Pensieve header: A unified verification notebook for the PPSA project.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\PPSA"];
NotebookOpen[wdir <> "\\MakeVSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is $\hbar$, and all “coupling constants” are proportional to it.

TD

```
$TℏD = 5; ℏ /: ℏ^d_. /; d > $TℏD := 0;
$TϵD = 2; (* Can't be ∞ at least because of Quesne. Can't be ≤
 1 at least because of the explicit ϵ^2 in S𝔻$g. *)
ϵ /: ϵ^d_. /; d > $TϵD := 0;
SetAttributes[SS, HoldAll];
SS[ℰ_] := Block[{ℏ, ϵ}, (* Shielded Series *)
  Collect[Normal@Series[ℰ /. {T_i_ → e^(ℏ t_i/2), T → e^(ℏ t/2)}, {ℏ, 0, $TℏD}], ℏ, Together] ]
Simp[ℰ_] := Collect[ℰ, _CU | _QU, Expand];
```

Differential polynomials (DP):

Utils

```
DP_(α_→D_x_, β_→D_y_)[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :→ c D[λ, {x, m}, {y, n}]]
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, cp, CE, pow,
    gs = Generators /. {opts}, cs = Centrals /. {opts}
   },
   gp = Alternatives @@ gs;
   gp = gp | gp_; (* generators pattern *)
   sr = Thread[gs → Range@Length@gs]; (* sorting rule *)
   cp = Alternatives @@ cs; (* centrals pattern *)
   CE[ℰ_] := Collect[ℰ, _U, Expand];
   U_{i_}[ℰ_] := ℰ /. {t : cp :> t_i, u_U :> Replace[u, x_ :> x_i, 1]};
   U_{i_}[NCM[]] := U[];
   B[U@(x_)_{i_}, U@(y_)_{i_}] := B[U@x_i, U@y_i] = U_i@B[U@x, U@y];
   B[U@(x_)_{i_}, U@(y_)_{j_}] /; i =!= j := 0;
   B[U@y_, U@x_] := CE[-B[U@x, U@y]];
   x_ ** U[] := x; U[] ** x_ := x;
   (a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, CE[a b (x ** y)]];
   (a_ * x_U) ** y_ := CE[a (x ** y)]; x_ ** (a_ * y_U) := CE[a (x ** y)];
   U[xx___, x_] ** U[y_, yy___] := If[OrderedQ[{x, y} /. sr],
     U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
   U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
   U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
   U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
   U@{} = U[];
   U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
   U@{l_, r___} := U@{Expand[l], r};
   U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
   𝕆_U[poly_, specs___] := Module[{sp, null, vs, us},
     sp = Replace[{specs}, l_List :> l_null, {1}];
     vs = Join @@ (First /@ sp);
     us = Join @@ (sp /. l_{s_} :> (l /. x_{i_} :> x_s));
     CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) :> c U@(us^p)
       ]] /. x_null :> x
    ];
   pow[ℰ_, 0] = U[]; pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
   𝕊_U[ℰ_, ss___Rule] := CE@Total[
     CoefficientRules[ℰ, First /@ {ss}] /.
      (p_ → c_) :> c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
   S_{i_}[c_. * u_U] := CE[(c /. S_i[U, Centrals]) DeleteCases[u, _i] **
      U_i[NCM @@ Reverse@Cases[u, x_i :> S@U@x]]];
  ]
```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, (g_ → img_) :→ (m[U[g]] = img), {1}];
  m[U[]] = V[];
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U :→ m[u]];)
```

Meta-Operations

QLImplementation

```
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing $sl_2^{\gamma\epsilon}$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[CU@a, CU@y] = -γ CU@y; B[CU@x, CU@a] = -γ CU@x;
B[CU@x, CU@y] = 2 ϵ CU@a - t CU[];
(S@CU@y = -CU@y; S@CU@a = -CU@a; S@CU@x = -CU@x;)
S_i_[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{0.578125, {(28 t² γ⁴ + 116 t γ⁵ ϵ + 120 γ⁶ ϵ²) CU[y, y, y, x, x] +
   <<23>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that S is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y_1, a_1, x_1}},
 Table[HL@Simp[S_1[z1 ** z2] - S_1[z2] ** S_1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y1, a1, x1}},
 Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Implementing $\mathcal{U}_{\gamma\epsilon;\hbar}$

With $q = e^{\hbar\gamma\epsilon}$, $A = e^{-\hbar\epsilon a}$, $T = e^{\hbar t/2}$, and $[f, g]_q := fg - qgf$, our algebra is $\mathcal{U}_{\gamma\epsilon;\hbar} = \langle t, y, a, x\rangle/\mathcal{R}$, where $\mathcal{R} = \left([t, *] = 0,\ [a, y] = -\gamma y,\ [x, y]_q = \left(1 - T^2 A^2\right)/\hbar,\ [x, a] = -\gamma x\right)$.

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
q = SS[e^(γ ϵ ħ)]; (*T=SS[e^(ħ t/2)];*)
B[QU@a, QU@y] = -γ QU@y; B[QU@x, QU@a] = -γ QU@x;
B[QU@x, QU@y] = (q - 1) QU@{y, x} + 𝕆QU[SS[(1 - T^2 e^(-2 ϵ a ħ)) / ħ], {a}];
(S@QU@y = 𝕆QU[SS[-T^-2 e^(ħ ϵ a) y], {a, y}];
 S@QU@a = -QU@a; S@QU@x = 𝕆QU[SS[-e^(ħ ϵ a) x], {a, x}];)
S_i_ [QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$$\Big\{\Big\{\{QU[y], QU[y]\} \to 0,\ \{QU[y], QU[a]\} \to \gamma\, QU[y],$$
$$\{QU[y], QU[x]\} \to \left(t + \frac{t^2 \hbar}{2} + \frac{t^3 \hbar^2}{6} + \frac{t^4 \hbar^3}{24}\right) QU[\,] + \left(-2 \epsilon - 2 t \epsilon \hbar - t^2 \epsilon \hbar^2 - \frac{1}{3} t^3 \epsilon \hbar^3\right) QU[a] +$$
$$\left(2 \epsilon^2 \hbar + 2 t \epsilon^2 \hbar^2 + t^2 \epsilon^2 \hbar^3\right) QU[a, a] + \left(-\gamma \epsilon \hbar - \frac{1}{2} \gamma^2 \epsilon^2 \hbar^2\right) QU[y, x]\Big\},$$
$$\{\{QU[a], QU[y]\} \to -\gamma\, QU[y],\ \{QU[a], QU[a]\} \to 0,\ \{QU[a], QU[x]\} \to \gamma\, QU[x]\},$$
$$\Big\{\{QU[x], QU[y]\} \to \left(-t - \frac{t^2 \hbar}{2} - \frac{t^3 \hbar^2}{6} - \frac{t^4 \hbar^3}{24}\right) QU[\,] + \left(2 \epsilon + 2 t \epsilon \hbar + t^2 \epsilon \hbar^2 + \frac{1}{3} t^3 \epsilon \hbar^3\right) QU[a] +$$
$$\left(-2 \epsilon^2 \hbar - 2 t \epsilon^2 \hbar^2 - t^2 \epsilon^2 \hbar^3\right) QU[a, a] + \left(\gamma \epsilon \hbar + \frac{1}{2} \gamma^2 \epsilon^2 \hbar^2\right) QU[y, x],$$
$$\{QU[x], QU[a]\} \to -\gamma\, QU[x],\ \{QU[x], QU[x]\} \to 0\Big\}\Big\}$$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ $TħD=5, $TϵD=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp]
  }] // Timing
```

$\{50.7656,$
$\{(28\,t^2\,\gamma^4 + 116\,t\,\gamma^5\,\epsilon + 120\,\gamma^6\,\epsilon^2 + 28\,t^3\,\gamma^4\,\hbar + \ll 5 \gg + 7\,t^5\,\gamma^4\,\hbar^3 + \frac{361}{3}\,t^4\,\gamma^5\,\epsilon\,\hbar^3 + \frac{2495}{3}\,t^3\,\gamma^6\,\epsilon^2\,\hbar^3)$
$\mathrm{QU}[y, \ll 3 \gg, x] + \ll 22 \gg, 0\}\}$

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y₁, a₁, x₁}},
 Table[{z1, z2} → HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas}]]
```

