Pensieve header: A unified verification notebook for the PPSA project.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\PPSA"];
NotebookOpen[wdir <> "\\MakeVSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is ℏ, and all “coupling constants” are proportional to it.

TD

```
$TℏD = 3; $TϵD = 2; ϵ /: ϵ^d_· /; d > $TϵD := 0;
(* $TϵD can't be ∞ at least because of Quesne. Can't be ≤
 1 at least because of the explicit ϵ² in S𝔻$g. *)
SetAttributes[{SS, SST}, HoldAll];
SS[ℰ_] := Block[{ℏ, ϵ}, (* Shielded Series *)
   Collect[Normal@Series[ℰ, {ℏ, 0, $TℏD}], ℏ, Together]];
SST[ℰ_] := Block[{ℏ, ϵ},
   Collect[Normal@Series[ℰ /. {T_i_ → ⅇ^(ℏ t_i/2), T → ⅇ^(ℏ t/2)}, {ℏ, 0, $TℏD}], ℏ, Together]];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op]; Simp[ℰ_] := Simp[ℰ, SS];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST];
```

Differential polynomials (DP):

Utils

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :→ c D[λ, {x, m}, {y, n}]]
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, cp, CE, pow,
    gs = Generators /. {opts}, cs = Centrals /. {opts}
  },
  gp = Alternatives @@ gs;
  gp = gp | gp_; (* generators pattern *)
  sr = Thread[gs → Range@Length@gs]; (* sorting rule *)
  cp = Alternatives @@ cs; (* centrals pattern *)
  CE[ℰ_] := Collect[ℰ, _U, (Expand[#] /. ħ^d_ /; d > $TħD ⧴ 0) &];
  U_{i_}[ℰ_] := ℰ /. {t : cp ⧴ t_i, u_U ⧴ Replace[u, x_ ⧴ x_i, 1]};
  U_{i_}[NCM[]] := U[];
  B[U@(x_)_{i_}, U@(y_)_{i_}] := B[U@x_i, U@y_i] = U_i@B[U@x, U@y];
  B[U@(x_)_{i_}, U@(y_)_{j_}] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** U[] := x; U[] ** x_ := x;
  (a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, CE[a b (x ** y)]];
  (a_ * x_U) ** y_ := CE[a (x ** y)]; x_ ** (a_ * y_U) := CE[a (x ** y)];
  U[xx___, x_] ** U[y_, yy___] := If[OrderedQ[{x, y} /. sr],
    U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{} = U[];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[poly_, specs___] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List ⧴ l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_{s_} ⧴ (l /. x_{i_} ⧴ x_s));
    CE[Total[
       CoefficientRules[poly, vs] /. (p_ → c_) ⧴ c U@(us^p)
      ]] /. x_null ⧴ x
    ];
  pow[ℰ_, 0] = U[]; pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
  𝕊_U[ℰ_, ss___Rule] := CE@Total[
     CoefficientRules[ℰ, First /@ {ss}] /.
      (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
  S_{i_}[c_. * u_U] := CE[(c /. S_i[U, Centrals]) DeleteCases[u, _i] **
     U_i[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]]];
 ]
```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, (g_ → img_) ⧴ (m[U[g]] = img), {1}];
  m[U[]] = V[];
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /.  u_U ⧴ m[u]];)
```

Meta-Operations

QLImplementation

```
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing $sl_2^{\gamma\epsilon}$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[CU@a, CU@y] = -γ CU@y; B[CU@x, CU@a] = -γ CU@x;
B[CU@x, CU@y] = 2 ϵ CU@a - t CU[];
(S@CU@y = -CU@y; S@CU@a = -CU@a; S@CU@x = -CU@x;)
S_i_[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

$\{0.703125, \{4 (7 t^2 \gamma^4 + 29 t \gamma^5 \epsilon + 30 \gamma^6 \epsilon^2)$ CU[y, y, y, x, x] +
≪30≫ + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}

Verifying that *S* is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y_1, a_1, x_1}},
 Table[HL@Simp[S_1[z1 ** z2] - S_1[z2] ** S_1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y1, a1, x1}},
 Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Implementing $\mathcal{U}_{\gamma\epsilon;\hbar}$

With $q = e^{\hbar\gamma\epsilon}$, $A = e^{-\hbar\epsilon a}$, $T = e^{\hbar t/2}$, and $[f, g]_q := fg - qgf$, our algebra is $\mathcal{U}_{\gamma\epsilon;\hbar} = \langle t, y, a, x\rangle/\mathcal{R}$, where $\mathcal{R} = \left([t, *] = 0,\ [a, y] = -\gamma y,\ [x, y]_q = \left(1 - T^2 A^2\right)/\hbar,\ [x, a] = -\gamma x\right)$.

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
q = SS[e^(γ ϵ ħ)]; (*T=SS[e^(ħ t/2)];*)
B[QU@a, QU@y] = -γ QU@y; B[QU@x, QU@a] = -γ QU@x;
B[QU@x, QU@y] = (q - 1) QU@{y, x} + 𝕆QU[SS[(1 - T^2 e^(-2 ϵ a ħ)) / ħ], {a}];
(S@QU@y = 𝕆QU[SS[-T^-2 e^(ħ ϵ a) y], {a, y}];
 S@QU@a = -QU@a; S@QU@x = 𝕆QU[SS[-e^(ħ ϵ a) x], {a, x}];)
Si_[QU, Centrals] = {ti → -ti, Ti → Ti^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$$\left\{\left\{\{QU[y], QU[y]\} \to 0,\ \{QU[y], QU[a]\} \to \gamma\, QU[y],\ \{QU[y], QU[x]\} \to \frac{\left(-1 + T^2\right) QU[\,]}{\hbar} - 2\,T^2 \epsilon\, QU[a] + 2\,T^2 \epsilon^2 \hbar\, QU[a, a] + \left(-\gamma \epsilon \hbar - \frac{1}{2}\gamma^2 \epsilon^2 \hbar^2\right) QU[y, x]\right\},\right.$$
$$\left\{\{QU[a], QU[y]\} \to -\gamma\, QU[y],\ \{QU[a], QU[a]\} \to 0,\ \{QU[a], QU[x]\} \to \gamma\, QU[x]\right\},$$
$$\left\{\{QU[x], QU[y]\} \to \frac{\left(1 - T^2\right) QU[\,]}{\hbar} + 2\,T^2 \epsilon\, QU[a] - 2\,T^2 \epsilon^2 \hbar\, QU[a, a] + \left(\gamma \epsilon \hbar + \frac{1}{2}\gamma^2 \epsilon^2 \hbar^2\right) QU[y, x],\right.$$
$$\left.\left.\{QU[x], QU[a]\} \to -\gamma\, QU[x],\ \{QU[x], QU[x]\} \to 0\right\}\right\}$$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ $TħD=5, $TϵD=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp]
  }] // Timing
```

```
{31.7813, {<<1>>, 0}}
```

Verifying that S is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

$\{\{\{QU[y_1], QU[y_1]\} \to 0, \{QU[y_1], QU[a_1]\} \to 0, \{QU[y_1], QU[x_1]\} \to 0\},$
$\{\{QU[a_1], QU[y_1]\} \to 0, \{QU[a_1], QU[a_1]\} \to 0, \{QU[a_1], QU[x_1]\} \to 0\},$
$\{\{QU[x_1], QU[y_1]\} \to 0, \{QU[x_1], QU[a_1]\} \to 0, \{QU[x_1], QU[x_1]\} \to 0\}\}$

Verifying that $\text{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ T\hbar$D=5, T\epsilon$D=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. {QU → CU, T → e^(ℏ t/2)}, ℏ → 0] - lhs] // HL
  }] // Timing
```

$\{32.7969,$
$\{2(8t^2\gamma^4 + 16t\gamma^5\epsilon)\,CU[y, y, y, x, x] + (8t\gamma^5\epsilon + 16\gamma^6\epsilon^2)\,CU[y, y, y, x, x] + \ll 106 \gg +$
$CU[y, y, y, y, y, a, a, a, a, x, x, x, x], (-8T^2\gamma^6\epsilon^2 + 8T^4\gamma^6\epsilon^2)\,QU[y, y, y, x, x] + \ll 489 \gg +$
$\left(\gamma\epsilon\hbar + \frac{15}{2}\gamma^2\epsilon^2\hbar^2\right) QU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0\}\}$

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -CU@x, a → -CU@a, x → -CU@y}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y → 𝕆QU[SS[-T^-1 e^(ℏ ϵ a) x], {a, x}],
   a → -QU@a, x → 𝕆QU[SS[-T^-1 e^(ℏ ϵ a) y], {a, y}]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

$\{CU[y] \to -CU[x] \to CU[y], CU[a] \to -CU[a] \to CU[a], CU[x] \to -CU[y] \to CU[x]\}$

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

$\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\}$

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Qθ[Qθ[z]]], {z, bas}] ]
```

