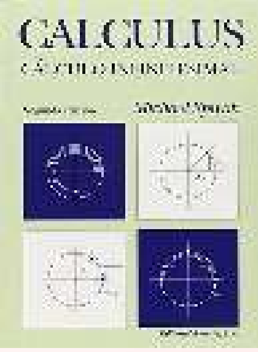
My Favourite First-Year Analysis Theorem

Handout, video, links at ωεβ:=http://drorbn.net/maa18/

Abstract. Whatever it may be, it should say something useful and exciting and it should not be *about* rigour, yet it should *demand* rigour. You can't guess. You probably think it the dreariest. You are wrong.

Contents Ⓢ

Several excerpts here are from Spivak's "Calculus" Ⓢ. I believe they fall under "fair use".



for every $\varepsilon > 0$ there is $\delta > 0$ such that, for all x, Ⓢ
if $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

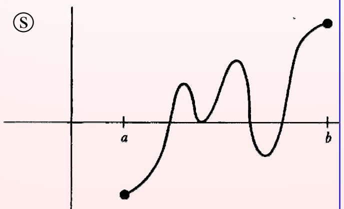
If f and g are continuous at a, then

(1) $f + g$ is continuous at a,

(2) $f \cdot g$ is continuous at a.

6 Continuous Functions.

If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$. Ⓢ



7 Three Hard Theorems.

11 Significance of the Derivative.

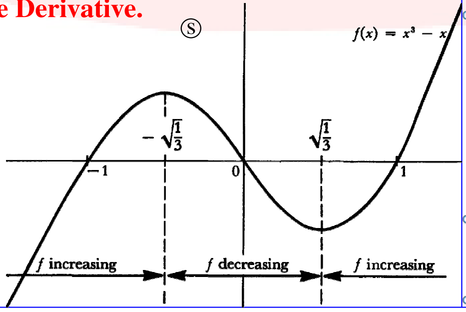
Ⓢ

$y=x^3-x$

$y'=3x^2-1$

$=(\sqrt{3}x+1)(\sqrt{3}x-1)$

$$=\begin{cases} >0 & x>\sqrt{1/3} \\ <0 & -\sqrt{1/3}<x<\sqrt{1/3} \\ >0 & x<-\sqrt{1/3} \end{cases}$$



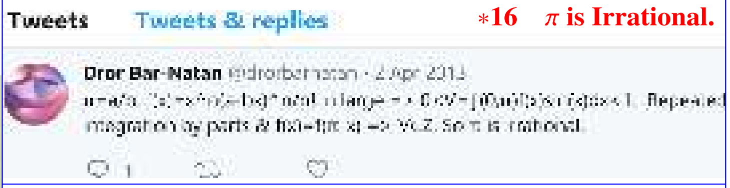
14 The Fundamental Theorem of Calculus.

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a).$$

Ⓢ

*16 π is Irrational.



Dror Bar-Natan

20 Approximation by Polynomial Functions.

Suppose that f is a function for which

$$f'(a), \dots, f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \le k \le n,$$

and define

$$P_{n,a}(x) = a_0 + a_1(x - a) + \cdots + a_n(x - a)^n.$$

Then

$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x - a)^n} = 0.$$

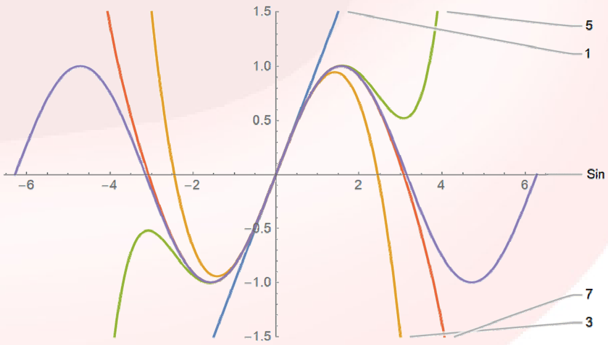
Ⓢ

For example for $f(x) = \sin(x)$ at $a = 0$, $f^{(k)} = \sin, \cos, -\sin, -\cos, \sin, \dots$, so