$\{\{\{\mathrm{QU}[y_1], \mathrm{QU}[y_1]\} \to 0, \{\mathrm{QU}[y_1], \mathrm{QU}[a_1]\} \to 0, \{\mathrm{QU}[y_1], \mathrm{QU}[x_1]\} \to 0\},$
$\{\{\mathrm{QU}[a_1], \mathrm{QU}[y_1]\} \to 0, \{\mathrm{QU}[a_1], \mathrm{QU}[a_1]\} \to 0, \{\mathrm{QU}[a_1], \mathrm{QU}[x_1]\} \to 0\},$
$\{\{\mathrm{QU}[x_1], \mathrm{QU}[y_1]\} \to 0, \{\mathrm{QU}[x_1], \mathrm{QU}[a_1]\} \to 0, \{\mathrm{QU}[x_1], \mathrm{QU}[x_1]\} \to 0\}\}$

Verifying that $\mathrm{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ \$TħD=5, \$TϵD=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. {QU → CU, T → e^(ħ t/2)}, ħ → 0] - lhs] // HL
  }] // Timing
```

$\{32.375, \{2\,(8\,t^2\,\gamma^4 + 16\,t\,\gamma^5\,\epsilon)\,\mathrm{CU}[y, y, y, x, x] +$
$(8\,t\,\gamma^5\,\epsilon + 16\,\gamma^6\,\epsilon^2)\,\mathrm{CU}[y, y, y, x, x] + \ll 106 \gg + \mathrm{CU}[y, y, y, y, y, a, a, a, a, x, x, x, x],$
$2\,(8\,t\,\gamma^6\,\epsilon^2\,\hbar + 12\,t^2\,\gamma^6\,\epsilon^2\,\hbar^2 + \frac{28}{3}\,t^3\,\gamma^6\,\epsilon^2\,\hbar^3)\,\mathrm{QU}[y, y, y, x, x] + \ll 566 \gg +$
$(\gamma\,\epsilon\,\hbar + \frac{15}{2}\,\gamma^2\,\epsilon^2\,\hbar^2)\,\mathrm{QU}[y, y, y, \ll 7 \gg, x, x, x], 0\}\}$

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -CU@x, a → -CU@a, x → -CU@y}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y → 𝕆_QU[SS[-T^-1 e^(ħ ϵ a) x], {a, x}],
   a → -QU@a, x → 𝕆_QU[SS[-T^-1 e^(ħ ϵ a) y], {a, y}]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}]]
```

$\{\mathrm{CU}[y] \to -\mathrm{CU}[x] \to \mathrm{CU}[y], \mathrm{CU}[a] \to -\mathrm{CU}[a] \to \mathrm{CU}[a], \mathrm{CU}[x] \to -\mathrm{CU}[y] \to \mathrm{CU}[x]\}$

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}]]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Qθ[Qθ[z]]], {z, bas}]]
```

$\{QU[y] \to \left(-1+\frac{t\hbar}{2}-\frac{t^2\hbar^2}{8}+\frac{t^3\hbar^3}{48}\right) QU[x] + \left(-\epsilon\hbar+\frac{1}{2}t\epsilon\hbar^2-\frac{1}{8}t^2\epsilon\hbar^3\right) QU[a, x] + \left(-\frac{1}{2}\epsilon^2\hbar^2+\frac{1}{4}t\epsilon^2\hbar^3\right) QU[a, a, x] \to QU[y], QU[a] \to -QU[a] \to QU[a],$

$QU[x] \to \left(-1+\frac{t\hbar}{2}+\gamma\epsilon\hbar-\frac{t^2\hbar^2}{8}-\frac{1}{2}t\gamma\epsilon\hbar^2-\frac{1}{2}\gamma^2\epsilon^2\hbar^2+\frac{t^3\hbar^3}{48}+\frac{1}{8}t^2\gamma\epsilon\hbar^3+\frac{1}{4}t\gamma^2\epsilon^2\hbar^3\right) QU[y] + \left(-\epsilon\hbar+\frac{1}{2}t\epsilon\hbar^2+\gamma\epsilon^2\hbar^2-\frac{1}{8}t^2\epsilon\hbar^3-\frac{1}{2}t\gamma\epsilon^2\hbar^3\right) QU[y, a] + \left(-\frac{1}{2}\epsilon^2\hbar^2+\frac{1}{4}t\epsilon^2\hbar^3\right) QU[y, a, a] \to QU[x]\}$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}]]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

$$\left\{\text{Short}[\text{tt} = \text{QU}[y] ** \text{Q}\theta[\text{QU}[y]] - \text{Q}\theta[\text{QU}[y]] ** \text{QU}[y]],\ \text{tt} \;/.\; t \to \frac{t}{\hbar}\right\}$$

$$\Big\{\ll 1\gg,\ -\left(\frac{t}{\hbar}+\frac{t^3}{24\hbar}+\frac{t^5}{1920\hbar}\right)\text{QU}[\,] - \left(-2\epsilon - t\epsilon - \frac{t^2\epsilon}{4} - \frac{t^3\epsilon}{24} - \frac{t^4\epsilon}{192} - \frac{t^5\epsilon}{1920}\right)\text{QU}[a] -$$
$$\left(t\epsilon + \frac{t^3\epsilon}{24} + \frac{t^5\epsilon}{1920}\right)\text{QU}[a] - \left(\frac{1}{2}t\epsilon^2\hbar + \frac{1}{48}t^3\epsilon^2\hbar\right)\text{QU}[a, a] -$$
$$\left(-2\epsilon^2\hbar - t\epsilon^2\hbar - \frac{1}{4}t^2\epsilon^2\hbar - \frac{1}{24}t^3\epsilon^2\hbar - \frac{1}{192}t^4\epsilon^2\hbar\right)\text{QU}[a, a] -$$
$$\left(2\epsilon^2\hbar + t\epsilon^2\hbar + \frac{1}{4}t^2\epsilon^2\hbar + \frac{1}{24}t^3\epsilon^2\hbar + \frac{1}{192}t^4\epsilon^2\hbar\right)\text{QU}[a, a] +$$
$$\left(-1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{48} - \frac{t^4}{384} + \frac{t^5}{3840}\right)\text{QU}[y, x] - \Big(\gamma\epsilon\hbar - \frac{1}{2}t\gamma\epsilon\hbar + \frac{1}{8}t^2\gamma\epsilon\hbar - \frac{1}{48}t^3\gamma\epsilon\hbar +$$
$$\frac{1}{384}t^4\gamma\epsilon\hbar + \gamma^2\epsilon^2\hbar^2 - \frac{1}{2}t\gamma^2\epsilon^2\hbar^2 + \frac{1}{8}t^2\gamma^2\epsilon^2\hbar^2 - \frac{1}{48}t^3\gamma^2\epsilon^2\hbar^2\Big)\text{QU}[y, x] -$$
$$\left(-\frac{1}{2}\gamma^2\epsilon^2\hbar^2 + \frac{1}{4}t\gamma^2\epsilon^2\hbar^2 - \frac{1}{16}t^2\gamma^2\epsilon^2\hbar^2 + \frac{1}{96}t^3\gamma^2\epsilon^2\hbar^2\right)\text{QU}[y, x] -$$
$$\Big(-1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{48} - \frac{t^4}{384} + \frac{t^5}{3840} - \gamma\epsilon\hbar + \frac{1}{2}t\gamma\epsilon\hbar - \frac{1}{8}t^2\gamma\epsilon\hbar + \frac{1}{48}t^3\gamma\epsilon\hbar -$$
$$\frac{1}{384}t^4\gamma\epsilon\hbar - \frac{1}{2}\gamma^2\epsilon^2\hbar^2 + \frac{1}{4}t\gamma^2\epsilon^2\hbar^2 - \frac{1}{16}t^2\gamma^2\epsilon^2\hbar^2 + \frac{1}{96}t^3\gamma^2\epsilon^2\hbar^2\Big)\text{QU}[y, x] +$$
$$\left(-\epsilon\hbar + \frac{t\epsilon\hbar}{2} - \frac{1}{8}t^2\epsilon\hbar + \frac{1}{48}t^3\epsilon\hbar - \frac{1}{384}t^4\epsilon\hbar\right)\text{QU}[y, a, x] -$$
$$\left(\gamma\epsilon^2\hbar^2 - \frac{1}{2}t\gamma\epsilon^2\hbar^2 + \frac{1}{8}t^2\gamma\epsilon^2\hbar^2 - \frac{1}{48}t^3\gamma\epsilon^2\hbar^2\right)\text{QU}[y, a, x] -$$
$$\left(-\epsilon\hbar + \frac{t\epsilon\hbar}{2} - \frac{1}{8}t^2\epsilon\hbar + \frac{1}{48}t^3\epsilon\hbar - \frac{1}{384}t^4\epsilon\hbar - \gamma\epsilon^2\hbar^2 + \frac{1}{2}t\gamma\epsilon^2\hbar^2 - \frac{1}{8}t^2\gamma\epsilon^2\hbar^2 + \frac{1}{48}t^3\gamma\epsilon^2\hbar^2\right)$$
$$\text{QU}[y, a, x]\Big\}$$

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\text{A}\mathbb{D}\$f = \frac{\gamma}{\hbar}e^{\hbar\left(\frac{t}{2}-(a+\gamma)\epsilon\right)}\left(\left(\text{Cosh}\left[\hbar\left(a\epsilon + \frac{\gamma\epsilon}{2} - \frac{t}{2}\right)\right] - \text{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\epsilon}{2}\right)^2 + \epsilon\omega}\right]\right)\Big/\left(\text{Sinh}\left[\frac{\gamma\epsilon\hbar}{2}\right]\left(a^2\epsilon + a\gamma\epsilon - a t - \omega\right)\right)\right);$$

Scaling behaviour of A𝔻$f:

$$\text{HL@Simplify}\left[\text{A}\mathbb{D}\$f == \left(\left(\text{A}\mathbb{D}\$f \;/.\; \gamma \to 1\right) \;/.\; \left\{\epsilon \to \gamma\epsilon,\ a \to \gamma^{-1}a,\ \omega \to \gamma^{-1}\omega\right\}\right)\right]$$

True

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

True

ADeq

