```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌐, ⌢}.
```

```
FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
```

```
AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm,
  hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.
```

```
AwCalculus` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.
```

```
Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

```
lhs = Rs[1, 2] ** Rs[1, 3] ** Rs[2, 3]; rhs = Rs[2, 3] ** Rs[1, 3] ** Rs[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

$$\left\{\mathrm{Es}\left[\left\langle 1 \to \mathrm{LS}[0,\ 0,\ 0,\ \ldots],\ 2 \to \mathrm{LS}\left[\overline{1},\ 0,\ 0,\ \ldots\right],\ 3 \to \mathrm{LS}\left[\overline{1}+\overline{2},\ \frac{\overline{1\,2}}{2},\ \frac{1}{12}\,\overline{1\,\overline{1\,2}}+\frac{1}{12}\,\overline{\overline{1\,2}\,2},\ \ldots\right]\right\rangle,\ \mathrm{CWS}[0,\ 0,\ 0,\ \ldots]\right],\ \mathrm{BS}[6\ \mathrm{True},\ \ldots]\right\}$$

```
x = LW["x"]; y = LW["y"]; z = LW["z"];
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V0 = Es[⟨x → α, y → β⟩, γ];
```

**$V_0$**

$$\mathrm{Es}\Big[\Big\langle \overline{x} \to \mathrm{LS}\big[\overline{x}\,\alpha s[x] + \overline{y}\,\alpha s[y],\ \overline{x\,y}\,\alpha s[x, y],\ \overline{x\,\overline{x\,y}}\,\alpha s[x, x, y] + \overline{\overline{x\,y}\,y}\,\alpha s[x, y, y],$$
$$\overline{x\,\overline{x\,\overline{x\,y}}}\,\alpha s[x, x, x, y] + \overline{x\,\overline{\overline{x\,y}\,y}}\,\alpha s[x, x, y, y] + \overline{\overline{\overline{x\,y}\,y}\,y}\,\alpha s[x, y, y, y],\ \ldots\big],$$
$$\overline{y} \to \mathrm{LS}\big[\overline{x}\,\beta s[x] + \overline{y}\,\beta s[y],\ \overline{x\,y}\,\beta s[x, y],\ \overline{x\,\overline{x\,y}}\,\beta s[x, x, y] + \overline{\overline{x\,y}\,y}\,\beta s[x, y, y],$$
$$\overline{x\,\overline{x\,\overline{x\,y}}}\,\beta s[x, x, x, y] + \overline{x\,\overline{\overline{x\,y}\,y}}\,\beta s[x, x, y, y] + \overline{\overline{\overline{x\,y}\,y}\,y}\,\beta s[x, y, y, y],\ \ldots\big]\Big\rangle,$$
$$\mathrm{CWS}[\widehat{x}\,\gamma s[x] + \widehat{y}\,\gamma s[y],\ \widehat{xx}\,\gamma s[x, x] + \widehat{xy}\,\gamma s[x, y] + \widehat{yy}\,\gamma s[y, y],$$
$$\widehat{xxx}\,\gamma s[x, x, x] + \widehat{xxy}\,\gamma s[x, x, y] + \widehat{xyy}\,\gamma s[x, y, y] + \widehat{yyy}\,\gamma s[y, y, y],$$
$$\widehat{xxxx}\,\gamma s[x, x, x, x] + \widehat{xxxy}\,\gamma s[x, x, x, y] + \widehat{xxyy}\,\gamma s[x, x, y, y] +$$
$$\widehat{xyxy}\,\gamma s[x, y, x, y] + \widehat{xyyy}\,\gamma s[x, y, y, y] + \widehat{yyyy}\,\gamma s[y, y, y, y],\ \ldots]\Big]$$

**(Rs[x, z] // dΔ[x, x, y])**

$$\mathrm{Es}[\langle \overline{x} \to \mathrm{LS}[0, 0, 0, 0, \ldots],\ \overline{y} \to \mathrm{LS}[0, 0, 0, 0, \ldots],\ \overline{z} \to \mathrm{LS}[\overline{x} + \overline{y}, 0, 0, 0, \ldots]\rangle,$$
$$\mathrm{CWS}[0, 0, 0, 0, \ldots]]$$