$\left\{QU[y] \to -\frac{QU[x]}{T} - \frac{\epsilon\hbar\,QU[a, x]}{T} - \frac{\epsilon^2\hbar^2\,QU[a, a, x]}{2T} \to QU[y], QU[a] \to -QU[a] \to QU[a],\right.$
$\left.QU[x] \to \left(-\frac{1}{T} + \frac{\gamma\epsilon\hbar}{T} - \frac{\gamma^2\epsilon^2\hbar^2}{2T}\right) QU[y] + \left(-\frac{\epsilon\hbar}{T} + \frac{\gamma\epsilon^2\hbar^2}{T}\right) QU[y, a] - \frac{\epsilon^2\hbar^2\,QU[y, a, a]}{2T} \to QU[x]\right\}$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

$$\left\{\text{Short}[tt = QU[y] ** Q\theta[QU[y]] - Q\theta[QU[y]] ** QU[y]],\ tt\ /.\ t \to \frac{t}{\hbar}\right\}$$

$$\Big\{-\left(-\frac{1}{T\hbar}+\frac{T}{\hbar}\right)QU[\,] + \ll 18 \gg,$$
$$-\left(-\frac{1}{T\hbar}+\frac{T}{\hbar}\right)QU[\,] + 2\,T\,\epsilon\,QU[a] - \left(-\frac{\epsilon}{T}+T\,\epsilon\right)QU[a] - \left(-\frac{\epsilon^2\hbar}{2\,T}+\frac{1}{2}T\,\epsilon^2\hbar\right)QU[a,a] - \frac{QU[y,x]}{T} +$$
$$\frac{\gamma^2\epsilon^2\hbar^2\,QU[y,x]}{2\,T} - \left(-\frac{1}{T}-\frac{\gamma\,\epsilon\,\hbar}{T}-\frac{\gamma^2\epsilon^2\hbar^2}{2\,T}\right)QU[y,x] - \left(\frac{\gamma\,\epsilon\,\hbar}{T}+\frac{\gamma^2\epsilon^2\hbar^2}{T}\right)QU[y,x] -$$
$$\frac{\epsilon\,\hbar\,QU[y,a,x]}{T} - \frac{\gamma\,\epsilon^2\hbar^2\,QU[y,a,x]}{T} - \left(-\frac{\epsilon\,\hbar}{T}-\frac{\gamma\,\epsilon^2\hbar^2}{T}\right)QU[y,a,x]\Big\}$$

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\mathbb{AD}\$f = \frac{\gamma}{\hbar}\,e^{\hbar\left(\frac{t}{2}-(a+\gamma)\,\epsilon\right)}\left(\left(\text{Cosh}\left[\hbar\left(a\,\epsilon+\frac{\gamma\,\epsilon}{2}-\frac{t}{2}\right)\right] - \text{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\,\epsilon}{2}\right)^2+\epsilon\,\omega}\,\right]\right)\Big/\left(\text{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(a^2\,\epsilon+a\,\gamma\,\epsilon-a\,t-\omega\right)\right)\right);$$

Scaling behaviour of AD$f:

```
HL@Simplify[AD$f == ((AD$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
```

```
True
```

```
HL@FullSimplify[
  AD$f == ((AD$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

```
True
```

ADeq

```
AD$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
DeclareMorphism[AD, QU → CU,
 {a → CU@a, x → CU@x, y → S_CU[SS[AD$f], a → CU[a], ω → AD$ω] ** CU@y}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

```
S𝔻$g = Sqrt[((Cosh[ℏ/2 Sqrt[t^2 + γ^2 ϵ^2 + 4 ϵ ϖ]] - Cosh[ℏ/2 (t - (2 a + γ) ϵ)]) /
    (Sinh[γ ϵ ℏ/2] (t (2 a + γ) - 2 a (a + γ) ϵ + 2 ϖ) ℏ / (2 γ)))];
```

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{S𝔻$P = (Cosh[ℏ ((ϵ-t)/2 + ϵ a)] - Cosh[ℏ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]]) /
    (ℏ Sinh[-ϵ ℏ/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
 Simplify[S𝔻$P == (S𝔻$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(S𝔻$P /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    S𝔻$g (S𝔻$g /. {a → -a - γ, t → -t})] // HL,
 S𝔻$Q = Simplify[S𝔻$P /. {a → c - 1/2}],
 Simplify[S𝔻$Q == (S𝔻$Q /. {c → -c, t → -t})] // HL,
 Simplify[S𝔻$g == FullSimplify[
    Sqrt[S𝔻$Q] /. c → a + 1/2 /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

```
{-(((Cosh[(a ϵ + 1/2 (-t + ϵ)) ℏ] - Cosh[Sqrt[1/4 (t^2 + ϵ^2) + ϵ ϖ] ℏ]) Csch[ϵ ℏ/2]) /
    ((t/2 + a (t - ϵ) - a^2 ϵ + ϖ) ℏ)), True, True,
 -((4 (Cosh[1/2 (t - 2 c ϵ) ℏ] - Cosh[1/2 Sqrt[t^2 + ϵ^2 + 4 ϵ ϖ] ℏ]) Csch[ϵ ℏ/2]) /
    ((4 c t + ϵ - 4 c^2 ϵ + 4 ϖ) ℏ)), True, True}
```

SDeq

```
S𝔻$f = FullSimplify[ⅇ^(ℏ (t/2-ϵ a)) (S𝔻$g /. {a → -a, t → -t})];
```

SDeq

```
S𝔻$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ CU[] / 2;
```

SDeq

```
DeclareMorphism[S𝔻, QU → CU, {a → CU@a,
  x → 𝕊CU[SS[S𝔻$f], a → CU[a], ϖ → S𝔻$ϖ] ** CU@x,
  y → 𝕊CU[SS[S𝔻$g] , a → CU[a], ϖ → S𝔻$ϖ] ** CU@y
 }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[S𝔻[z]] == S𝔻[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

```
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[S𝔻[z1 ** z2] - S𝔻[z1] ** S𝔻[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

R in QU.

Quesne's formula:

Quesne

$$\mathbb{e}_{q\_,n\_}[x\_] := \mathrm{Exp}\left[\sum_{k=1}^{n} \frac{(1-q)^k}{k\,(1-q^k)} x^k\right];\ \mathbb{e}_{q\_}[x\_] := \mathbb{e}_{q,\$T\epsilon D}[x]$$

```
Table[Together@SeriesCoefficient[𝕖ρ,5[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1,\ 1,\ \frac{1}{1+\rho},\ \frac{1}{(1+\rho)\,(1+\rho+\rho^2)},\ \frac{1}{(1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)},\ 1\Big/\left((1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)\,(1+\rho+\rho^2+\rho^3+\rho^4)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, ρ] SeriesCoefficient[𝕖ρ,5[x], {x, 0, n}]], {n, 0, 5}]
```

```
{1, 1, 1, 1, 1, 1}
```

R

$$\mathrm{QU}[R_{i\_,j\_}] := \mathbb{O}_{QU}\left[SS\left[\mathbb{e}^{\hbar b_1 a_2}\, \mathbb{e}_q[\hbar\, y_1 x_2] \text{ /. } b_1 \to \gamma^{-1}\,(\epsilon\, a_1 - t_i)\right],\ \{y_1, a_1\}_i,\ \{a_2, x_2\}_j\right];$$
$$\mathrm{QU}\left[R^{-1}_{i\_,j\_}\right] := S_j\text{@}\mathrm{QU}[R_{i,j}];$$

```
QU[R3,4] // Short
```

$$\mathrm{QU}[\,] + \frac{\epsilon\,\hbar\,\mathrm{QU}[a_3, a_4]}{\gamma} + \hbar\,\mathrm{QU}[y_3, x_4] + \ll 20 \gg + \frac{\hbar^3\,\mathrm{QU}[y_3, a_4, a_4, x_4]\,t_3^2}{2\,\gamma^2} - \frac{\hbar^3\,\mathrm{QU}[a_4, a_4, a_4]\,t_3^3}{6\,\gamma^3}$$

Verifying R2 (~2 secs @ T\hbar$D=4, T\epsilon$D=2):

$$\mathrm{QU}\left[R_{1,2} ** R^{-1}_{1,2}\right] \text{ // Simp // HL // Timing}$$

```
{0.6875, QU[]}
```

Verifying R3 (~156 secs @ $Tħ$D=4, $Tϵ$D=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$\{12.0156,$

$\left\{\mathrm{QU}[\,] + \frac{\epsilon\hbar\,\mathrm{QU}[a_1, a_2]}{\gamma} + \ll 455 \gg + \mathrm{QU}[y_1, y_1, y_1, x_3, x_3, x_3]\left(\frac{\hbar^3}{6} - \frac{1}{2}\hbar^3 T_2^2 + \frac{1}{2}\hbar^3 T_2^4 - \frac{1}{6}\hbar^3 T_2^6\right), 0\right\}\}$

The representation ρ

rho

```
{ρ@(CU | QU)@y, ρ@(CU | QU)@a} = {(0 0; ϵ 0), (γ 0; 0 0)};
ρ@CU@x = (0 γ; 0 0); ρ@QU@x = SS@(0  (1 - e^(-γ ϵ ħ)) / (ϵ ħ); 0  0);
ρ[e^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] :=
 (ℰ /. {t → γ ϵ, T → e^(ħ γ ϵ/2), (U : CU | QU)[u___] :→ Dot[(1 0; 0 1), Sequence @@ (ρ /@ U /@ {u})]})
```

Verifying that ρ represents CU and QU:

```
Table[ρ[z1 ** z2] == ρ[z1].ρ[z2] // SS // HL,
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

The Classical Logos $C\Lambda$

Lemma 3C. To degree k,
$\mathbb{O}_{\mathrm{CU}}\left(e^{\eta y+\xi x+\delta y x} \mid x y\right) = \mathbb{O}_{\mathrm{CU}}\left(v\, e^{v(-t\xi\eta+\eta y+\xi x+\delta y x)}\, C\Lambda_k(\epsilon, \gamma, y, a, x, \eta, \xi, \delta) \mid y a x\right)$, with $v = (1+t\delta)^{-1}$ and where $C\Lambda_k(\epsilon, \gamma, y, a, x, \eta, \xi, \delta)$ is a fixed polynomial of degree at most $4k$ in y, $\sqrt{a}$, x, η, ξ, with scalar coefficients.

Comment. Even better, $\log(C\Lambda_k)$ is of degree at most $2k+2$ in said variables.

