Pensieve header: Exp relative to am, bm, cm, dm.

Follows code in Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Startup

In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
(*Once[<< KnotTheory`];*)
Once[<< "../Profile/Profile.m";];
<< "Engine-Speedy.m";
<< "Objects.m";
$k = 3;
HL[ℰ_] := Style[ℰ, Background → Green];
```

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: April 2020. Original version: July 1994.
```

» Warning: On Sep 4 2019 I swapped the operations
$\in$ and η. Some incompatibilities may arise in older notebooks.

Exponentials as needed.

Task. Define $\mathrm{Exp}_{m,i,k}[P]$ to compute $e^{\mathbb{O}(P)}$ to ϵ^k in the using the $\mathrm{mm}_{i,i\to i}$ multiplication, where P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\lambda\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\lambda P_0}\,F(\lambda)\right)$, then $F(\lambda=0)=1$ and we have:

$\mathbb{O}\left(e^{\lambda P_0}(P_0\,F(\lambda)+\partial_\lambda F)\right) = \mathbb{O}\left(\partial_\lambda e^{\lambda P_0}\,F(\lambda)\right) = \partial_\lambda\,\mathbb{O}\left(e^{\lambda P_0}\,F(\lambda)\right) = \partial_\lambda e^{\lambda\,\mathbb{O}(P)} = e^{\lambda\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\lambda P_0}\,F(\lambda)\right)\mathbb{O}(P)$.

This is a linear ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k\,\varphi$ we find that $F_0 = 1$ and solve for φ.

In[•]:=
```
(* Bug: The first line is valid only if 𝕆[e^P0]==e^𝕆[P0]. *)
Exp_mm_,i_,0[P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[LQ /. (x | y)_i → 0, LQ /. (b | a | t)_i → 0, 1]];
Exp_mm_,i_,k_[P_] := Block[{$k = k},
  Module[{P0, λ, φ, F, j, rhs, err = 0, pows, at0, atλ},
   P0 = Normal@P /. ϵ → 0;
   F = Normal@Last@Exp_mm,i,k-1[λ P];
   (*Unary*)While[
    If[err =!= 0,
     pows = Echo[First /@ CoefficientRules[err, {y_i, b_i, a_i, x_i}]];
     F += Sum[ϵ^k φ_js[λ] Times @@ {y_i, b_i, a_i, x_i}^js, {js, pows}];
     rhs = Normal@Last@mm_i,j→i[
         𝔼_{}→{i}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F]_k sσ_i→j@𝔼_{}→{i}[0, 0, P]_k];
     err = CF[(∂_λ F) + P0 F - rhs];
     at0 = Table[φ_js[0] == 0, {js, pows}];
     atλ = (# == 0) & /@ (pows /. CoefficientRules[err, {y_i, b_i, a_i, x_i}]);
     F = F /. DSolve[And @@ (at0 ⋃ atλ), Table[φ_js[λ], {js, pows}], λ][[1]]
    ];
    rhs = Normal@Last@mm_i,j→i[
        𝔼_{}→{i}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0, F]_k sσ_i→j@𝔼_{}→{i}[0, 0, P]_k];
    err = CF[(∂_λ F) + P0 F - rhs];
    err =!= 0
   ];
   𝔼_{}→{i}[P0 /. (x | y)_i → 0, P0 /. (b | a | t)_i → 0, F + O[ϵ]^(k+1) /. λ → 1]]
 ]
```

In[•]:= `Exp_dm,1,2[ξ (x_1 + ϵ y_1)]`

» `{{1, 0, 0, 0}, {0, 0, 0, 0}}`

» `{{2, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}}`

Out[•]=
$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\ \xi\,x_1,\ 1+\left(-\frac{\xi^2\left(-1+B_1\right)}{2\,\hbar}+\xi\,y_1\right)\epsilon+\left(\frac{\xi^4\left(-1+B_1\right)^2}{8\,\hbar^2}+\frac{1}{2}\,\xi^2\,a_1\,B_1-\frac{1}{6}\,\gamma\,\xi^3\left(-1+3\,B_1\right)x_1-\frac{\xi^3\left(-1+B_1\right)y_1}{2\,\hbar}+\frac{1}{2}\,\gamma\,\xi^2\,\hbar\,x_1\,y_1+\frac{1}{2}\,\xi^2\,y_1^2\right)\epsilon^2+O[\epsilon]^3\right]$$