**$V_0$ ** (Rs[x, z] // dΔ[x, x, y])**

$$\mathrm{Es}\Big[\Big\langle \overline{x} \to \mathrm{LS}\big[\overline{x}\,\alpha s[x] + \overline{y}\,\alpha s[y],\ \overline{x\,y}\,\alpha s[x, y],\ \overline{x\,\overline{x\,y}}\,\alpha s[x, x, y] + \overline{\overline{x\,y}\,y}\,\alpha s[x, y, y],$$
$$\overline{x\,\overline{x\,\overline{x\,y}}}\,\alpha s[x, x, x, y] + \overline{x\,\overline{\overline{x\,y}\,y}}\,\alpha s[x, x, y, y] + \overline{\overline{\overline{x\,y}\,y}\,y}\,\alpha s[x, y, y, y],\ \ldots\big],$$
$$\overline{y} \to \mathrm{LS}\big[\overline{x}\,\beta s[x] + \overline{y}\,\beta s[y],\ \overline{x\,y}\,\beta s[x, y],\ \overline{x\,\overline{x\,y}}\,\beta s[x, x, y] + \overline{\overline{x\,y}\,y}\,\beta s[x, y, y],$$
$$\overline{x\,\overline{x\,\overline{x\,y}}}\,\beta s[x, x, x, y] + \overline{x\,\overline{\overline{x\,y}\,y}}\,\beta s[x, x, y, y] + \overline{\overline{\overline{x\,y}\,y}\,y}\,\beta s[x, y, y, y],\ \ldots\big],$$
$$\overline{z} \to \mathrm{LS}[\overline{x} + \overline{y}, 0, 0, 0, \ldots]\Big\rangle,\ \mathrm{CWS}[\widehat{x}\,\gamma s[x] + \widehat{y}\,\gamma s[y],$$
$$\widehat{xx}\,\gamma s[x, x] + \widehat{xy}\,\gamma s[x, y] + \widehat{yy}\,\gamma s[y, y],$$
$$\widehat{xxx}\,\gamma s[x, x, x] + \widehat{xxy}\,\gamma s[x, x, y] + \widehat{xyy}\,\gamma s[x, y, y] + \widehat{yyy}\,\gamma s[y, y, y],$$
$$\widehat{xxxx}\,\gamma s[x, x, x, x] + \widehat{xxxy}\,\gamma s[x, x, x, y] + \widehat{xxyy}\,\gamma s[x, x, y, y] +$$
$$\widehat{xyxy}\,\gamma s[x, y, x, y] + \widehat{xyyy}\,\gamma s[x, y, y, y] + \widehat{yyyy}\,\gamma s[y, y, y, y],\ \ldots]\Big]$$

```
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
```

```
R4Eqn = V0 ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V0 ;
UnitarityEqn = V0 ** (V0 // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]];
CapEqn = (V0 ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]);
```

**R4Eqn**

$$\mathrm{BS}\Big[2\,\mathrm{True},\ 0 == -\frac{\overline{x\,y}}{2} - \overline{x\,y}\,\alpha s[y] + \overline{x\,y}\,\beta s[x],\ 0 == -\frac{\overline{x\,y}}{2} - \overline{x\,y}\,\alpha s[y] + \overline{x\,y}\,\beta s[x]\ \&\&$$

$$0 == \frac{1}{12}\,\overline{x\,\overline{x\,y}} + \frac{1}{12}\,\overline{\overline{x\,y}\,y} + \frac{1}{2}\,\overline{\overline{x\,y}\,y}\,\alpha s[y] - \frac{1}{2}\,\overline{x\,\overline{x\,y}}\,\alpha s[x]\,\alpha s[y] + \frac{1}{2}\,\overline{\overline{x\,y}\,y}\,\alpha s[y]^2 -$$