```
eqn = ρ[e^(ξ CU@x)].ρ[e^(η CU@y)] == ρ[e^(d CU@y)].ρ[e^(c (t CU[] - 2 ϵ CU@a))].ρ[e^(b CU@x)]
```

$\{\{1+\gamma\epsilon\eta\xi, \gamma\xi\}, \{\epsilon\eta, 1\}\} == \{\{e^{-c\gamma\epsilon}, b\, e^{-c\gamma\epsilon}\gamma\}, \{d\, e^{-c\gamma\epsilon}\epsilon, e^{c\gamma\epsilon} + b\, d\, e^{-c\gamma\epsilon}\gamma\epsilon\}\}$

```
sol = Solve[Thread[Flatten /@ eqn], {d, b, c}][[1]] /. C[1] → 0
```

$\left\{d \to \frac{\eta}{1+\gamma\epsilon\eta\xi},\ b \to \frac{\xi}{1+\gamma\epsilon\eta\xi},\ c \to \frac{\mathrm{Log}\left[\frac{1}{1+\gamma\epsilon\eta\xi}\right]}{\gamma\epsilon}\right\}$

Proof of Lemma 3C. We know that $\mathbb{O}_{\mathrm{CU}}\left(e^{\xi x+\eta y} \mid x y\right) = \mathbb{O}_{\mathrm{CU}}\left(e^{ct+ay-2\epsilon ca+bx} \mid y a x\right)$, with $\left\{d \to \frac{\eta}{1+\gamma\epsilon\eta\xi}, b \to \frac{\xi}{1+\gamma\epsilon\eta\xi}, c \to \frac{\mathrm{Log}[1+\gamma\epsilon\eta\xi]}{-\gamma\epsilon}\right\}$. Expanding in ϵ we get
$\mathbb{O}_{\mathrm{CU}}\left(e^{\xi x+\eta y} \mid x y\right) = \mathbb{O}_{\mathrm{CU}}\left(\lambda_\epsilon(\xi, \eta)\, e^{\eta y+\xi x-\eta\xi t} \mid y a x\right) = \mathbb{O}_{\mathrm{CU}}\left(\lambda_\epsilon(\partial_x, \partial_y)\, e^{\eta y+\xi x-\eta\xi t} \mid y a x\right)$ and so
$\mathbb{O}_{\mathrm{CU}}\left(e^{\eta y+\xi x+\delta y x} \mid x y\right) = \mathbb{O}\left(\lambda_\epsilon(\partial_x, \partial_y)\, e^{\delta\partial_\eta\partial_\xi}\, e^{\eta y+\xi x-\eta\xi t} \mid y a x\right) = \mathbb{O}\left(\lambda_\epsilon(\partial_x, \partial_y)\, v\, e^{v(-t\xi\eta+\eta y+\xi x+\delta y x)} \mid y a x\right)$.

Logos

```
SSϵ[ℰ_] := Block[{ϵ}, Collect[Normal@Series[ℰ, {ϵ, 0, $TϵD}], ϵ, Together]];
(* Shielded ϵ-Series *)
CΛ[t1_, y1_, a1_, x1_, ξ1_, η1_, δ_] := Module[
  {eqn, d, b, c, sol, λ, q, ν, ξ, η},
  eqn = ρ[ⅇ^(ξ CU@x)].ρ[ⅇ^(η CU@y)] == ρ[ⅇ^(d CU@y)].ρ[ⅇ^(c (t CU[] - 2 ϵ CU@a))].ρ[ⅇ^(b CU@x)];
  sol = Solve[Thread[Flatten /@ eqn], {d, b, c}][[1]] /. C[1] → 0;
  λ = Simplify[ⅇ^(-η y-ξ x+η ξ t) SSϵ[ⅇ^(c t + d y - 2 ϵ c a + b x) /. sol]];
  q = ⅇ^(ν (-t ξ η+η y+ξ x+δ y x));
  Collect[ν q^-1 DP_{ξ→Dx,η→Dy}[λ][q] /. ν → (1 + t δ)^-1, ϵ, Simplify] /.
   {t → t1, y → y1, a → a1, x → x1, ξ → ξ1, η → η1}
  ];
```

```
CΛ[t, y, a, x, ξ, η, δ]
```

$$
\frac{1}{1+t\,\delta}+\frac{1}{24\left(1+t\,\delta\right)^9}
$$

$$
\epsilon^2\Big(48\,a^2\left(1+t\,\delta\right)^4\left(2\,\delta^2\left(1+t\,\delta\right)^2+4\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)-
$$

$$
24\,a\,\gamma\left(1+t\,\delta\right)^4\left(2\,\delta^2\left(1+t\,\delta\right)^2+4\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)-
$$

$$
48\,a\,y\,\gamma\left(1+t\,\delta\right)^3(x\,\delta+\eta)
$$

$$
\left(6\,\delta^2\left(1+t\,\delta\right)^2+6\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)+24\,y\,\gamma^2\left(1+t\,\delta\right)^3
$$

$$
(x\,\delta+\eta)\left(6\,\delta^2\left(1+t\,\delta\right)^2+6\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)-48\,a\,x\,\gamma
$$

$$
\left(1+t\,\delta\right)^3(y\,\delta+\xi)\left(6\,\delta^2\left(1+t\,\delta\right)^2+6\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)+
$$

$$
24\,x\,\gamma^2\left(1+t\,\delta\right)^3(y\,\delta+\xi)\left(6\,\delta^2\left(1+t\,\delta\right)^2+6\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.
$$

$$
\left.(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)+12\,y^2\,\gamma^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)^2
$$

$$
\left(12\,\delta^2\left(1+t\,\delta\right)^2+8\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)+12\,x^2\,\gamma^2
$$

$$
\left(1+t\,\delta\right)^2(y\,\delta+\xi)^2\left(12\,\delta^2\left(1+t\,\delta\right)^2+8\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)+
$$

$$
24\,a\,t\,\gamma\left(1+t\,\delta\right)^2\left(6\,\delta^3\left(1+t\,\delta\right)^3+18\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.
$$

$$
\left.9\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)-
$$

$$
8\,t\left(\gamma+t\,\gamma\,\delta\right)^2\left(6\,\delta^3\left(1+t\,\delta\right)^3+18\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.
$$

$$
\left.9\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)+
$$

$$
24\,x\,y\left(\gamma+t\,\gamma\,\delta\right)^2\left(6\,\delta^3\left(1+t\,\delta\right)^3+18\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.
$$

$$
\left.9\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)-
$$

$$
12\,t\,y\,\gamma^2\left(1+t\,\delta\right)(x\,\delta+\eta)\left(24\,\delta^3\left(1+t\,\delta\right)^3+36\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.
$$

$$
\left.12\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)-
$$

$$
12\,t\,x\,\gamma^2\left(1+t\,\delta\right)(y\,\delta+\xi)\left(24\,\delta^3\left(1+t\,\delta\right)^3+36\,\delta^2\left(1+t\,\delta\right)^2(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.
$$

$$
\left.12\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\right)+
$$

$$
3\,t^2\,\gamma^2\left(24\,\delta^4\left(1+t\,\delta\right)^4+96\,\delta^3\left(1+t\,\delta\right)^3(x\,\delta+\eta)\,(y\,\delta+\xi)+72\,\delta^2\left(1+t\,\delta\right)^2\right.
$$

$$
\left.(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+16\,\delta\left(1+t\,\delta\right)(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3+(x\,\delta+\eta)^4\,(y\,\delta+\xi)^4\right)\Big)+
$$

$$
\frac{1}{2\left(1+t\,\delta\right)^5}\,\epsilon\,\Big(4\,a\left(1+t\,\delta\right)^2\left(\left(t+x\,y\right)\delta^2+\eta\,\xi+\delta\left(1+y\,\eta+x\,\xi\right)\right)+
$$

$$
\gamma\left(2\,t^3\,\delta^4+4\,t^2\,\delta^2\left(\delta-x\,y\,\delta^2+\eta\,\xi\right)-2\left(y\,\eta\left(\delta\left(2+y\,\eta\right)+\eta\,\xi\right)+x^2\,\delta\left(2\,y^2\,\delta^2+3\,y\,\delta\,\xi+\xi^2\right)+\right.\right.
$$

$$
\left.x\left(3\,y^2\,\delta^2\,\eta+4\,y\,\delta\,(\delta+\eta\,\xi)+\xi\left(2\,\delta+\eta\,\xi\right)\right)\right)-t\left(3\,x^2\,y^2\,\delta^4-4\,\delta\,\eta\,\xi-\eta^2\,\xi^2+\right.
$$

$$
\left.\left.4\,x\,y\,\delta^3\left(3+y\,\eta+x\,\xi\right)+\delta^2\left(-2+y^2\,\eta^2+4\,x\,\xi+x^2\,\xi^2+4\,y\,(\eta+x\,\eta\,\xi)\right)\right)\right)\Big)
$$