$$a_k = \begin{cases} \frac{(-1)^{(k-1)/2}}{k!} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

```
In[1]:= a_k_ := { (-1)^((k-1)/2) / k!   OddQ[k]
                  0                      EvenQ[k] ;
        Plot[Evaluate@Append[
            Table[Labeled[Sum[a_k x^k, {k, 0, n}], n], {n, {1, 3, 5, 7}}],
            Labeled[Sin[x], Sin]
          ], {x, -2π, 2π}, PlotRange → {-1.5, 1.5}]
```

Out[2]=



```
In[3]:= Column@Table[k → N[a_k 157^k], {k, {0, 3, 9, 13, 29, 35, 157, 223, 457}}]

        0 → 0.
        3 → -644982.
        9 → 1.59711 × 10^14
        13 → 5.65477 × 10^18
Out[3]= 29 → 5.42689 × 10^32
        35 → -6.95433 × 10^36
        157 → 4.86366 × 10^66
        223 → -1.94045 × 10^61
        457 → 4.87404 × 10^-16

In[4]:= {N[Sum[a_k 157^k, {k, 0, 457}]], Sum[N[a_k 157^k], {k, 0, 457}]}

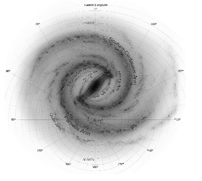
Out[4]= {-0.0795485, 5.10624 × 10^50}

In[5]:= N@Sin[157]

Out[5]= -0.0795485
```

Some sizes (in multiples of the diameter of a Hydrogen atom:

A red blood cell	1.56×10^5
The CN Tower	1.11×10^{13}
The rings of Saturn	5.6×10^{18}
The Milky Way galaxy	1.89×10^{31}
The observable universe	1.76×10^{37}



Do Not Turn Over Until Instructed

Dror Bar-Natan: Talks: Matemale-1804:

Solvable Approximations of the Quantum sl_2 Portfolio

See also [BV]

With Roland van der Veen

ωεβ:=http://drorbn.net/mm18/

Our Main Theorem (loosely stated). Everything that matters in the quantum sl_2 portfolio can be continuously expressed in terms of docile perturbed Gaussians using solvable approximations. ○

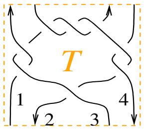
Our Main Points.

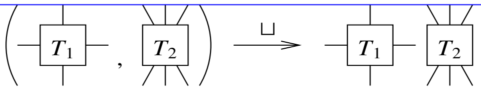
- What's the "quantum sl_2 portfolio"?
- What in it "matters" and why? (the most important question)
- What's "solvable approximation"? What's "continuously"?
- What are "docile perturbed Gaussians"?
- Why do they matter? (2^{nd} most important)
- How proven? (docile)
- How implemented? (sacred; the work of unsung heroes)
- Some context and background.
- What's next?

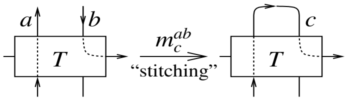
The quantum sl_2 Portfolio includes a classical universal enveloping algebra CU, its quantization QU, their tensor powers $CU^{\otimes S}$ and $QU^{\otimes S}$ with the "tensor operations" ⊗, their products m^{ij}_k, coproducts Δ^i_{jk} and antipodes S_i, their Cartan automophisms $C\theta\colon CU \to CU$ and $Q\theta\colon QU \to QU$, the "dequantizators" $A\mathbb{D}\colon QU \to CU$ and $S\mathbb{D}\colon QU \to CU$, and most importantly, the R-matrix R and the Drinfel'd element s. All this in any PBW basis, and change of basis maps are included.

$$\begin{tikzcd} R, s \in \{QU^{\otimes S}\} \arrow[loop above, "{\otimes, m^{ij}_k, \Delta^i_{jk}, S_i, \theta}"] \arrow[r, "{A\mathbb{D}, S\mathbb{D}}"] & \{CU^{\otimes S}\} \arrow[loop above, "{\otimes, m^{ij}_k, \Delta^i_{jk}, S_i, \theta}"] \end{tikzcd}$$

(v-)Tangles.