In[•]:= `dS_1[𝔼_{}→{1}[0, 0, #]] & /@ {y_1, x_1}`

Out[•]=
$$\left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -\frac{y_1}{B_1}+\frac{\gamma\,\hbar\,y_1\,\epsilon}{B_1}-\frac{\left(\gamma^2\,\hbar^2\,y_1\right)\epsilon^2}{2\,B_1}+\frac{\gamma^3\,\hbar^3\,y_1\,\epsilon^3}{6\,B_1}+O[\epsilon]^4\right],\right.$$
$$\left.\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1-\hbar\,a_1\,x_1\,\epsilon-\frac{1}{2}\left(\hbar^2\,a_1^2\,x_1\right)\epsilon^2-\frac{1}{6}\left(\hbar^3\,a_1^3\,x_1\right)\epsilon^3+O[\epsilon]^4\right]\right\}$$

In[•]:=
```
Timing@{lhs = 𝔼_{1}→{1}[0, ξ_1 x_1, 1] // dS_1,
  rhs = Exp_dm,1,$k[ξ_1 Last@dS_1[𝔼_{}→{1}[0, 0, x_1]]] /. {{} → {1}}; HL[lhs ≡ rhs]}
```

```
» {{0, 0, 1, 1}, {0, 0, 0, 2}}

» {{0, 0, 2, 2}, {0, 0, 2, 1}, {0, 0, 1, 3}, {0, 0, 1, 2}, {0, 0, 0, 4}, {0, 0, 0, 3}, {0, 0, 0, 2}}

» {{0, 0, 3, 3}, {0, 0, 3, 2}, {0, 0, 3, 1}, {0, 0, 2, 4}, {0, 0, 2, 3},
   {0, 0, 2, 2}, {0, 0, 1, 5}, {0, 0, 1, 4}, {0, 0, 1, 3}, {0, 0, 1, 2},
   {0, 0, 0, 6}, {0, 0, 0, 5}, {0, 0, 0, 4}, {0, 0, 0, 3}, {0, 0, 0, 2}}
```

Out[•]=

$$\Big\{3.53125,\ \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0,\ -x_1\,\xi_1,\ 1+\Big(-\hbar\,a_1\,x_1\,\xi_1-\frac{1}{2}\,\gamma\,\hbar\,x_1^2\,\xi_1^2\Big)\,\epsilon+\Big(-\frac{1}{2}\,\hbar^2\,a_1^2\,x_1\,\xi_1+\frac{1}{4}\,\gamma^2\,\hbar^2\,x_1^2\,\xi_1^2-\gamma\,\hbar^2\,a_1\,x_1^2\,\xi_1^2+\frac{1}{2}\,\hbar^2\,a_1^2\,x_1^2\,\xi_1^2-\frac{1}{2}\,\gamma^2\,\hbar^2\,x_1^3\,\xi_1^3+\frac{1}{2}\,\gamma\,\hbar^2\,a_1\,x_1^3\,\xi_1^3+\frac{1}{8}\,\gamma^2\,\hbar^2\,x_1^4\,\xi_1^4\Big)\,\epsilon^2+\Big(-\frac{1}{6}\,\hbar^3\,a_1^3\,x_1\,\xi_1-\frac{1}{12}\,\gamma^3\,\hbar^3\,x_1^2\,\xi_1^2+\frac{1}{2}\,\gamma^2\,\hbar^3\,a_1\,x_1^2\,\xi_1^2-\gamma\,\hbar^3\,a_1^2\,x_1^2\,\xi_1^2+\frac{1}{2}\,\hbar^3\,a_1^3\,x_1^2\,\xi_1^2+\frac{2}{3}\,\gamma^3\,\hbar^3\,x_1^3\,\xi_1^3-\frac{7}{4}\,\gamma^2\,\hbar^3\,a_1\,x_1^3\,\xi_1^3+\frac{5}{4}\,\gamma\,\hbar^3\,a_1^2\,x_1^3\,\xi_1^3-\frac{1}{6}\,\hbar^3\,a_1^3\,x_1^3\,\xi_1^3-\frac{19}{24}\,\gamma^3\,\hbar^3\,x_1^4\,\xi_1^4+\gamma^2\,\hbar^3\,a_1\,x_1^4\,\xi_1^4-\frac{1}{4}\,\gamma\,\hbar^3\,a_1^2\,x_1^4\,\xi_1^4+\frac{1}{4}\,\gamma^3\,\hbar^3\,x_1^5\,\xi_1^5-\frac{1}{8}\,\gamma^2\,\hbar^3\,a_1\,x_1^5\,\xi_1^5-\frac{1}{48}\,\gamma^3\,\hbar^3\,x_1^6\,\xi_1^6\Big)\,\epsilon^3+O[\epsilon]^4\Big],\ \text{True}\Big\}\Big\}$$