```
{Short[lhs = 𝕆CU[SS[ℯ^(ℏ (ξ x+η y+δ x y))], {x, y}], 5], HL[lhs ==
   𝕆CU[SS[ℯ^(ℏ ν (ξ x+η y+δ x y-t ℏ ξ η)) CΛ[t, y, a, x, ℏ ξ, ℏ η, ℏ δ] /. ν → (1 + ℏ t δ)^-1], {y, a, x}]]}
```

$$\{(1 - t\delta\hbar + t^2\delta^2\hbar^2 + t\gamma\delta^2\epsilon\hbar^2 - t\eta\xi\hbar^2 - t^3\delta^3\hbar^3 - 3t^2\gamma\delta^3\epsilon\hbar^3 - 2t\gamma^2\delta^3\epsilon^2\hbar^3 + 2t^2\delta\eta\xi\hbar^3 + 2t\gamma\delta\epsilon\eta\xi\hbar^3)\,\mathrm{CU}[\,] + (2\delta\epsilon\hbar - 4t\delta^2\epsilon\hbar^2 - 2\gamma\delta^2\epsilon^2\hbar^2 + 2\epsilon\eta\xi\hbar^2 + 6t^2\delta^3\epsilon\hbar^3 + 12t\gamma\delta^3\epsilon^2\hbar^3 - 8t\delta\epsilon\eta\xi\hbar^3 - 4\gamma\delta\epsilon^2\eta\xi\hbar^3)\,\mathrm{CU}[a] + (\xi\hbar - 2t\delta\xi\hbar^2 - 2\gamma\delta\epsilon\xi\hbar^2 + 3t^2\delta^2\xi\hbar^3 + 9t\gamma\delta^2\epsilon\xi\hbar^3 + 6\gamma^2\delta^2\epsilon^2\xi\hbar^3 - t\eta\xi^2\hbar^3 - \gamma\epsilon\eta\xi^2\hbar^3)\,\mathrm{CU}[x] + \ll 23 \gg + \frac{1}{2}\delta^2\xi\hbar^3\,\mathrm{CU}[y,y,x,x,x] + \frac{1}{2}\delta^2\eta\hbar^3\,\mathrm{CU}[y,y,y,x,x] + \frac{1}{6}\delta^3\hbar^3\,\mathrm{CU}[y,y,y,x,x,x],\ \mathrm{True}\}$$

C𝕆 and Swaps

Swaps from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
SetAttributes[C𝕆, Orderless];
CU@C𝕆[specs___, 𝔼[L_, Q_, P_]] := 𝕆CU[SS[ℯ^(L+Q) P], specs]
```

```
CU@C𝕆[𝔼[ℏ t1 a2, ℏ t1^-1 (ℯ^t1 - 1) y1 x2, 1 + ϵ x1 y2], {y1, x1}1, {x2, a2, y2}2] // Short
```

$$\mathrm{CU}[\,] + \ll 27 \gg + \mathrm{CU}[y_1, x_1]\left(-\gamma\epsilon\hbar^2 t_2 + e^{t_1}\gamma\epsilon\hbar^2 t_2 + \frac{\epsilon\hbar \ll 1 \gg}{t_1} - \frac{\ll 1 \gg}{\ll 1 \gg} + \frac{1}{2}\gamma^2\epsilon\hbar^3 t_1 t_2 - \frac{1}{2}e^{t_1}\gamma^2\epsilon\hbar^3 t_1 t_2\right)$$

```
HL[ρ[ℯ^(ξ CU@x)].ρ[ℯ^(α CU@a)] == ρ[ℯ^(α CU@a)].ρ[ℯ^(ℯ^(-γ α) ξ CU@x)]]
```

True

SW

```
SW_{x_i_, a_j_}[C𝕆[{lh___, x_i_, a_j_, rh___}_s_, more___, 𝔼[L_, Q_, P_]]] :=
 C𝕆[{lh, a_j, x_i, rh}_s, more,
  With[{q = ℯ^(-γ α) ξ x_i + α a_j},
   𝔼[L, ℯ^(-γ α) ξ x_i + (Q /. x_i → 0), ℯ^-q DP_{x_i→D_ξ, a_j→D_α}[P][ℯ^q]] /. {α → ∂_{a_j} L, ξ → ∂_{x_i} Q}]]
```

```
c𝕠 = C𝕆[𝔼[ℏ t1 a2, ℏ t1^-1 (ℯ^t1 - 1) y1 x2, 1 + ϵ x1 y2], {y1, x1}1, {x2, a2, y2}2]
```

$$\mathrm{C}\mathbb{O}\left[\{y_1, x_1\}_1, \{x_2, a_2, y_2\}_2, \mathbb{E}\left[\hbar a_2 t_1, \frac{(-1 + e^{t_1})\hbar x_2 y_1}{t_1}, 1 + \epsilon x_1 y_2\right]\right]$$

```
SW_{x2,a2}[c𝕠]
```

$$\mathrm{C}\mathbb{O}\left[\{y_1, x_1\}_1, \{a_2, x_2, y_2\}_2, \mathbb{E}\left[\hbar a_2 t_1, \frac{e^{-\gamma\hbar t_1}(-1 + e^{t_1})\hbar x_2 y_1}{t_1}, 1 + \epsilon x_1 y_2\right]\right]$$

```
With[{cω = CO[{y1, x1}1, {x2, a2, y2}2, E[ħ t1 a2, ħ t1^-1 (e^t1 - 1) y1 x2, 1 + ϵ x1 y2]]},
 HL[CU[cω] == CU[cω // SW_{x2,a2}]]]
```

True

```
With[{cω = CO[{y1, a1, x1}1, {x2, a2, y2}2,
    E[ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2), ħ (γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]]},
 {CU[cω] // Short, HL[CU[cω] == CU[cω // SW_{x2,a2}]]}
]
```

$\{\mathrm{CU}[a_1, a_1, a_1, a_1]\left(\frac{1}{6}\epsilon\hbar^3 l_1 l_{11}^3 t_1^3 + \frac{1}{2}\epsilon\hbar^3 l_1 l_{11}^2 l_{21} t_1^2 t_2 + \frac{1}{2}\epsilon \ll 5\gg t_2^2 + \frac{1}{6}\epsilon\hbar^3 l_1 l_{21}^3 t_2^3\right) + \ll 181\gg, \mathrm{True}\}$

SW

```
SW_{xi_,yj_→k_}[CO[{lh___, xi_, yj_, rh___}s_, more___, E[L_, Q_, P_]]] :=
 CO[{lh, yk, ak, xk, rh}s, more,
  With[{q = ν (ξ xk + η yk + δ xk yk - tk ξ η)},
   E[L, q + (Q /. xi | yj → 0), e^-q DP_{xi→Dξ,yj→Dη}[P][CΛ[tk, yk, ak, xk, ξ, η, δ] e^q]] /.
    ν → (1 + tk δ)^-1 /. {ξ → (∂_{xi} Q /. yj → 0), η → (∂_{yj} Q /. xi → 0), δ → ∂_{xi,yj} Q}]]
```

```
With[{cω = CO[{x1, y1}1, {x2, a2, y2}2,
    E[ħ (l12 t1 a2 + l22 t2 a2), ħ (γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     1 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]]},
 {CU[cω] // Short, HL[CU[cω] == CU[cω // SW_{x1,y1→1}]]}
]
```

$\{12\epsilon^2\hbar^3\mathrm{CU}[y_1, a_1, a_1, x_1]\gamma_{11}^3 + \ll 159\gg + \mathrm{CU}[\,]\left(\ll 301\gg + \epsilon\hbar^3 p_{11} t_1 t_2^3\gamma_{22}^3 + 4\epsilon\hbar^3 p_{22} t_2^4\gamma_{22}^3\right), \mathrm{True}\}$

The Quantum Logos QΛ

Goal 1: In QU, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$.