```
A𝔻$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
DeclareMorphism[A𝔻, QU → CU,
 {a → CU@a, x → CU@x, y → 𝕊_CU[SS[A𝔻$f], a → CU[a], ω → A𝔻$ω] ** CU@y}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$\mathrm{S𝔻\$g} = \sqrt{\left(\left(\mathrm{Cosh}\left[\frac{\hbar}{2}\sqrt{t^2+\gamma^2\epsilon^2+4\epsilon\varpi}\right]-\mathrm{Cosh}\left[\frac{\hbar}{2}\left(t-(2a+\gamma)\epsilon\right)\right]\right)\Big/\left(\mathrm{Sinh}\left[\frac{\gamma\epsilon\hbar}{2}\right]\left(t(2a+\gamma)-2a(a+\gamma)\epsilon+2\varpi\right)\hbar/(2\gamma)\right)\right)};$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{SD$P = (Cosh[ħ ((ϵ-t)/2 + ϵ a)] - Cosh[ħ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]]) / (ħ Sinh[-ϵ ħ/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 Simplify[SD$g == FullSimplify[
    Sqrt[SD$Q] /. c → a + 1/2 /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}\,(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}\,(t^2+\epsilon^2)+\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a\,(t-\epsilon)-a^2\,\epsilon+\varpi\right)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\right.$$

$$\left.-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}\,(t-2\,c\,\epsilon)\,\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\,\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(4\,c\,t+\epsilon-4\,c^2\,\epsilon+4\,\varpi\right)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True}\right\}$$

SDeq

```
SD$f = FullSimplify[e^(ħ (t/2-ϵ a)) (SD$g /. {a → -a, t → -t})];
```

SDeq

```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ CU[]/2;
```

SDeq

```
DeclareMorphism[SD, QU → CU, {a → CU@a,
  x → S_CU[SS[SD$f], a → CU[a], ϖ → SD$ϖ] ** CU@x,
  y → S_CU[SS[SD$g] , a → CU[a], ϖ → SD$ϖ] ** CU@y
 }]
```

Verifying the θ-symmetry:

```
Table[HL@Simplify[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

```
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@Simp[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

R in QU.

Quesne's formula:

Quesne

$$\mathbb{e}_{q\_,n\_}[x\_] := \text{Exp}\left[\sum_{k=1}^{n} \frac{(1-q)^k}{k\,(1-q^k)} x^k\right];\ \mathbb{e}_{q\_}[x\_] := \mathbb{e}_{q,\$T\epsilon D}[x]$$

```
Table[Together@SeriesCoefficient[e_{ρ,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1,\ 1,\ \frac{1}{1+\rho},\ \frac{1}{(1+\rho)\,(1+\rho+\rho^2)},\ \frac{1}{(1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)},\ 1\Big/\left((1+\rho)^2\,(1+\rho^2)\,(1+\rho+\rho^2)\,(1+\rho+\rho^2+\rho^3+\rho^4)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, ρ] SeriesCoefficient[e_{ρ,5}[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

R

$$\text{QU}[R_{i\_,j\_}] := \mathbb{O}_{QU}\left[\text{SS}\left[\mathbb{e}^{\hbar b_1 a_2}\,\mathbb{e}_q[\hbar\, y_1 x_2] \text{ /. } b_1 \to \gamma^{-1}(\epsilon\, a_1 - t_i)\right],\ \{y_1, a_1\}_i,\ \{a_2, x_2\}_j\right];$$
$$\text{QU}\left[R^{-1}_{i\_,j\_}\right] := S_j\text{@QU}[R_{i,j}];$$