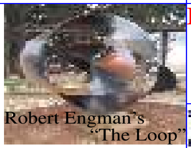






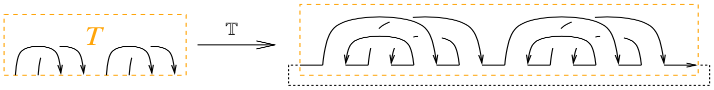
(meta-associativity: $m^{ab}_x /\!\!/ m^{xc}_y = m^{bc}_x /\!\!/ m^{ax}_y$) (tangles are generated by ↗ and ↖)

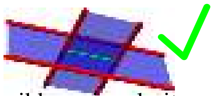
Strand doubling: $\xrightarrow{\quad a \quad}\ \xrightarrow{\Delta^a_{bc}}\ \begin{matrix}\xrightarrow{\quad b \quad}\\ \xrightarrow{\quad c \quad}\end{matrix}$ Strand reversal: $\xrightarrow{\quad a \quad}\ \xrightarrow{S_a}\ \xleftarrow{\quad a \quad}$



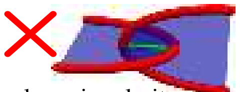
Robert Engman's "The Loop"

Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.
Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via



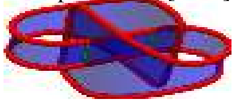


a ribbon singularity

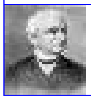


a clasp singularity

example [BN2]

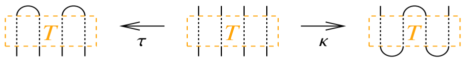


A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,
Conjecture. Some slice knots are not ribbon.
Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)



> "God created the knots, all else in topology is the work of mortals."
>
> Leopold Kronecker (modified)

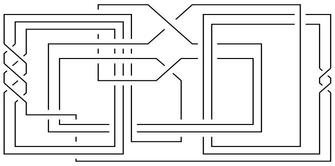
www.katlas.org



Gompf, Scharlemann, Thompson [GST]

$$\begin{tikzcd} & U \in \mathcal{T}_n & & 1 \in \mathcal{A}_n \\ \mathcal{T}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] \arrow[rr, "z"] & & \mathcal{A}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] & \\ & \text{ribbon } K \in \mathcal{T}_1 & & z(K) \in \mathcal{R} \subseteq \mathcal{A}_1 \end{tikzcd}$$

with $\mathcal{R} := \kappa(\tau^{-1}(1))$



Faster is better, leaner is meaner!

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has
$$\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A \end{array}\right\} \text{ with } R_S := \mathbb{Z}(\{t_a\colon a \in S\}):$$

$$\left({}_a\nearrow_b, {}_b\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1 - t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}\right)$$

(Roland: "add to A the product of column b and row a, divide by $(1 - A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

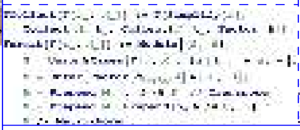
$$\begin{tikzcd} \begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \arrow[r, "q\Delta^a_{bc}", "{\substack{\mu := T_a - 1 \\ \nu := \alpha - \sigma_a \\ T_a \mapsto T_b T_c}}"'] \arrow[d, "dS^a"', "T_a \to T_a^{-1}"] & \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b - 1)T_c\nu/\mu & (T_b - 1)T_c\theta/\mu \\ c & (T_c - 1)\nu/\mu & (\alpha - \sigma_a T_a - \nu T_c)/\mu & (T_c - 1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array}\right) \\ \left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array}\right) & \end{tikzcd}$$

Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left({}_a\nearrow_b, {}_b\nwarrow_a\right) = (a \to 1, b \to t_a^{\pm 1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma /\!\!/ m^{ab}_c = (\sigma \setminus \{a, b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a, t_b \to t_c}$.