First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum.

```
adx[ℰ_] := Simp[QU@x ** ℰ - ℰ ** QU@x];
G = Simp[NestList[adx, QU@y, $TϵD + 1].Table[ξ^k / k!, {k, 0, $TϵD + 1}]]
```

$$\frac{(\xi - T^2\xi)\,\mathrm{QU}[\,]}{\hbar} + 2T^2\epsilon\xi\,\mathrm{QU}[a] + \left(\frac{1}{2}(\gamma\epsilon\xi^2 - 3T^2\gamma\epsilon\xi^2) + \frac{1}{4}(\gamma^2\epsilon^2\xi^2 - 5T^2\gamma^2\epsilon^2\xi^2)\hbar\right)\mathrm{QU}[x] + \mathrm{QU}[y] - 2T^2\epsilon^2\xi\hbar\,\mathrm{QU}[a, a] + 3T^2\gamma\epsilon^2\xi^2\hbar\,\mathrm{QU}[a, x] + \frac{1}{6}(\gamma^2\epsilon^2\xi^3 - 7T^2\gamma^2\epsilon^2\xi^3)\hbar\,\mathrm{QU}[x, x] + \left(\gamma\epsilon\xi\hbar + \frac{1}{2}\gamma^2\epsilon^2\xi\hbar^2\right)\mathrm{QU}[y, x] + \frac{1}{2}\gamma^2\epsilon^2\xi^2\hbar^2\,\mathrm{QU}[y, x, x]$$

```
G /. ϵ → 0
```

$$\frac{(\xi - T^2\xi)\,\mathrm{QU}[\,]}{\hbar} + \mathrm{QU}[y]$$

Now F satisfies the ODE $\partial_\eta F = \partial_\eta(e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. We set it up

and solve:

```
F = Sum[f_{l,i,j,k}[η] ϵ^l QU@{y^i, a^j, x^k},
  {l, 0, $TϵD}, {i, 0, l}, {j, 0, l}, {k, 0, Min[l, 2 l - i - j]}]
```

$\mathrm{QU}[\,]\, f_{0,0,0,0}[\eta] + \epsilon\, \mathrm{QU}[\,]\, f_{1,0,0,0}[\eta] + \epsilon\, \mathrm{QU}[x]\, f_{1,0,0,1}[\eta] + \epsilon\, \mathrm{QU}[a]\, f_{1,0,1,0}[\eta] + \epsilon\, \mathrm{QU}[a, x]\, f_{1,0,1,1}[\eta] + \epsilon\, \mathrm{QU}[y]\, f_{1,1,0,0}[\eta] + \epsilon\, \mathrm{QU}[y, x]\, f_{1,1,0,1}[\eta] + \epsilon\, \mathrm{QU}[y, a]\, f_{1,1,1,0}[\eta] + \epsilon^2\, \mathrm{QU}[\,]\, f_{2,0,0,0}[\eta] + \epsilon^2\, \mathrm{QU}[x]\, f_{2,0,0,1}[\eta] + \epsilon^2\, \mathrm{QU}[x, x]\, f_{2,0,0,2}[\eta] + \epsilon^2\, \mathrm{QU}[a]\, f_{2,0,1,0}[\eta] + \epsilon^2\, \mathrm{QU}[a, x]\, f_{2,0,1,1}[\eta] + \epsilon^2\, \mathrm{QU}[a, x, x]\, f_{2,0,1,2}[\eta] + \epsilon^2\, \mathrm{QU}[a, a]\, f_{2,0,2,0}[\eta] + \epsilon^2\, \mathrm{QU}[a, a, x]\, f_{2,0,2,1}[\eta] + \epsilon^2\, \mathrm{QU}[a, a, x, x]\, f_{2,0,2,2}[\eta] + \epsilon^2\, \mathrm{QU}[y]\, f_{2,1,0,0}[\eta] + \epsilon^2\, \mathrm{QU}[y, x]\, f_{2,1,0,1}[\eta] + \epsilon^2\, \mathrm{QU}[y, x, x]\, f_{2,1,0,2}[\eta] + \epsilon^2\, \mathrm{QU}[y, a]\, f_{2,1,1,0}[\eta] + \epsilon^2\, \mathrm{QU}[y, a, x]\, f_{2,1,1,1}[\eta] + \epsilon^2\, \mathrm{QU}[y, a, x, x]\, f_{2,1,1,2}[\eta] + \epsilon^2\, \mathrm{QU}[y, a, a]\, f_{2,1,2,0}[\eta] + \epsilon^2\, \mathrm{QU}[y, a, a, x]\, f_{2,1,2,1}[\eta] + \epsilon^2\, \mathrm{QU}[y, y]\, f_{2,2,0,0}[\eta] + \epsilon^2\, \mathrm{QU}[y, y, x]\, f_{2,2,0,1}[\eta] + \epsilon^2\, \mathrm{QU}[y, y, x, x]\, f_{2,2,0,2}[\eta] + \epsilon^2\, \mathrm{QU}[y, y, a]\, f_{2,2,1,0}[\eta] + \epsilon^2\, \mathrm{QU}[y, y, a, x]\, f_{2,2,1,1}[\eta] + \epsilon^2\, \mathrm{QU}[y, y, a, a]\, f_{2,2,2,0}[\eta]$

```
unowns = Cases[F, f___[η], ∞]
```

$\{f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta], f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], f_{2,0,0,0}[\eta], f_{2,0,0,1}[\eta], f_{2,0,0,2}[\eta], f_{2,0,1,0}[\eta], f_{2,0,1,1}[\eta], f_{2,0,1,2}[\eta], f_{2,0,2,0}[\eta], f_{2,0,2,1}[\eta], f_{2,0,2,2}[\eta], f_{2,1,0,0}[\eta], f_{2,1,0,1}[\eta], f_{2,1,0,2}[\eta], f_{2,1,1,0}[\eta], f_{2,1,1,1}[\eta], f_{2,1,1,2}[\eta], f_{2,1,2,0}[\eta], f_{2,1,2,1}[\eta], f_{2,2,0,0}[\eta], f_{2,2,0,1}[\eta], f_{2,2,0,2}[\eta], f_{2,2,1,0}[\eta], f_{2,2,1,1}[\eta], f_{2,2,2,0}[\eta]\}$

```
bas = Union @@ Table[ϵ^l Cases[Coefficient[F, ϵ, l], _QU, ∞], {l, 0, $TϵD}]
```

$\{\mathrm{QU}[\,], \epsilon\, \mathrm{QU}[\,], \epsilon^2\, \mathrm{QU}[\,], \epsilon\, \mathrm{QU}[a], \epsilon^2\, \mathrm{QU}[a], \epsilon\, \mathrm{QU}[x], \epsilon^2\, \mathrm{QU}[x], \epsilon\, \mathrm{QU}[y], \epsilon^2\, \mathrm{QU}[y], \epsilon^2\, \mathrm{QU}[a, a], \epsilon\, \mathrm{QU}[a, x], \epsilon^2\, \mathrm{QU}[a, x], \epsilon^2\, \mathrm{QU}[x, x], \epsilon\, \mathrm{QU}[y, a], \epsilon^2\, \mathrm{QU}[y, a], \epsilon\, \mathrm{QU}[y, x], \epsilon^2\, \mathrm{QU}[y, x], \epsilon^2\, \mathrm{QU}[y, y], \epsilon^2\, \mathrm{QU}[a, a, x], \epsilon^2\, \mathrm{QU}[a, x, x], \epsilon^2\, \mathrm{QU}[y, a, a], \epsilon^2\, \mathrm{QU}[y, a, x], \epsilon^2\, \mathrm{QU}[y, x, x], \epsilon^2\, \mathrm{QU}[y, y, a], \epsilon^2\, \mathrm{QU}[y, y, x], \epsilon^2\, \mathrm{QU}[a, a, x, x], \epsilon^2\, \mathrm{QU}[y, a, a, x], \epsilon^2\, \mathrm{QU}[y, a, x, x], \epsilon^2\, \mathrm{QU}[y, y, a, a], \epsilon^2\, \mathrm{QU}[y, y, a, x], \epsilon^2\, \mathrm{QU}[y, y, x, x]\}$

```
Short[eqns = Flatten[{(Coefficient[F - QU[], #] /. η → 0) == 0,
     Expand[Coefficient[Simp[F ** G - QU[y] ** F - ∂_η F], #]] == 0} & /@ bas], 8]
```