```
QU[R_{3,4}] // Short
```

$$\text{QU}[\,] + \frac{\epsilon\,\hbar\,\text{QU}[a_3, a_4]}{\gamma} + \hbar\,\text{QU}[y_3, x_4] + \ll 20 \gg + \frac{\hbar^3\,\text{QU}[y_3, a_4, a_4, x_4]\,t_3^2}{2\,\gamma^2} - \frac{\hbar^3\,\text{QU}[a_4, a_4, a_4]\,t_3^3}{6\,\gamma^3}$$

Verifying R2 (~2 secs @ $TℏD=4, $TϵD=2):

```
QU[R_{1,2} ** R^{-1}_{1,2}] // Simp // HL // Timing
```

{0.390625, QU[]}

Verifying R3 (~156 secs @ $TℏD=4, $TϵD=2):

```
{Short[lhs = QU[R_{1,2} ** R_{1,3} ** R_{2,3}]], HL@Simp[lhs - QU[R_{2,3} ** R_{1,3} ** R_{1,2}]]} // Timing
```

$$\left\{2.46875,\ \left\{\text{QU}[\,] + \frac{\epsilon\,\hbar\,\text{QU}[a_1, a_2]}{\gamma} + \ll 347 \gg + \text{QU}[y_1, a_2, x_3]\,(2\,\epsilon\,\hbar^2 + 2\,\epsilon\,\hbar^3\,t_2) + \text{QU}[y_1, x_3]\left(-\hbar^2\,t_2 - \frac{1}{2}\hbar^3\,t_2^2\right),\ 0\right\}\right\}$$

The representation ρ

rho

```
{ρ@(CU | QU)@y, ρ@(CU | QU)@a} = {(0 0; ϵ 0), (γ 0; 0 0)};
ρ@CU@x = (0 γ; 0 0); ρ@QU@x = SS@(0 (1 - e^(-γ ϵ ħ)) / (ϵ ħ); 0 0);
ρ[e^𝓔_] := MatrixExp[ρ[𝓔]];
ρ[𝓔_] := (𝓔 /. {t → γ ϵ, (U : CU | QU)[u___] :> Dot[(1 0; 0 1), Sequence @@ (ρ /@ U /@ {u})]})
```

Verifying that ρ represents CU and QU:

```
Table[ρ[z1 ** z2] == ρ[z1].ρ[z2] // Simplify // HL,
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

The Classical Logos $C\Lambda$

Lemma 3C. To degree k,
$\mathbb{O}_{CU}\left(e^{\eta y+\xi x+\delta y x} \mid x y\right) = \mathbb{O}_{CU}\left(v\, e^{v(-t\xi\eta+\eta y+\xi x+\delta y x)}\, C\Lambda_k(\epsilon, \gamma, y, a, x, \eta, \xi, \delta) \mid y a x\right)$, with $v = (1+t\delta)^{-1}$ and where $C\Lambda_k(\epsilon, \gamma, y, a, x, \eta, \xi, \delta)$ is a fixed polynomial of degree at most $4k$ in y, $\sqrt{a}$, x, η, ξ, with scalar coefficients.

Comment. Even better, $\log(C\Lambda_k)$ is of degree at most $2k+2$ in said variables.

```
eqn = ρ[e^(ξ CU@x)].ρ[e^(η CU@y)] == ρ[e^(d CU@y)].ρ[e^(c (t CU[] - 2 ϵ CU@a))].ρ[e^(b CU@x)]
```

$$\{\{1+\gamma\epsilon\eta\xi, \gamma\xi\}, \{\epsilon\eta, 1\}\} == \left\{\left\{e^{-c\gamma\epsilon}, b\, e^{-c\gamma\epsilon}\gamma\right\}, \left\{d\, e^{-c\gamma\epsilon}\epsilon, e^{c\gamma\epsilon}+b\,d\, e^{-c\gamma\epsilon}\gamma\epsilon\right\}\right\}$$

```
sol = Solve[Thread[Flatten /@ eqn], {d, b, c}][[1]] /. C[1] → 0
```

$$\left\{d \to \frac{\eta}{1+\gamma\epsilon\eta\xi}, b \to \frac{\xi}{1+\gamma\epsilon\eta\xi}, c \to \frac{\mathrm{Log}\left[\frac{1}{1+\gamma\epsilon\eta\xi}\right]}{\gamma\epsilon}\right\}$$

Proof of Lemma 3C. We know that $\mathbb{O}_{CU}\left(e^{\xi x+\eta y} \mid x y\right) = \mathbb{O}_{CU}\left(e^{ct+ay-2\epsilon c a+bx} \mid y a x\right)$, with $\left\{d \to \frac{\eta}{1+\gamma\epsilon\eta\xi}, b \to \frac{\xi}{1+\gamma\epsilon\eta\xi}, c \to \frac{\mathrm{Log}[1+\gamma\epsilon\eta\xi]}{-\gamma\epsilon}\right\}$. Expanding in ϵ we get
$\mathbb{O}_{CU}\left(e^{\xi x+\eta y} \mid x y\right) = \mathbb{O}_{CU}\left(\lambda_\epsilon(\xi, \eta)\, e^{\eta y+\xi x-\eta\xi t} \mid y a x\right) = \mathbb{O}_{CU}\left(\lambda_\epsilon(\partial_x, \partial_y)\, e^{\eta y+\xi x-\eta\xi t} \mid y a x\right)$ and so
$\mathbb{O}_{CU}\left(e^{\eta y+\xi x+\delta y x} \mid x y\right) = \mathbb{O}\left(\lambda_\epsilon(\partial_x, \partial_y)\, e^{\delta\partial_\eta\partial_\xi}\, e^{\eta y+\xi x-\eta\xi t} \mid y a x\right) = \mathbb{O}\left(\lambda_\epsilon(\partial_x, \partial_y)\, v\, e^{v(-t\xi\eta+\eta y+\xi x+\delta y x)} \mid y a x\right)$.

Logos

```
SSϵ[ℰ_] := Block[{ϵ}, Collect[Normal@Series[ℰ, {ϵ, 0, $TϵD}], ϵ, Together]];
(* Shielded ϵ-Series *)
CΛ[t1_, y1_, a1_, x1_, ξ1_, η1_, δ_] := Module[
   {eqn, d, b, c, sol, λ, q, ν, ξ, η},
   eqn = ρ[e^(ξ CU@x)].ρ[e^(η CU@y)] == ρ[e^(d CU@y)].ρ[e^(c (t CU[] - 2 ϵ CU@a))].ρ[e^(b CU@x)];
   sol = Solve[Thread[Flatten /@ eqn], {d, b, c}][[1]] /. C[1] → 0;
   λ = Simplify[e^(-η y-ξ x+η ξ t) SSϵ[e^(c t + d y - 2 ϵ c a + b x) /. sol]];
   q = e^(ν (-t ξ η+η y+ξ x+δ y x));
   Collect[ν q^-1 DP_(ξ→D_x,η→D_y)[λ][q] /. ν → (1 + t δ)^-1, ϵ, Simplify] /.
    {t → t1, y → y1, a → a1, x → x1, ξ → ξ1, η → η1}
  ];
```