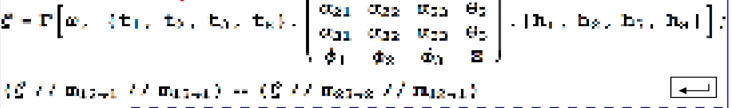
Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Implementation key idea: ωεβ/AlexDemo

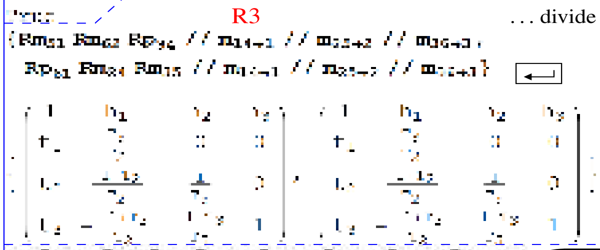
$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

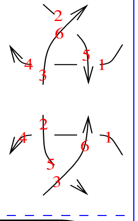


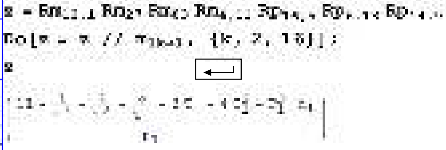
Meta-Associativity Runs.

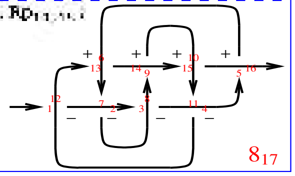


R3 . . . divide and conquer!









Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/

Dror Bar-Natan: Talks: LesDiablerets-1708:

The Dogma is Wrong

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. Preliminary writeup [BV1], fuller writeup [BV2]. More at ωεβ/talks.

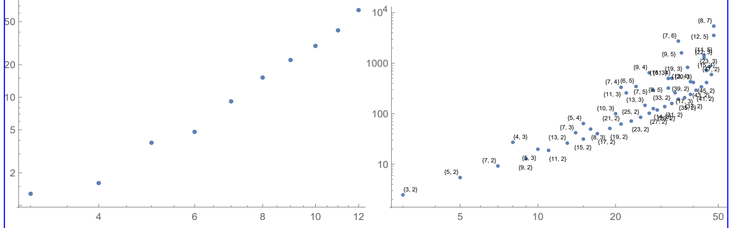
Happy Birthday Anton!

ωεβ:=http://drorbn.net/ld17/

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

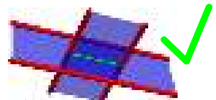
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



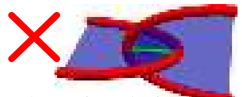
Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \leq 2g - 1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Ribbon Knots.

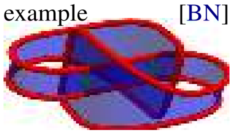


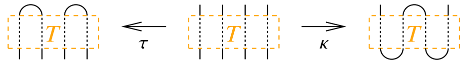
a ribbon singularity



a clasp singularity

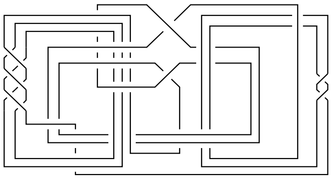
example [BN]





Gompf, Scharlemann, Thompson [GST]

$$\begin{tikzcd} & U \in \mathcal{T}_n & & 1 \in \mathcal{A}_n \\ \mathcal{T}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] \arrow[rr, "z"] & & \mathcal{A}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] & \\ & \text{ribbon } K \in \mathcal{T}_1 & & z(K) \in \mathcal{R} \subseteq \mathcal{A}_1 \end{tikzcd}$$



[Vo]: Works for Alexander!

with $\mathcal{R} := \kappa(\tau^{-1}(1))$

$A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$

$\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$

Faster is better, leaner is meaner!