$\Big\{-1 + f_{0,0,0,0}[0] + \epsilon\, f_{1,0,0,0}[0] + \epsilon^2 f_{2,0,0,0}[0] == 0,$
$\frac{\xi f_{0,0,0,0}[\eta]}{\hbar} - \frac{T^2 \xi f_{0,0,0,0}[\eta]}{\hbar} + \frac{\epsilon \xi f_{1,0,0,0}[\eta]}{\hbar} - \frac{T^2 \epsilon \xi f_{1,0,0,0}[\eta]}{\hbar} + \frac{\epsilon f_{1,0,0,1}[\eta]}{\hbar} - \frac{T^2 \epsilon f_{1,0,0,1}[\eta]}{\hbar} + \frac{\epsilon^2 \xi f_{2,0,0,0}[\eta]}{\hbar} - \frac{T^2 \epsilon^2 \xi f_{2,0,0,0}[\eta]}{\hbar} + \frac{\epsilon^2 f_{2,0,0,1}[\eta]}{\hbar} - \frac{T^2 \epsilon^2 f_{2,0,0,1}[\eta]}{\hbar} - f_{0,0,0,0}'[\eta] - \epsilon f_{1,0,0,0}'[\eta] - \epsilon^2 f_{2,0,0,0}'[\eta] == 0, f_{1,0,0,0}[0] == 0,$
$\frac{\xi f_{1,0,0,0}[\eta]}{\hbar} - \frac{T^2 \xi f_{1,0,0,0}[\eta]}{\hbar} + \frac{f_{1,0,0,1}[\eta]}{\hbar} - \frac{T^2 f_{1,0,0,1}[\eta]}{\hbar} - f_{1,0,0,0}'[\eta] == 0,$
$f_{2,0,0,0}[0] == 0, \ll 53 \gg, f_{2,2,1,1}[0] == 0,$
$\gamma \xi \hbar f_{1,1,1,0}[\eta] - 2 \gamma f_{2,1,2,1}[\eta] + \frac{\xi f_{2,2,1,1}[\eta]}{\hbar} - \frac{T^2 \xi f_{2,2,1,1}[\eta]}{\hbar} - f_{2,2,1,1}'[\eta] == 0,$
$f_{2,2,0,2}[0] == 0, \gamma \xi \hbar f_{1,1,0,1}[\eta] - \gamma f_{2,1,1,2}[\eta] + \frac{\xi f_{2,2,0,2}[\eta]}{\hbar} - \frac{T^2 \xi f_{2,2,0,2}[\eta]}{\hbar} - f_{2,2,0,2}'[\eta] == 0\Big\}$

```
{sol} = DSolve[eqns, unowns, η]
```

$$\Big\{\Big\{f_{0,0,0,0}[\eta] \to e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}},\ f_{1,0,0,0}[\eta] \to \frac{e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\left(-1+T^2\right)\left(-1+3T^2\right)\gamma\,\eta^2\,\xi^2}{4\hbar},$$

$$f_{1,0,0,1}[\eta] \to -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\left(-1+3T^2\right)\gamma\,\eta\,\xi^2,\ f_{1,0,1,0}[\eta] \to 2\,e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}T^2\,\eta\,\xi,$$

$$f_{1,0,1,1}[\eta] \to 0,\ f_{1,1,0,0}[\eta] \to -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\left(-1+3T^2\right)\gamma\,\eta^2\,\xi,\ f_{1,1,0,1}[\eta] \to e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma\,\eta\,\xi\,\hbar,$$

$$f_{1,1,1,0}[\eta] \to 0,\ f_{2,0,0,0}[\eta] \to \frac{1}{288\hbar^2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\left(-1+T^2\right)\gamma^2\,\eta^2\,\xi^2\left(-9\eta^2\xi^2+63T^2\eta^2\xi^2-\right.$$
$$\left.135T^4\eta^2\xi^2+81T^6\eta^2\xi^2-40\eta\,\xi\,\hbar+272T^2\eta\,\xi\,\hbar-328T^4\eta\,\xi\,\hbar-36\hbar^2+180T^2\hbar^2\right),$$

$$f_{2,0,0,1}[\eta] \to -\frac{1}{24\hbar}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\,\eta\,\xi^2\left(-3\eta^2\xi^2+21T^2\eta^2\xi^2-45T^4\eta^2\xi^2+\right.$$
$$\left.27T^6\eta^2\xi^2-10\eta\,\xi\,\hbar+68T^2\eta\,\xi\,\hbar-82T^4\eta\,\xi\,\hbar-6\hbar^2+30T^2\hbar^2\right),$$

$$f_{2,0,0,2}[\eta] \to \frac{1}{24}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\,\eta\,\xi^3\left(3\eta\,\xi-18T^2\eta\,\xi+27T^4\eta\,\xi+4\hbar-28T^2\hbar\right),$$

$$f_{2,0,1,0}[\eta] \to \frac{1}{2\hbar}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}T^2\gamma\,\eta^2\,\xi^2\left(\eta\,\xi-4T^2\eta\,\xi+3T^4\eta\,\xi+4\hbar-6T^2\hbar\right),$$

$$f_{2,0,1,1}[\eta] \to -e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}T^2\gamma\,\eta\,\xi^2\left(-\eta\,\xi+3T^2\eta\,\xi-3\hbar\right),\ f_{2,0,1,2}[\eta] \to 0,$$

$$f_{2,0,2,0}[\eta] \to 2\,e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}T^2\,\eta\,\xi\left(T^2\eta\,\xi-\hbar\right),\ f_{2,0,2,1}[\eta] \to 0,$$

$$f_{2,0,2,2}[\eta] \to 0,\ f_{2,1,0,0}[\eta] \to -\frac{1}{24\hbar}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\,\eta^2\,\xi\left(-3\eta^2\xi^2+21T^2\eta^2\xi^2-\right.$$
$$\left.45T^4\eta^2\xi^2+27T^6\eta^2\xi^2-10\eta\,\xi\,\hbar+68T^2\eta\,\xi\,\hbar-82T^4\eta\,\xi\,\hbar-6\hbar^2+30T^2\hbar^2\right),$$

$$f_{2,1,0,1}[\eta] \to \frac{1}{4}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\,\eta\,\xi\left(2\eta^2\xi^2-10T^2\eta^2\xi^2+12T^4\eta^2\xi^2+5\eta\,\xi\,\hbar-21T^2\eta\,\xi\,\hbar+2\hbar^2\right),$$

$$f_{2,1,0,2}[\eta] \to -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\,\eta\,\xi^2\left(-\eta\,\xi+3T^2\eta\,\xi-\hbar\right)\hbar,$$

$$f_{2,1,1,0}[\eta] \to -e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}T^2\gamma\,\eta^2\,\xi\left(-\eta\,\xi+3T^2\eta\,\xi-3\hbar\right),$$

$$f_{2,1,1,1}[\eta] \to 2\,e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}T^2\gamma\,\eta^2\,\xi^2\,\hbar,\ f_{2,1,1,2}[\eta] \to 0,\ f_{2,1,2,0}[\eta] \to 0,\ f_{2,1,2,1}[\eta] \to 0,$$

$$f_{2,2,0,0}[\eta] \to \frac{1}{24}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\,\eta^3\,\xi\left(3\eta\,\xi-18T^2\eta\,\xi+27T^4\eta\,\xi+4\hbar-28T^2\hbar\right),$$

$$f_{2,2,0,1}[\eta] \to -\frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\,\eta^2\,\xi\left(-\eta\,\xi+3T^2\eta\,\xi-\hbar\right)\hbar,$$

$$f_{2,2,0,2}[\eta] \to \frac{1}{2}e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\gamma^2\,\eta^2\,\xi^2\,\hbar^2,\ f_{2,2,1,0}[\eta] \to 0,\ f_{2,2,1,1}[\eta] \to 0,\ f_{2,2,2,0}[\eta] \to 0\Big\}\Big\}$$

```
Union@Cases[sol, e-, ∞]
```

$$\left\{e^{-\frac{\eta(-\xi+T^2\xi)}{\hbar}}\right\}$$

```
FF = Collect[F /. sol /. {𝕖- → 1, QU → Times}, ϵ, Simplify]
```

$$1+\frac{1}{4\hbar}\epsilon\,\eta\,\xi\left(8\,a\,T^2\,\hbar+\left(-1+3\,T^2\right)\gamma\,\eta\left(\left(-1+T^2\right)\xi-2\,y\,\hbar\right)+2\,x\,\gamma\,\hbar\left(\xi-3\,T^2\,\xi+2\,y\,\hbar\right)\right)+$$
$$\frac{1}{288\,\hbar^2}\epsilon^2\,\eta\,\xi\left(576\,a^2\,T^2\left(T^2\,\eta\,\xi-\hbar\right)\hbar^2+144\,a\,T^2\,\gamma\,\hbar\left(6\,x\,\xi\,\hbar^2+\left(-1+3\,T^2\right)\eta^2\,\xi\left(\left(-1+T^2\right)\xi-2\,y\,\hbar\right)+2\,\eta\,\hbar\left(\left(x-3\,T^2\,x\right)\xi^2+3\,y\,\hbar+\xi\left(2-3\,T^2+2\,x\,y\,\hbar\right)\right)\right)+\right.$$
$$\gamma^2\left(9\left(1-3\,T^2\right)^2\eta^3\,\xi\left(\xi-T^2\,\xi+2\,y\,\hbar\right)^2+24\,x\,\hbar^3\left(2\left(1-7\,T^2\right)x\,\xi^2+6\,y\,\hbar+\xi\left(3-15\,T^2+6\,x\,y\,\hbar\right)\right)+\right.$$
$$12\,\eta\,\hbar^2\left(3\left(1-3\,T^2\right)^2x^2\,\xi^3+6\,y\,\hbar\left(1-5\,T^2+2\,x\,y\,\hbar\right)+\right.$$
$$3\,\xi\left(1+5\,T^4+10\,x\,y\,\hbar+4\,x^2\,y^2\,\hbar^2-6\,T^2\left(1+7\,x\,y\,\hbar\right)\right)+$$
$$\left.2\,x\,\xi^2\left(5+41\,T^4+6\,x\,y\,\hbar-2\,T^2\left(17+9\,x\,y\,\hbar\right)\right)\right)-4\,\eta^2\,\hbar\left(9\left(1-3\,T^2\right)^2\left(-1+T^2\right)x\,\xi^3+\right.$$
$$12\left(-1+7\,T^2\right)y^2\,\hbar^2+6\,y\,\xi\,\hbar\left(-5-41\,T^4-6\,x\,y\,\hbar+2\,T^2\left(17+9\,x\,y\,\hbar\right)\right)+$$
$$\left.\left.\left.2\,\xi^2\left(-5+41\,T^6-18\,x\,y\,\hbar-3\,T^4\left(25+36\,x\,y\,\hbar\right)+T^2\left(39+90\,x\,y\,\hbar\right)\right)\right)\right)\right)$$

```
{Short[lhs = SimpT@𝕆QU[SS[𝕖^(ℏ (ξ x+η y))], {x, y}], 5],
 HL[lhs == SimpT@𝕆QU[SS[𝕖^(ℏ (ξ x+η y+(1-T²) ξ η)) (FF /. {ξ → ℏ ξ, η → ℏ η})], {y, a, x}]]}
```