CΛ[t, y, a, x, ξ, η, δ]

$$\frac{1}{1+t\,\delta}+\frac{1}{24\left(1+t\,\delta\right)^{9}}$$

$$\epsilon^{2}\left(48\,a^{2}\left(1+t\,\delta\right)^{4}\left(2\,\delta^{2}\left(1+t\,\delta\right)^{2}+4\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}\right)-\right.$$
$$24\,a\,\gamma\left(1+t\,\delta\right)^{4}\left(2\,\delta^{2}\left(1+t\,\delta\right)^{2}+4\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}\right)-$$
$$48\,a\,y\,\gamma\left(1+t\,\delta\right)^{3}\,(x\,\delta+\eta)$$
$$\left(6\,\delta^{2}\left(1+t\,\delta\right)^{2}+6\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}\right)+24\,y\,\gamma^{2}\left(1+t\,\delta\right)^{3}$$
$$(x\,\delta+\eta)\left(6\,\delta^{2}\left(1+t\,\delta\right)^{2}+6\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}\right)-48\,a\,x\,\gamma$$
$$\left(1+t\,\delta\right)^{3}\,(y\,\delta+\xi)\left(6\,\delta^{2}\left(1+t\,\delta\right)^{2}+6\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}\right)+$$
$$24\,x\,\gamma^{2}\left(1+t\,\delta\right)^{3}\,(y\,\delta+\xi)\left(6\,\delta^{2}\left(1+t\,\delta\right)^{2}+6\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}\right)+12\,y^{2}\,\gamma^{2}\left(1+t\,\delta\right)^{2}\,(x\,\delta+\eta)^{2}$$
$$\left(12\,\delta^{2}\left(1+t\,\delta\right)^{2}+8\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}\right)+12\,x^{2}\,\gamma^{2}$$
$$\left(1+t\,\delta\right)^{2}\,(y\,\delta+\xi)^{2}\left(12\,\delta^{2}\left(1+t\,\delta\right)^{2}+8\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}\right)+$$
$$24\,a\,t\,\gamma\left(1+t\,\delta\right)^{2}\left(6\,\delta^{3}\left(1+t\,\delta\right)^{3}+18\,\delta^{2}\left(1+t\,\delta\right)^{2}\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.9\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}+(x\,\delta+\eta)^{3}\,(y\,\delta+\xi)^{3}\right)-$$
$$8\,t\left(\gamma+t\,\gamma\,\delta\right)^{2}\left(6\,\delta^{3}\left(1+t\,\delta\right)^{3}+18\,\delta^{2}\left(1+t\,\delta\right)^{2}\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.9\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}+(x\,\delta+\eta)^{3}\,(y\,\delta+\xi)^{3}\right)+$$
$$24\,x\,y\left(\gamma+t\,\gamma\,\delta\right)^{2}\left(6\,\delta^{3}\left(1+t\,\delta\right)^{3}+18\,\delta^{2}\left(1+t\,\delta\right)^{2}\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.9\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}+(x\,\delta+\eta)^{3}\,(y\,\delta+\xi)^{3}\right)-$$
$$12\,t\,y\,\gamma^{2}\left(1+t\,\delta\right)\,(x\,\delta+\eta)\left(24\,\delta^{3}\left(1+t\,\delta\right)^{3}+36\,\delta^{2}\left(1+t\,\delta\right)^{2}\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.12\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}+(x\,\delta+\eta)^{3}\,(y\,\delta+\xi)^{3}\right)-$$
$$12\,t\,x\,\gamma^{2}\left(1+t\,\delta\right)\,(y\,\delta+\xi)\left(24\,\delta^{3}\left(1+t\,\delta\right)^{3}+36\,\delta^{2}\left(1+t\,\delta\right)^{2}\,(x\,\delta+\eta)\,(y\,\delta+\xi)+\right.$$
$$\left.12\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}+(x\,\delta+\eta)^{3}\,(y\,\delta+\xi)^{3}\right)+$$
$$3\,t^{2}\,\gamma^{2}\left(24\,\delta^{4}\left(1+t\,\delta\right)^{4}+96\,\delta^{3}\left(1+t\,\delta\right)^{3}\,(x\,\delta+\eta)\,(y\,\delta+\xi)+72\,\delta^{2}\left(1+t\,\delta\right)^{2}\right.$$
$$\left.\left.(x\,\delta+\eta)^{2}\,(y\,\delta+\xi)^{2}+16\,\delta\left(1+t\,\delta\right)\,(x\,\delta+\eta)^{3}\,(y\,\delta+\xi)^{3}+(x\,\delta+\eta)^{4}\,(y\,\delta+\xi)^{4}\right)\right)+$$
$$\frac{1}{2\left(1+t\,\delta\right)^{5}}\,\epsilon\left(4\,a\left(1+t\,\delta\right)^{2}\left(\left(t+x\,y\right)\,\delta^{2}+\eta\,\xi+\delta\left(1+y\,\eta+x\,\xi\right)\right)+\right.$$
$$\gamma\left(2\,t^{3}\,\delta^{4}+4\,t^{2}\,\delta^{2}\left(\delta-x\,y\,\delta^{2}+\eta\,\xi\right)-2\left(y\,\eta\left(\delta\left(2+y\,\eta\right)+\eta\,\xi\right)+x^{2}\,\delta\left(2\,y^{2}\,\delta^{2}+3\,y\,\delta\,\xi+\xi^{2}\right)+\right.\right.$$
$$\left.x\left(3\,y^{2}\,\delta^{2}\,\eta+4\,y\,\delta\,(\delta+\eta\,\xi)+\xi\left(2\,\delta+\eta\,\xi\right)\right)\right)-t\left(3\,x^{2}\,y^{2}\,\delta^{4}-4\,\delta\,\eta\,\xi-\eta^{2}\,\xi^{2}+\right.$$
$$\left.\left.\left.4\,x\,y\,\delta^{3}\left(3+y\,\eta+x\,\xi\right)+\delta^{2}\left(-2+y^{2}\,\eta^{2}+4\,x\,\xi+x^{2}\,\xi^{2}+4\,y\,(\eta+x\,\eta\,\xi)\right)\right)\right)\right)$$

```
{Short[lhs = 𝕆CU[SS[ℯ^(ℏ (ξ x+η y+δ x y))], {x, y}], 5], HL[lhs ==
   𝕆CU[SS[ℯ^(ℏ ν (ξ x+η y+δ x y-t ℏ ξ η)) CΛ[t, y, a, x, ℏ ξ, ℏ η, ℏ δ] /. ν → (1 + ℏ t δ)^-1], {y, a, x}]]}
```

$\{(1 - t\,\delta\,\hbar + t^2\,\delta^2\,\hbar^2 + t\,\gamma\,\delta^2\,\epsilon\,\hbar^2 - t\,\eta\,\xi\,\hbar^2 - t^3\,\delta^3\,\hbar^3 - 3\,t^2\,\gamma\,\delta^3\,\epsilon\,\hbar^3 - 2\,t\,\gamma^2\,\delta^3\,\epsilon^2\,\hbar^3 + 2\,t^2\,\delta\,\eta\,\xi\,\hbar^3 + 2\,t\,\gamma\,\delta\,\epsilon\,\eta\,\xi\,\hbar^3)\,\mathrm{CU}[\,] + (2\,\delta\,\epsilon\,\hbar - 4\,t\,\delta^2\,\epsilon\,\hbar^2 - 2\,\gamma\,\delta^2\,\epsilon^2\,\hbar^2 + 2\,\epsilon\,\eta\,\xi\,\hbar^2 + 6\,t^2\,\delta^3\,\epsilon\,\hbar^3 + 12\,t\,\gamma\,\delta^3\,\epsilon^2\,\hbar^3 - 8\,t\,\delta\,\epsilon\,\eta\,\xi\,\hbar^3 - 4\,\gamma\,\delta\,\epsilon^2\,\eta\,\xi\,\hbar^3)\,\mathrm{CU}[a] + (\xi\,\hbar - 2\,t\,\delta\,\xi\,\hbar^2 - 2\,\gamma\,\delta\,\epsilon\,\xi\,\hbar^2 + 3\,t^2\,\delta^2\,\xi\,\hbar^3 + 9\,t\,\gamma\,\delta^2\,\epsilon\,\xi\,\hbar^3 + 6\,\gamma^2\,\delta^2\,\epsilon^2\,\xi\,\hbar^3 - t\,\eta\,\xi^2\,\hbar^3 - \gamma\,\epsilon\,\eta\,\xi^2\,\hbar^3)\,\mathrm{CU}[x] + \ll 23 \gg + \frac{1}{2}\,\delta^2\,\xi\,\hbar^3\,\mathrm{CU}[y, y, x, x, x] + \frac{1}{2}\,\delta^2\,\eta\,\hbar^3\,\mathrm{CU}[y, y, y, x, x] + \frac{1}{6}\,\delta^3\,\hbar^3\,\mathrm{CU}[y, y, y, x, x, x],\ \mathrm{True}\}$