$$\overline{x\,\overline{x\,y}}\,\alpha s[x,y] - \frac{1}{2}\,\overline{x\,\overline{x\,y}}\,\beta s[x] + \frac{1}{2}\,\overline{x\,\overline{x\,y}}\,\beta s[x]^2 - \frac{1}{2}\,\overline{\overline{x\,y}\,y}\,\beta s[x]\,\beta s[y] + \overline{\overline{x\,y}\,y}\,\beta s[x,y],$$

$$0 == -\frac{\overline{x\,y}}{2} - \overline{x\,y}\,\alpha s[y] + \overline{x\,y}\,\beta s[x]\ \&\&\ 0 == \frac{1}{12}\,\overline{x\,\overline{x\,y}} + \frac{1}{12}\,\overline{\overline{x\,y}\,y} + \frac{1}{2}\,\overline{\overline{x\,y}\,y}\,\alpha s[y] -$$

$$\frac{1}{2}\,\overline{x\,\overline{x\,y}}\,\alpha s[x]\,\alpha s[y] + \frac{1}{2}\,\overline{\overline{x\,y}\,y}\,\alpha s[y]^2 - \overline{x\,\overline{x\,y}}\,\alpha s[x,y] - \frac{1}{2}\,\overline{x\,\overline{x\,y}}\,\beta s[x] +$$

$$\frac{1}{2}\,\overline{x\,\overline{x\,y}}\,\beta s[x]^2 - \frac{1}{2}\,\overline{\overline{x\,y}\,y}\,\beta s[x]\,\beta s[y] + \overline{\overline{x\,y}\,y}\,\beta s[x,y]\ \&\&$$

$$0 == -\frac{1}{24}\,\overline{x\,\overline{\overline{x\,y}\,y}} - \frac{1}{12}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\alpha s[y] - \frac{1}{12}\,\overline{\overline{\overline{x\,y}\,y}\,y}\,\alpha s[y] + \frac{1}{4}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\alpha s[x]\,\alpha s[y] -$$

$$\frac{1}{6}\,\overline{x\,\overline{x\,\overline{x\,y}}}\,\alpha s[x]^2\,\alpha s[y] - \frac{1}{4}\,\overline{\overline{\overline{x\,y}\,y}\,y}\,\alpha s[y]^2 + \frac{1}{3}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\alpha s[x]\,\alpha s[y]^2 - \frac{1}{6}\,\overline{\overline{\overline{x\,y}\,y}\,y}\,\alpha s[y]^3 +$$

$$\frac{1}{2}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\alpha s[x,y] - \frac{1}{2}\,\overline{x\,\overline{x\,\overline{x\,y}}}\,\alpha s[x]\,\alpha s[x,y] + \frac{1}{2}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\alpha s[y]\,\alpha s[x,y] -$$

$$\overline{x\,\overline{x\,\overline{x\,y}}}\,\alpha s[x,x,y] - \overline{x\,\overline{\overline{x\,y}\,y}}\,\alpha s[x,y,y] + \frac{1}{12}\,\overline{x\,\overline{x\,\overline{x\,y}}}\,\beta s[x] + \frac{1}{12}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\beta s[x] -$$

$$\frac{1}{4}\,\overline{x\,\overline{x\,\overline{x\,y}}}\,\beta s[x]^2 + \frac{1}{6}\,\overline{x\,\overline{x\,\overline{x\,y}}}\,\beta s[x]^3 + \frac{1}{4}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\beta s[x]\,\beta s[y] - \frac{1}{3}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\beta s[x]^2\,\beta s[y] +$$

$$\frac{1}{6}\,\overline{\overline{\overline{x\,y}\,y}\,y}\,\beta s[x]\,\beta s[y]^2 - \frac{1}{2}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\beta s[x,y] + \frac{1}{2}\,\overline{x\,\overline{\overline{x\,y}\,y}}\,\beta s[x]\,\beta s[x,y] -$$

$$\frac{1}{2}\,\overline{\overline{\overline{x\,y}\,y}\,y}\,\beta s[y]\,\beta s[x,y] + \overline{x\,\overline{\overline{x\,y}\,y}}\,\beta s[x,x,y] + \overline{\overline{\overline{x\,y}\,y}\,y}\,\beta s[x,y,y],\ \ldots\Big]$$