In[•]:=
```
Timing@{lhs = 𝔼{1}→{1}[0, η1 y1, 1] // dS1,
  rhs = Exp_{dm,1,$k}[η Last@dS1[𝔼{}→{1}[0, 0, y1]]] /. {η → η1, {} → {1}}, HL[lhs ≡ rhs]}
```

```
» {{2, 0, 0, 0}, {1, 0, 0, 0}}

» {{4, 0, 0, 0}, {3, 0, 0, 0}, {2, 0, 0, 0}, {1, 0, 0, 0}}

» {{6, 0, 0, 0}, {5, 0, 0, 0}, {4, 0, 0, 0}, {3, 0, 0, 0}, {2, 0, 0, 0}, {1, 0, 0, 0}}
```

Out[•]=

$$\Big\{3.29688,\ \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0,\ -\frac{y_1\,\eta_1}{B_1},\ 1+\Big(\frac{\gamma\,\hbar\,y_1\,\eta_1}{B_1}-\frac{\gamma\,\hbar\,y_1^2\,\eta_1^2}{2\,B_1^2}\Big)\,\epsilon+\Big(-\frac{\gamma^2\,\hbar^2\,y_1\,\eta_1}{2\,B_1}+\frac{7\,\gamma^2\,\hbar^2\,y_1^2\,\eta_1^2}{4\,B_1^2}-\frac{\gamma^2\,\hbar^2\,y_1^3\,\eta_1^3}{B_1^3}+\frac{\gamma^2\,\hbar^2\,y_1^4\,\eta_1^4}{8\,B_1^4}\Big)\,\epsilon^2+\Big(\frac{\gamma^3\,\hbar^3\,y_1\,\eta_1}{6\,B_1}-\frac{25\,\gamma^3\,\hbar^3\,y_1^2\,\eta_1^2}{12\,B_1^2}+\frac{23\,\gamma^3\,\hbar^3\,y_1^3\,\eta_1^3}{6\,B_1^3}-\frac{49\,\gamma^3\,\hbar^3\,y_1^4\,\eta_1^4}{24\,B_1^4}+\frac{3\,\gamma^3\,\hbar^3\,y_1^5\,\eta_1^5}{8\,B_1^5}-\frac{\gamma^3\,\hbar^3\,y_1^6\,\eta_1^6}{48\,B_1^6}\Big)\,\epsilon^3+O[\epsilon]^4\Big],\ \mathbb{E}_{\{1\}\to\{1\}}\Big[0,\ -\frac{y_1\,\eta_1}{B_1},\ 1-\frac{\left(\gamma\,\hbar\,\left(-2\,B_1\,y_1\,\eta_1+y_1^2\,\eta_1^2\right)\right)\,\epsilon}{2\,B_1^2}+\frac{\gamma^2\,\hbar^2\,\left(-4\,B_1^3\,y_1\,\eta_1+14\,B_1^2\,y_1^2\,\eta_1^2-8\,B_1\,y_1^3\,\eta_1^3+y_1^4\,\eta_1^4\right)\,\epsilon^2}{8\,B_1^4}-\frac{\left(\gamma^3\,\hbar^3\,\left(-8\,B_1^5\,y_1\,\eta_1+100\,B_1^4\,y_1^2\,\eta_1^2-184\,B_1^3\,y_1^3\,\eta_1^3+98\,B_1^2\,y_1^4\,\eta_1^4-18\,B_1\,y_1^5\,\eta_1^5+y_1^6\,\eta_1^6\right)\right)\,\epsilon^3}{48\,B_1^6}+O[\epsilon]^4\Big],\ \text{True}\Big\}\Big\}$$

In[•]:=
```
Timing@{lhs = 𝔼{1}→{1}[0, ξ1 x1, 1] // cS1,
  rhs = Exp_{cm,1,$k}[ξ1 Last@cS1[𝔼{}→{1}[0, 0, x1]]] /. {} → {1}; HL[lhs ≡ rhs]}
```

Out[•]=

$$\Big\{3.67188,\ \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0,\ -x_1\,\xi_1,\ 1+O[\epsilon]^4\Big],\ \text{True}\Big\}\Big\}$$

In[•]:=
```
Timing@{lhs = 𝔼{1}→{1}[0, η1 y1, 1] // cS1,
  rhs = Exp_{cm,1,$k}[η1 Last@cS1[𝔼{}→{1}[0, 0, y1]]] /. {} → {1}; HL[lhs ≡ rhs]}
```

Out[•]=

$$\Big\{0.296875,\ \Big\{\mathbb{E}_{\{1\}\to\{1\}}\Big[0,\ -y_1\,\eta_1,\ 1+O[\epsilon]^4\Big],\ \text{True}\Big\}\Big\}$$