C𝕆 and Swaps

Swaps from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
SetAttributes[C𝕆, Orderless];
CU@C𝕆[specs___, 𝔼[L_, Q_, P_]] := 𝕆CU[SS[ℯ^(L+Q) P], specs]
```

```
CU@C𝕆[𝔼[ℏ t1 a2, ℏ t1^-1 (ℯ^t1 - 1) y1 x2, 1 + ϵ x1 y2], {y1, x1}1, {x2, a2, y2}2] // Short
```

$\mathrm{CU}[\,] + \ll 27 \gg + \mathrm{CU}[y_1, x_1]\left(-\gamma\,\epsilon\,\hbar^2\,t_2 + e^{t_1}\,\gamma\,\epsilon\,\hbar^2\,t_2 + \frac{\epsilon\,\hbar \ll 1 \gg}{t_1} - \frac{\ll 1 \gg}{\ll 1 \gg} + \frac{1}{2}\,\gamma^2\,\epsilon\,\hbar^3\,t_1\,t_2 - \frac{1}{2}\,e^{t_1}\,\gamma^2\,\epsilon\,\hbar^3\,t_1\,t_2\right)$

```
HL[ρ[ℯ^(ξ CU@x)].ρ[ℯ^(α CU@a)] == ρ[ℯ^(α CU@a)].ρ[ℯ^(ℯ^(-γ α) ξ CU@x)]]
```

True

SW

```
SW_{x_i_, a_j_}[C𝕆[{lh___, x_i_, a_j_, rh___}_s_, more___, 𝔼[L_, Q_, P_]]] :=
 C𝕆[{lh, a_j, x_i, rh}_s, more,
  With[{q = ℯ^(-γ α) ξ x_i + α a_j},
   𝔼[L, ℯ^(-γ α) ξ x_i + (Q /. x_i → 0), ℯ^-q DP_{x_i→D_ξ, a_j→D_α}[P][ℯ^q]] /. {α → ∂_{a_j} L, ξ → ∂_{x_i} Q}]]
```

```
c𝕆 = C𝕆[𝔼[ℏ t1 a2, ℏ t1^-1 (ℯ^t1 - 1) y1 x2, 1 + ϵ x1 y2], {y1, x1}1, {x2, a2, y2}2]
```

$\mathrm{C}\mathbb{O}\left[\{y_1, x_1\}_1, \{x_2, a_2, y_2\}_2, \mathbb{E}\left[\hbar\,a_2\,t_1, \frac{(-1 + e^{t_1})\,\hbar\,x_2\,y_1}{t_1}, 1 + \epsilon\,x_1\,y_2\right]\right]$

```
SW_{x2,a2}[c𝕆]
```

$\mathrm{C}\mathbb{O}\left[\{y_1, x_1\}_1, \{a_2, x_2, y_2\}_2, \mathbb{E}\left[\hbar\,a_2\,t_1, \frac{e^{-\gamma\,\hbar\,t_1}\,(-1 + e^{t_1})\,\hbar\,x_2\,y_1}{t_1}, 1 + \epsilon\,x_1\,y_2\right]\right]$

```
With[{cω = Cω[{y₁, x₁}₁, {x₂, a₂, y₂}₂, 𝔼[ℏ t₁ a₂, ℏ t₁^-1 (e^t₁ - 1) y₁ x₂, 1 + ϵ x₁ y₂]]},
 HL[CU[cω] == CU[cω // SW_{x₂,a₂}]]]
```

True

```
With[{cω = Cω[{y₁, a₁, x₁}₁, {x₂, a₂, y₂}₂,
    𝔼[ℏ (l₁₁ t₁ a₁ + l₁₂ t₁ a₂ + l₂₁ t₂ a₁ + l₂₂ t₂ a₂), ℏ (γ₁₁ x₁ y₁ + γ₁₂ x₁ y₂ + γ₂₁ x₂ y₁ + γ₂₂ x₂ y₂),
     1 + ϵ (l₁ a₁ + l₂ a₂ + p₁₁ x₁ y₁ + p₁₂ x₁ y₂ + p₂₁ x₂ y₁ + p₂₂ x₂ y₂)]]},
 {CU[cω] // Short, HL[CU[cω] == CU[cω // SW_{x₂,a₂}]]}
]
```

$\left\{\mathrm{CU}[a_1, a_1, a_1, a_1]\left(\frac{1}{6}\epsilon\hbar^3 l_1 l_{11}^3 t_1^3 + \frac{1}{2}\epsilon\hbar^3 l_1 l_{11}^2 l_{21} t_1^2 t_2 + \frac{1}{2}\epsilon \ll 5\gg t_2^2 + \frac{1}{6}\epsilon\hbar^3 l_1 l_{21}^3 t_2^3\right) + \ll 181\gg, \text{True}\right\}$

SW

```
SW_{x_i_,y_j_→k_}[Cω[{lh___, x_i_, y_j_, rh___}_s_, more___, 𝔼[L_, Q_, P_]]] :=
 Cω[{lh, y_k, a_k, x_k, rh}_s, more,
  With[{q = ν (ξ x_k + η y_k + δ x_k y_k - t_k ξ η)},
   𝔼[L, q + (Q /. x_i | y_j → 0), e^-q DP_{x_i→D_ξ, y_j→D_η}[P][CΛ[t_k, y_k, a_k, x_k, ξ, η, δ] e^q]] /.
    ν → (1 + t_k δ)^-1 /. {ξ → (∂_{x_i} Q /. y_j → 0), η → (∂_{y_j} Q /. x_i → 0), δ → ∂_{x_i,y_j} Q}]]
```

```
With[{cω = Cω[{x₁, y₁}₁, {x₂, a₂, y₂}₂,
    𝔼[ℏ (l₁₂ t₁ a₂ + l₂₂ t₂ a₂), ℏ (γ₁₁ x₁ y₁ + γ₁₂ x₁ y₂ + γ₂₁ x₂ y₁ + γ₂₂ x₂ y₂),
     1 + ϵ (l₂ a₂ + p₁₁ x₁ y₁ + p₁₂ x₁ y₂ + p₂₁ x₂ y₁ + p₂₂ x₂ y₂)]]},
 {CU[cω] // Short, HL[CU[cω] == CU[cω // SW_{x₁,y₁→1}]]}
]
```

$\left\{12\epsilon^2\hbar^3\,\mathrm{CU}[y_1, a_1, a_1, x_1]\,\gamma_{11}^3 + \ll 159\gg + \mathrm{CU}[\,]\left(\ll 301\gg + \epsilon\hbar^3 p_{11} t_1 t_2^3 \gamma_{22}^3 + 4\epsilon\hbar^3 p_{22} t_2^4 \gamma_{22}^3\right), \text{True}\right\}$

The Quantum Logos QΛ

Goal 1: In QU, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$.

First compute $G = e^{\xi x}\, y\, e^{-\xi x}$, a finite sum.