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of *poly* in $\mathcal{S}(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(a_1^3 y_1 a_2 e^{y_3} x_3^9 \mid x_3 a_1 \otimes y_1 y_3 a_2\right) = x^9 a^3 \otimes y e^y a \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

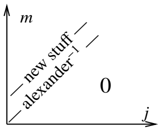
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

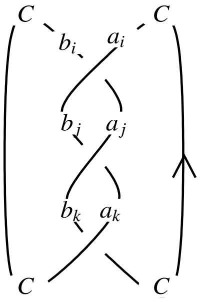
$$\left.\frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot \omega(K)(e^\hbar) = 1$.



"Above diagonal" we have **Rozansky's Theorem** [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})\omega(K)(q^d)} \left(1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

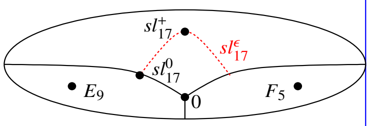
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

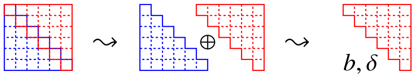
The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

$$\begin{tikzcd} \{\mathcal{F}_S\} \arrow[loop left, "m_k^{ij}"] \arrow[r, "\mathbb{E}"] & \{U^{\otimes S}\} \arrow[loop right, "m_k^{ij}"] \end{tikzcd}$$

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon / (\epsilon^{k+1} = 0)$.



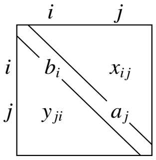
Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b\colon ◹ \otimes ◹ \to ◹$

$b(◺) \rightsquigarrow \delta\colon ◹ \to ◹ \otimes ◹$

Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon ◺$, and $[◹, ◺] = ◺ + \epsilon ◹$. In detail, it is



$[x_{ij}, x_{kl}] = \delta_{jk} x_{il} - \delta_{li} x_{kj} \qquad [y_{ij}, y_{kl}] = \epsilon\delta_{jk} y_{il} - \epsilon\delta_{li} y_{kj}$

$[x_{ij}, y_{kl}] = \delta_{jk}(\epsilon\delta_{j<k} x_{il} + \delta_{il}(b_i + \epsilon a_i)/2 + \delta_{i>l} y_{il})$
$\qquad -\delta_{li}(\epsilon\delta_{k<j} x_{kj} + \delta_{kj}(b_j + \epsilon a_j)/2 + \delta_{k>j} y_{kj})$

$[a_i, x_{jk}] = (\delta_{ij} - \delta_{ik}) x_{jk} \qquad [b_i, x_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) x_{jk}$

$[a_i, y_{jk}] = (\delta_{ij} - \delta_{ik}) y_{jk} \qquad [b_i, y_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) y_{jk}$

The Main sl_2 Theorem. Let $\mathfrak{g}^\epsilon = \langle t, y, a, x \rangle / ([t, \cdot] = 0, [a, x] = x, [a, y] = -y, [x, y] = t - 2\epsilon a)$ and let $\mathfrak{g}_k = \mathfrak{g}^\epsilon / (\epsilon^{k+1} = 0)$. The $\mathfrak{g}_k$-invariant of any S-component tangle K can be written in the form $Z(K) = \mathbb{O}\left(\omega \mathbb{e}^{L+Q+P} \colon \bigotimes_{i\in S} y_i a_i x_i\right)$, where ω is a scalar (a rational function in the variables t_i and their exponentials $T_i := \mathbb{e}^{t_i}$), where $L = \sum l_{ij} t_i a_j$ is a quadratic in t_i and a_j with integer coefficients l_{ij}, where $Q = \sum q_{ij} y_i x_j$ is a quadratic in the variables y_i and x_j with scalar coefficients q_{ij}, and where P is a polynomial in $\{\epsilon, y_i, a_i, x_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{y_i, \sqrt{a_i}, x_i\}$. Furthermore, after setting $t_i = t$ and $T_i = T$ for all i, the invariant $Z(K)$ is poly-time computable.