$$\left\{\left(1-t\,\eta\,\xi\,\hbar^2-\frac{1}{2}t^2\,\eta\,\xi\,\hbar^3\right)QU[\,]+\left(2\,\epsilon\,\eta\,\xi\,\hbar^2+2\,t\,\epsilon\,\eta\,\xi\,\hbar^3\right)QU[a]+\right.$$
$$\left(\xi\,\hbar+\left(-t\,\eta\,\xi^2-\gamma\,\epsilon\,\eta\,\xi^2\right)\hbar^3\right)QU[x]+\left(\eta\,\hbar+\left(-t\,\eta^2\,\xi-\gamma\,\epsilon\,\eta^2\,\xi\right)\hbar^3\right)QU[y]-$$
$$2\,\epsilon^2\,\eta\,\xi\,\hbar^3\,QU[a,a]+2\,\epsilon\,\eta\,\xi^2\,\hbar^3\,QU[a,x]+\frac{1}{2}\xi^2\,\hbar^2\,QU[x,x]+2\,\epsilon\,\eta^2\,\xi\,\hbar^3\,QU[y,a]+$$
$$\left(\eta\,\xi\,\hbar^2+\gamma\,\epsilon\,\eta\,\xi\,\hbar^3\right)QU[y,x]+\frac{1}{2}\eta^2\,\hbar^2\,QU[y,y]+\frac{1}{6}\xi^3\,\hbar^3\,QU[x,x,x]+$$
$$\left.\frac{1}{2}\eta\,\xi^2\,\hbar^3\,QU[y,x,x]+\frac{1}{2}\eta^2\,\xi\,\hbar^3\,QU[y,y,x]+\frac{1}{6}\eta^3\,\hbar^3\,QU[y,y,y],\ \text{True}\right\}$$

Logos

```
QΛ[T_, y1_, a1_, x1_, ξ1_, η1_, δ_] := Module[
   {adx, G, F, f, unowns, bas, eqns, sol, λ, q, ν, ξ, η, t},
   adx[ℰ_] := Simp[QU@x ** ℰ - ℰ ** QU@x];
   G = Simp[NestList[adx, QU@y, $TϵD + 1].Table[ξ^k / k!, {k, 0, $TϵD + 1}]];
   F = Sum[f_{l,i,j,k}[η] ϵ^l QU@{y^i, a^j, x^k},
     {l, 0, $TϵD}, {i, 0, l}, {j, 0, l}, {k, 0, Min[l, 2 l - i - j]}];
   unowns = Cases[F, f___[η], ∞];
   bas = Union @@ Table[ϵ^l Cases[Coefficient[F, ϵ, l], _QU, ∞], {l, 0, $TϵD}];
   eqns = Flatten[{(Coefficient[F - QU[], #] /. η → 0) == 0,
        Expand[Coefficient[Simp[F ** G - QU[y] ** F - ∂_η F], #]] == 0} & /@ bas];
   {sol} = DSolve[eqns, unowns, η];
   λ = Collect[F /. sol /. {𝕖- → 1, QU → Times}, ϵ, Simplify];
   q = 𝕖^(ν (-t ξ η+η y+ξ x+δ y x));
   Collect[ν q^-1 DP_{ξ→D_x, η→D_y}[λ][q] /. ν → (1 + t δ)^-1 /. t → (T² - 1) / ℏ, ϵ, Simplify] /.
    {y → y1, a → a1, x → x1, ξ → ξ1, η → η1}
  ];
```

```
QΛ[T, y, a, x, ξ, η, δ]
```

$$\frac{\hbar}{\left(-1+T^2\right)\delta+\hbar}+\frac{1}{4\left(\left(-1+T^2\right)\delta+\hbar\right)^5}\epsilon\hbar^2\Big(8aT^2\left(\left(-1+T^2\right)\delta+\hbar\right)^2\left(\eta\xi\hbar+\delta\left(1+y\eta+x\xi\right)\hbar+\delta^2\left(-1+T^2+xy\hbar\right)\right)+\gamma\Big(\eta\xi\hbar^2\left(\left(-1+3T^2\right)\eta\left(\left(-1+T^2\right)\xi-2y\hbar\right)+2x\hbar\left(\xi-3T^2\xi+2y\hbar\right)\right)+\left(-1+T^2\right)\delta^4\left(-2+6T^6-x^2y^2\hbar^2-2T^4(7+4xy\hbar)+T^2\left(10+8xy\hbar-5x^2y^2\hbar^2\right)\right)-4\delta^3\hbar\left(1-3T^6+x^2y^2\hbar^2+T^4\left(7+2xy\left(3+y\eta\right)\hbar+2x^2y\xi\hbar\right)+T^2\left(-5-2xy\left(3+y\eta\right)\hbar+x^2y\hbar\left(-2\xi+y\hbar\right)\right)\right)+2\delta\hbar^2\left(\left(1-3T^2\right)y^2\eta^2\hbar+2\eta\left(\xi+3T^4\xi-4T^2\xi\left(1+xy\hbar\right)+y\hbar\left(1-3T^2+xy\hbar\right)\right)+x\hbar\left(\left(x-3T^2x\right)\xi^2+2y\hbar+\xi\left(2-6T^2+2xy\hbar\right)\right)\right)-\delta^2\hbar\Big(\left(1-4T^2+3T^4\right)y^2\eta^2\hbar+\hbar\Big(-2+3T^4\left(-2+4x\xi+x^2\xi^2\right)+4x(\xi+y\hbar)+x^2\left(\xi^2+2y\xi\hbar-4y^2\hbar^2\right)-2T^2\left(-4+x\left(8\xi-6y\hbar\right)+x^2\xi\left(2\xi-5y\hbar\right)\right)\Big)+2\eta\left(-2\left(-1+T^2\right)\xi\left(1+3T^4-2T^2\left(2+xy\hbar\right)\right)+y\hbar\left(2+6T^4+xy\hbar+T^2\left(-8+5xy\hbar\right)\right)\right)\Big)\Big)\Big)$$

```
{Short[lhs = SimpT@𝕆_QU[SS[e^(ℏ (ξ x+η y+δ x y))], {x, y}], 5],
 rhs = SimpT@𝕆_QU[SS[
     e^(ℏ ν (ξ x+η y+δ x y-(T^2-1) ξ η)) QΛ[T, y, a, x, ℏ ξ, ℏ η, ℏ δ] /. ν → (1 + (T^2 - 1) δ)^-1], {y, a, x}];
 HL[Simplify[lhs == rhs]]}
```

$$\Big\{\Big(1-t\delta\hbar+\frac{1}{2}\left(-t^2\delta+2t^2\delta^2+2t\gamma\delta^2\epsilon-2t\eta\xi\right)\hbar^2+\frac{1}{6}\left(-t^3\delta+6t^3\delta^2-6t^3\delta^3+12t^2\gamma\delta^2\epsilon-18t^2\gamma\delta^3\epsilon+6t\gamma^2\delta^2\epsilon^2-12t\gamma^2\delta^3\epsilon^2-3t^2\eta\xi+12t^2\delta\eta\xi+12t\gamma\delta\epsilon\eta\xi\right)\hbar^3\Big)$$
$$\mathrm{QU}[\,]+\Big(2\delta\epsilon\hbar-2\left(-t\delta\epsilon+2t\delta^2\epsilon+\gamma\delta^2\epsilon^2-\epsilon\eta\xi\right)\hbar^2+\left(t^2\delta\epsilon-6t^2\delta^2\epsilon+6t^2\delta^3\epsilon-8t\gamma\delta^2\epsilon^2+12t\gamma\delta^3\epsilon^2+2t\epsilon\eta\xi-8t\delta\epsilon\eta\xi-4\gamma\delta\epsilon^2\eta\xi\right)\hbar^3\Big)$$
$$\mathrm{QU}[a]+\ll 25\gg+\frac{1}{2}\delta^2\eta\hbar^3\,\mathrm{QU}[y,y,y,x,x]+\frac{1}{6}\delta^3\hbar^3\,\mathrm{QU}[y,y,y,x,x,x],\ \mathrm{True}\Big\}$$