```
adx[ℰ_] := Simp[QU@x ** ℰ - ℰ ** QU@x];
G = Simp[NestList[adx, QU@y, $TϵD + 1].Table[ξ^k / k !, {k, 0, $TϵD + 1}]]
```

$$\left(-t\,\xi-\frac{1}{2}t^2\xi\hbar-\frac{1}{6}t^3\xi\hbar^2-\frac{1}{24}t^4\xi\hbar^3-\frac{1}{120}t^5\xi\hbar^4-\frac{1}{720}t^6\xi\hbar^5\right)\mathrm{QU}[\,]+$$
$$\left(2\epsilon\xi+2t\epsilon\xi\hbar+t^2\epsilon\xi\hbar^2+\frac{1}{3}t^3\epsilon\xi\hbar^3+\frac{1}{12}t^4\epsilon\xi\hbar^4+\frac{1}{60}t^5\epsilon\xi\hbar^5\right)\mathrm{QU}[a]+$$
$$\left(-\gamma\epsilon\xi^2-\frac{3}{2}t\gamma\epsilon\xi^2\hbar-\gamma^2\epsilon^2\xi^2\hbar-\frac{3}{4}t^2\gamma\epsilon\xi^2\hbar^2-\frac{5}{4}t\gamma^2\epsilon^2\xi^2\hbar^2-\frac{1}{4}t^3\gamma\epsilon\xi^2\hbar^3-\frac{5}{8}t^2\gamma^2\epsilon^2\xi^2\hbar^3-\right.$$
$$\left.\frac{1}{16}t^4\gamma\epsilon\xi^2\hbar^4-\frac{5}{24}t^3\gamma^2\epsilon^2\xi^2\hbar^4-\frac{1}{80}t^5\gamma\epsilon\xi^2\hbar^5-\frac{5}{96}t^4\gamma^2\epsilon^2\xi^2\hbar^5\right)\mathrm{QU}[x]+$$
$$\mathrm{QU}[y]+\left(-2\epsilon^2\xi\hbar-2t\epsilon^2\xi\hbar^2-t^2\epsilon^2\xi\hbar^3-\frac{1}{3}t^3\epsilon^2\xi\hbar^4-\frac{1}{12}t^4\epsilon^2\xi\hbar^5\right)\mathrm{QU}[a,a]+$$
$$\left(3\gamma\epsilon^2\xi^2\hbar+3t\gamma\epsilon^2\xi^2\hbar^2+\frac{3}{2}t^2\gamma\epsilon^2\xi^2\hbar^3+\frac{1}{2}t^3\gamma\epsilon^2\xi^2\hbar^4+\frac{1}{8}t^4\gamma\epsilon^2\xi^2\hbar^5\right)\mathrm{QU}[a,x]+$$
$$\left(-\gamma^2\epsilon^2\xi^3\hbar-\frac{7}{6}t\gamma^2\epsilon^2\xi^3\hbar^2-\frac{7}{12}t^2\gamma^2\epsilon^2\xi^3\hbar^3-\frac{7}{36}t^3\gamma^2\epsilon^2\xi^3\hbar^4-\frac{7}{144}t^4\gamma^2\epsilon^2\xi^3\hbar^5\right)\mathrm{QU}[x,x]+$$
$$\left(\gamma\epsilon\xi\hbar+\frac{1}{2}\gamma^2\epsilon^2\xi\hbar^2\right)\mathrm{QU}[y,x]+\frac{1}{2}\gamma^2\epsilon^2\xi^2\hbar^2\,\mathrm{QU}[y,x,x]$$

```
G /. ϵ → 0
```

$$\left(-t\,\xi-\frac{1}{2}t^2\xi\hbar-\frac{1}{6}t^3\xi\hbar^2-\frac{1}{24}t^4\xi\hbar^3-\frac{1}{120}t^5\xi\hbar^4-\frac{1}{720}t^6\xi\hbar^5\right)\mathrm{QU}[\,]+\mathrm{QU}[y]$$

Now F satisfies the ODE $\partial_\eta F=\partial_\eta\left(e^{-\eta y}e^{\eta G}\right)=-yF+FG$ with initial conditions $F(\eta=0)=1$. We set it up and solve:

```
F = Sum[f_{l,i,j,k}[η] ϵ^l QU@{y^i, a^j, x^k},
  {l, 0, $TϵD}, {i, 0, l}, {j, 0, l}, {k, 0, Min[l, 2 l - i - j]}]
```

$$\mathrm{QU}[\,]f_{0,0,0,0}[\eta]+\epsilon\,\mathrm{QU}[\,]f_{1,0,0,0}[\eta]+\epsilon\,\mathrm{QU}[x]f_{1,0,0,1}[\eta]+\epsilon\,\mathrm{QU}[a]f_{1,0,1,0}[\eta]+$$
$$\epsilon\,\mathrm{QU}[a,x]f_{1,0,1,1}[\eta]+\epsilon\,\mathrm{QU}[y]f_{1,1,0,0}[\eta]+\epsilon\,\mathrm{QU}[y,x]f_{1,1,0,1}[\eta]+$$
$$\epsilon\,\mathrm{QU}[y,a]f_{1,1,1,0}[\eta]+\epsilon^2\mathrm{QU}[\,]f_{2,0,0,0}[\eta]+\epsilon^2\mathrm{QU}[x]f_{2,0,0,1}[\eta]+$$
$$\epsilon^2\mathrm{QU}[x,x]f_{2,0,0,2}[\eta]+\epsilon^2\mathrm{QU}[a]f_{2,0,1,0}[\eta]+\epsilon^2\mathrm{QU}[a,x]f_{2,0,1,1}[\eta]+$$
$$\epsilon^2\mathrm{QU}[a,x,x]f_{2,0,1,2}[\eta]+\epsilon^2\mathrm{QU}[a,a]f_{2,0,2,0}[\eta]+\epsilon^2\mathrm{QU}[a,a,x]f_{2,0,2,1}[\eta]+$$
$$\epsilon^2\mathrm{QU}[a,a,x,x]f_{2,0,2,2}[\eta]+\epsilon^2\mathrm{QU}[y]f_{2,1,0,0}[\eta]+\epsilon^2\mathrm{QU}[y,x]f_{2,1,0,1}[\eta]+$$
$$\epsilon^2\mathrm{QU}[y,x,x]f_{2,1,0,2}[\eta]+\epsilon^2\mathrm{QU}[y,a]f_{2,1,1,0}[\eta]+\epsilon^2\mathrm{QU}[y,a,x]f_{2,1,1,1}[\eta]+$$
$$\epsilon^2\mathrm{QU}[y,a,x,x]f_{2,1,1,2}[\eta]+\epsilon^2\mathrm{QU}[y,a,a]f_{2,1,2,0}[\eta]+\epsilon^2\mathrm{QU}[y,a,a,x]f_{2,1,2,1}[\eta]+$$
$$\epsilon^2\mathrm{QU}[y,y]f_{2,2,0,0}[\eta]+\epsilon^2\mathrm{QU}[y,y,x]f_{2,2,0,1}[\eta]+\epsilon^2\mathrm{QU}[y,y,x,x]f_{2,2,0,2}[\eta]+$$
$$\epsilon^2\mathrm{QU}[y,y,a]f_{2,2,1,0}[\eta]+\epsilon^2\mathrm{QU}[y,y,a,x]f_{2,2,1,1}[\eta]+\epsilon^2\mathrm{QU}[y,y,a,a]f_{2,2,2,0}[\eta]$$