Dror Bar-Natan: Talks: McGill-1702: Joint with Roland van der Veen

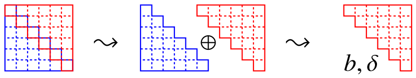
ωεβ:=http://drorbn.net/McGill-1702/

What else can you do with solvable approximations?

Thanks for the invitation!

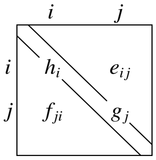
Abstract. Recently, Roland van der Veen and myself found that there are sequences of solvable Lie algebras "converging" to any given semi-simple Lie algebra (such as sl_2 or sl_3 or $E8$). Certain computations are much easier in solvable Lie algebras; in particular, using solvable approximations we can compute in polynomial time certain projections (originally discussed by Rozansky) of the knot invariants arising from the Chern-Simons-Witten topological quantum field theory. This provides us with the first strong knot invariants that are computable for truly large knots.
But sl_2 and sl_3 and similar algebras occur in physics (and in mathematics) in many other places, beyond the Chern-Simons-Witten theory. Do solvable approximations have further applications?

Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b\colon ◹ \otimes ◹ \to ◹$
$b(◺) \rightsquigarrow \delta\colon ◹ \to ◹ \otimes ◹$

Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon ◺$, and $[◹, ◺] = ◺ + \epsilon ◹$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk}f_{il} - \epsilon\delta_{li}f_{kj}$$
$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k}e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l}f_{il}) - \delta_{li}(\epsilon\delta_{k<j}e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j}f_{kj})$$
$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik})e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})e_{jk}$$
$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik})f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})f_{jk}$$

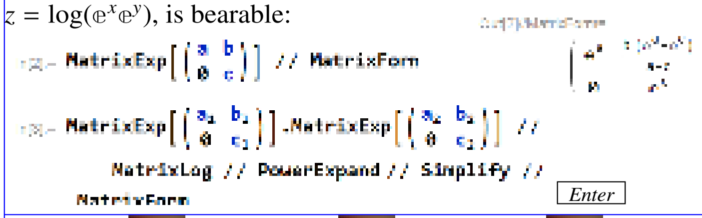
Solvable Approximation. At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n. By rescaling at $\epsilon \neq 0$, gl_n^ϵ is independent of ϵ. We let gl_n^k be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of gl_n!
Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, …, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.
Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

Why are "solvable algebras" any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

```
MatrixExp[( a b
            c d )] // FullSimplify // MatrixForm
```

Enter

Yet in solvable algebras, exponentiation is fine and even BCH, $z = \log(e^x e^y)$, is bearable:



```
MatrixExp[( a b
            0 c )] // MatrixForm
```

```
MatrixExp[( a1 b1
            0  c1 )].MatrixExp[( a2 b2
                                 0  c2 )] //
    MatrixLog // PowerExpand // Simplify //
  MatrixForm
```

Enter

Question. What else can you do with solvable approximation? Chern-Simons-Witten theory is often "solved" using ideas from conformal field theory and using quantization of various moduli spaces. Does it make sense to use solvable approximation there too? Elsewhere in physics? Elsewhere in mathematics?
See Also. Talks at George Washington University [ωεβ/gwu], Indiana [ωεβ/ind], and Les Diablerets [ωεβ/ld], and a University of Toronto "Algebraic Knot Theory" class [ωεβ/akt].

Chern-Simons-Witten. Given a knot $\gamma(t)$ in $\mathbb{R}^3$ and a metrized Lie algebra $\mathfrak{g}$, set $Z(\gamma) :=$

$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, e^{ik\,cs(A)} PExp_\gamma(A),$$

where $cs(A) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(AdA + \frac{2}{3}A^3\right)$ and