Stitching Direct

```
MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].MatrixExp[η2 ρ[CU@y]].
  MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] // Simplify // MatrixForm
```

$$\begin{pmatrix} e^{\gamma(\alpha_1+\alpha_2)}\left(1+\gamma\epsilon\eta_2\xi_1\right) & e^{\gamma\alpha_1}\gamma\left(e^{\gamma\alpha_2}\xi_2+\xi_1\left(1+e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2\right)\right) \\ e^{\gamma\alpha_2}\epsilon\left(\eta_2+e^{\gamma\alpha_1}\eta_1\left(1+\gamma\epsilon\eta_2\xi_1\right)\right) & 1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1+e^{\gamma\alpha_2}\gamma\epsilon\left(\eta_2+e^{\gamma\alpha_1}\eta_1\left(1+\gamma\epsilon\eta_2\xi_1\right)\right)\xi_2 \end{pmatrix}$$

```
eqn = MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].
  MatrixExp[η2 ρ[CU@y]].MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] ==
 e^(τ0 ϵ γ) MatrixExp[η0 ρ[CU@y]].MatrixExp[α0 ρ[CU@a]].MatrixExp[ξ0 ρ[CU@x]]
```

$$\Big\{\Big\{e^{\gamma\alpha_2}\left(e^{\gamma\alpha_1}+e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1\right),\ e^{\gamma\alpha_1}\gamma\xi_1+e^{\gamma\alpha_2}\gamma\left(e^{\gamma\alpha_1}+e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1\right)\xi_2\Big\},\ \Big\{e^{\gamma\alpha_2}\left(e^{\gamma\alpha_1}\epsilon\eta_1+\epsilon\eta_2\left(1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1\right)\right),\ 1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1+e^{\gamma\alpha_2}\gamma\left(e^{\gamma\alpha_1}\epsilon\eta_1+\epsilon\eta_2\left(1+e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1\right)\right)\xi_2\Big\}\Big\}==$$
$$\Big\{\Big\{e^{\alpha0\gamma+\gamma\epsilon\tau0},\ e^{\alpha0\gamma+\gamma\epsilon\tau0}\gamma\xi0\Big\},\ \Big\{e^{\alpha0\gamma+\gamma\epsilon\tau0}\epsilon\eta0,\ e^{\gamma\epsilon\tau0}\left(1+e^{\alpha0\gamma}\gamma\epsilon\eta0\xi0\right)\Big\}\Big\}$$

```
sol = Block[{ϵ}, Solve[Thread[Flatten /@ eqn], {τ0, η0, α0, ξ0}]]〚1〛
```

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is $\mathrm{Log}[e^{\gamma(\alpha 0+\epsilon\,\tau 0)}] - \mathrm{Log}[e^{\gamma\,\alpha_2}(e^{\gamma\,\mathrm{Subscript}[\ll 2\gg]} + e^{\mathrm{Times}[\ll 2\gg]}\,\gamma\,\epsilon\,\eta_2\,\xi_1)] == 0$.

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Solve: Equations may not give solutions for all "solve" variables.

$$\Big\{\tau 0 \to \frac{1}{\gamma\epsilon}\left(-\mathrm{Log}\left[e^{\alpha 0\,\gamma}\right] + \mathrm{Log}\left[e^{\gamma\alpha_1+\gamma\alpha_2} + e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_2\xi_1\right]\right),\ \eta 0 \to \frac{1}{\gamma\epsilon\left(\xi_1 + e^{\gamma\alpha_2}\xi_2 + e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_1\xi_2\right)}$$
$$e^{-\gamma\alpha_1}\Bigg(\frac{1}{2} + \frac{1}{2}e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\xi_2 + \frac{1}{2}e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2\xi_1\xi_2 -$$
$$\frac{1}{2}\sqrt{\Big(\left(-1 - e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\xi_2 - e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2\xi_1\xi_2\right)^2 +}$$
$$4\,e^{-\alpha 0\,\gamma+\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\Big(-e^{\gamma\alpha_1}\eta_1\xi_1 - \eta_2\xi_1 - e^{\gamma\alpha_1}\gamma\epsilon\eta_1\eta_2\xi_1^2 - e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\xi_2 - e^{\gamma\alpha_2}\eta_2\xi_2 -$$
$$2\,e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\eta_2\xi_1\xi_2 - e^{\gamma\alpha_2}\gamma\epsilon\eta_2^2\xi_1\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2^2\xi_1^2\xi_2\Big)\Big)\Bigg),$$
$$\xi 0 \to \frac{1}{e^{\gamma\alpha_1}\eta_1 + \eta_2 + e^{\gamma\alpha_1}\gamma\epsilon\eta_1\eta_2\xi_1}\,e^{-\gamma\alpha_2}\Bigg(\frac{1}{2\gamma\epsilon} + \frac{1}{2}e^{\gamma\alpha_1}\eta_1\xi_1 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\xi_2 +$$
$$\frac{1}{2}e^{\gamma\alpha_2}\eta_2\xi_2 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\eta_2\xi_1\xi_2 -$$
$$\frac{1}{2\gamma\epsilon}\Bigg(\sqrt{\Big(\left(-1 - e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\xi_2 - e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2\xi_1\xi_2\right)^2 +}$$
$$4\,e^{-\alpha 0\,\gamma+\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\Big(-e^{\gamma\alpha_1}\eta_1\xi_1 - \eta_2\xi_1 - e^{\gamma\alpha_1}\gamma\epsilon\eta_1\eta_2\xi_1^2 - e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\xi_2 - e^{\gamma\alpha_2}\eta_2\xi_2 -$$
$$2\,e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\eta_2\xi_1\xi_2 - e^{\gamma\alpha_2}\gamma\epsilon\eta_2^2\xi_1\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2^2\xi_1^2\xi_2\Big)\Big)\Bigg)\Bigg)\Big\}$$

```
eqn = MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].
    MatrixExp[η2 ρ[CU@y]].MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] ==
  T0 MatrixExp[η0 ρ[CU@y]].MatrixExp[α0 ρ[CU@a]].MatrixExp[ξ0 ρ[CU@x]]
```

$$\Big\{\Big\{e^{\gamma\alpha_2}\left(e^{\gamma\alpha_1} + e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1\right),\ e^{\gamma\alpha_1}\gamma\xi_1 + e^{\gamma\alpha_2}\gamma\left(e^{\gamma\alpha_1} + e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1\right)\xi_2\Big\},$$
$$\Big\{e^{\gamma\alpha_2}\left(e^{\gamma\alpha_1}\epsilon\eta_1 + \epsilon\eta_2\left(1 + e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1\right)\right),$$
$$1 + e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1 + e^{\gamma\alpha_2}\gamma\left(e^{\gamma\alpha_1}\epsilon\eta_1 + \epsilon\eta_2\left(1 + e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1\right)\right)\xi_2\Big\}\Big\} ==$$
$$\Big\{\Big\{e^{\alpha 0\,\gamma}\,T0,\ e^{\alpha 0\,\gamma}\,T0\,\gamma\,\xi 0\Big\},\ \Big\{e^{\alpha 0\,\gamma}\,T0\,\epsilon\,\eta 0,\ T0\left(1 + e^{\alpha 0\,\gamma}\gamma\epsilon\,\eta 0\,\xi 0\right)\Big\}\Big\}$$

```
sol = Block[{ϵ}, Solve[Thread[Flatten /@ eqn], {T0, η0, α0, ξ0}]]〚1〛
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\Big\{T0 \to \frac{1}{1 + \gamma\epsilon\eta_2\xi_1},\ \eta 0 \to \frac{\eta_1 + e^{-\gamma\alpha_1}\eta_2 + \gamma\epsilon\eta_1\eta_2\xi_1}{1 + \gamma\epsilon\eta_2\xi_1},$$
$$\alpha 0 \to \frac{\mathrm{Log}\left[e^{\gamma\alpha_1+\gamma\alpha_2}\left(1 + \gamma\epsilon\eta_2\xi_1\right)^2\right]}{\gamma},\ \xi 0 \to \frac{e^{-\gamma\alpha_2}\xi_1 + \xi_2 + \gamma\epsilon\eta_2\xi_1\xi_2}{1 + \gamma\epsilon\eta_2\xi_1}\Big\}$$

𝔼

𝔼[L, Q, P] means $e^{\hbar(L+Q)}P$, where L is linear in the a's, Q is a combination of $x_i\,y_j$, and P is a perturbation polynomial. It should be interpreted via 𝕆[𝔼[...], $\{x_1,\ a_1,\ y_1\}_i$, ...] (with some default for

direct interpretation), or likewise via $Q\mathbb{O}[\mathbb{E}[\ldots], \{x_1, a_1, y_1\}_i, \ldots]$. In themselves, $C\mathbb{O}$ and $Q\mathbb{O}$ should have an interpretation in CU/QU by casting.