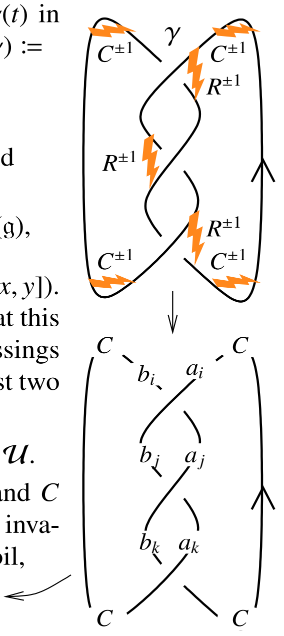
$$PExp_\gamma(A) := \prod_0^1 \exp(\gamma^* A) \in \mathcal{U} = \hat{\mathcal{U}}(\mathfrak{g}),$$

and $\mathcal{U}(\mathfrak{g}) := \langle\text{words in } \mathfrak{g}\rangle/(xy - yx = [x, y])$.
In a favourable gauge, one may hope that this computation will localize near the crossings and the bends, and all will depend on just two quantities,

$$R = \sum a_i \otimes b_i \in \mathcal{U} \otimes \mathcal{U} \quad \text{and} \quad C \in \mathcal{U}.$$

This was never done formally, yet R and C can be "guessed" and all "quantum knot invariants" arise in this way. So for the trefoil,

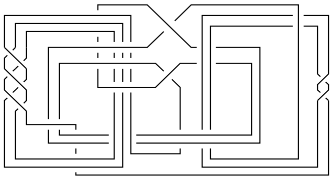
$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$



But Z lives in $\mathcal{U}$, a complicated space. How do you extract information out of it?
Solution 1, Representation Theory. Choose a finite dimensional representation ρ of $\mathfrak{g}$ in some vector space V. By luck and the wisdom of Drinfel'd and Jimbo, $\rho(R) \in V^* \otimes V^* \otimes V \otimes V$ and $\rho(C) \in V^* \otimes V$ are computable, so Z is computable too. But in exponential time!

Ribbon=Slice?

Gompf, Scharlemann, Thompson



Solution 2, Solvable Approximation. Work directly in $\hat{\mathcal{U}}(\mathfrak{g}_k)$, where $\mathfrak{g}_k = sl_2^k$ (or a similar algebra); everything is expressible using low-degree polynomials in a small number of variables, hence everything is poly-time computable!

Example 0. Take $\mathfrak{g}_0 = sl_2^0 = \mathbb{Q}\langle h, e, l, f\rangle$, with h central and $[f, l] = f$, $[e, l] = -e$, $[e, f] = h$. In it, using normal orderings,

$$R = \mathbb{O}\left(\exp\left(hl + \frac{e^h - 1}{h}ef\right) \mid e \otimes lf\right), \quad \text{and,}$$

$$\mathbb{O}\left(e^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu\delta ef} \mid ef\right) \quad \text{with } \nu = (1 + h\delta)^{-1}.$$

Example 1. Take $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$ and $\mathfrak{g}_1 = sl_2^1 = R\langle h, e, l, f\rangle$, with h central and $[f, l] = f$, $[e, l] = -e$, $[e, f] = h - 2\epsilon l$. In it,

$$\mathbb{O}\left(e^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\delta\Lambda/2)e^{\nu\delta ef} \mid elf\right), \quad \text{where } \Lambda \text{ is}$$

$$4\nu^3\delta^2e^2f^2 + 3\nu^3\delta^3he^2f^2 + 8\nu^2\delta ef + 4\nu^2\delta^2hef + 4\nu\delta elf - 2\nu\delta h + 4l.$$

Fact. Setting $h_i = h$ (for all i) and $t = e^h$, the $\mathfrak{g}_1$ invariant of any tangle T can be written in the form

$$Z_{\mathfrak{g}_1}(T) = \mathbb{O}\left(\omega^{-1}e^{hL+\omega^{-1}Q}(1 + \epsilon\omega^{-4}P) \mid \bigotimes_i e_i l_i f_i\right),$$

where L is linear, Q quadratic, and P quartic in the $\{e_i, l_i, f_i\}$ with ω and all coefficients polynomials in t. Furthermore, everything is poly-time computable.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/McGill-1702/`