```
unowns = Cases[F, f___[η], ∞]
```

$$\{f_{0,0,0,0}[\eta],f_{1,0,0,0}[\eta],f_{1,0,0,1}[\eta],f_{1,0,1,0}[\eta],f_{1,0,1,1}[\eta],f_{1,1,0,0}[\eta],f_{1,1,0,1}[\eta],f_{1,1,1,0}[\eta],$$
$$f_{2,0,0,0}[\eta],f_{2,0,0,1}[\eta],f_{2,0,0,2}[\eta],f_{2,0,1,0}[\eta],f_{2,0,1,1}[\eta],f_{2,0,1,2}[\eta],f_{2,0,2,0}[\eta],f_{2,0,2,1}[\eta],$$
$$f_{2,0,2,2}[\eta],f_{2,1,0,0}[\eta],f_{2,1,0,1}[\eta],f_{2,1,0,2}[\eta],f_{2,1,1,0}[\eta],f_{2,1,1,1}[\eta],f_{2,1,1,2}[\eta],f_{2,1,2,0}[\eta],$$
$$f_{2,1,2,1}[\eta],f_{2,2,0,0}[\eta],f_{2,2,0,1}[\eta],f_{2,2,0,2}[\eta],f_{2,2,1,0}[\eta],f_{2,2,1,1}[\eta],f_{2,2,2,0}[\eta]\}$$

```
bas = Union @@ Table[ϵ^l Cases[Coefficient[F, ϵ, l], _QU, ∞], {l, 0, $TϵD}]
```

$$\{QU[], \epsilon\, QU[], \epsilon^2 QU[], \epsilon\, QU[a], \epsilon^2 QU[a], \epsilon\, QU[x], \epsilon^2 QU[x], \epsilon\, QU[y], \epsilon^2 QU[y], \epsilon^2 QU[a, a], \epsilon\, QU[a, x], \epsilon^2 QU[a, x], \epsilon^2 QU[x, x], \epsilon\, QU[y, a], \epsilon^2 QU[y, a], \epsilon\, QU[y, x], \epsilon^2 QU[y, x], \epsilon^2 QU[y, y], \epsilon^2 QU[a, a, x], \epsilon^2 QU[a, x, x], \epsilon^2 QU[y, a, a], \epsilon^2 QU[y, a, x], \epsilon^2 QU[y, x, x], \epsilon^2 QU[y, y, a], \epsilon^2 QU[y, y, x], \epsilon^2 QU[a, a, x, x], \epsilon^2 QU[y, a, a, x], \epsilon^2 QU[y, a, x, x], \epsilon^2 QU[y, y, a, a], \epsilon^2 QU[y, y, a, x], \epsilon^2 QU[y, y, x, x]\}$$

```
Short[eqns = Flatten[{(Coefficient[F - QU[], #] /. η → 0) == 0,
     Expand[Coefficient[Simp[F ** G - QU[y] ** F - ∂_η F], #]] == 0} & /@ bas], 8]
```

$$\Big\{-1 + f_{0,0,0,0}[0] + \epsilon\, f_{1,0,0,0}[0] + \epsilon^2 f_{2,0,0,0}[0] == 0, \ll 60 \gg, \gamma\xi\hbar\, f_{1,1,0,1}[\eta] - \gamma\, f_{2,1,1,2}[\eta] - t\,\xi\, f_{2,2,0,2}[\eta] - \frac{1}{2} t^2 \xi\hbar\, f_{2,2,0,2}[\eta] - \frac{1}{6} t^3 \xi\hbar^2 f_{2,2,0,2}[\eta] - \frac{1}{24} t^4 \xi\hbar^3 f_{2,2,0,2}[\eta] - \frac{1}{120} t^5 \xi\hbar^4 f_{2,2,0,2}[\eta] - \frac{1}{720} t^6 \xi\hbar^5 f_{2,2,0,2}[\eta] - f_{2,2,0,2}'[\eta] == 0\Big\}$$

```
{sol} = DSolve[eqns, unowns, η]
```

$$\Big\{\Big\{f_{0,0,0,0}[\eta] \to e^{-\frac{1}{720}\eta\left(720 t\xi+360 t^2\xi\hbar+120 t^3\xi\hbar^2+30 t^4\xi\hbar^3+6 t^5\xi\hbar^4+t^6\xi\hbar^5\right)},$$

$$f_{1,0,0,0}[\eta] \to \frac{1}{115200} e^{-\frac{1}{720}\eta\left(720 t\xi+360 t^2\xi\hbar+120 t^3\xi\hbar^2+30 t^4\xi\hbar^3+6 t^5\xi\hbar^4+t^6\xi\hbar^5\right)} t\gamma\eta^2\xi^2 \left(80 + 120 t\hbar + 60 t^2\hbar^2 + 20 t^3\hbar^3 + 5 t^4\hbar^4 + t^5\hbar^5\right)\left(720 + 360 t\hbar + 120 t^2\hbar^2 + 30 t^3\hbar^3 + 6 t^4\hbar^4 + t^5\hbar^5\right),$$

$$f_{1,0,0,1}[\eta] \to -\frac{1}{80} e^{-\frac{1}{720}\eta\left(720 t\xi+360 t^2\xi\hbar+120 t^3\xi\hbar^2+30 t^4\xi\hbar^3+6 t^5\xi\hbar^4+t^6\xi\hbar^5\right)} \gamma\eta\xi^2 \left(80 + 120 t\hbar + 60 t^2\hbar^2 + 20 t^3\hbar^3 + 5 t^4\hbar^4 + t^5\hbar^5\right),$$

$$f_{1,0,1,0}[\eta] \to \frac{1}{60} e^{-\frac{1}{720}\eta\left(720 t\xi+360 t^2\xi\hbar+120 t^3\xi\hbar^2+30 t^4\xi\hbar^3+6 t^5\xi\hbar^4+t^6\xi\hbar^5\right)} \eta\xi \left(120 + 120 t\hbar + 60 t^2\hbar^2 + 20 t^3\hbar^3 + 5 t^4\hbar^4 + t^5\hbar^5\right), f_{1,0,1,1}[\eta] \to 0,$$

$$f_{1,1,0,0}[\eta] \to -\frac{1}{80} e^{-\frac{1}{720}\eta\left(720 t\xi+360 t^2\xi\hbar+120 t^3\xi\hbar^2+30 t^4\xi\hbar^3+6 t^5\xi\hbar^4+t^6\xi\hbar^5\right)} \gamma\eta^2\xi \left(80 + 120 t\hbar + 60 t^2\hbar^2 + 20 t^3\hbar^3 + 5 t^4\hbar^4 + t^5\hbar^5\right),$$

$$f_{1,1,0,1}[\eta] \to e^{-\frac{1}{720}\eta\left(720 t\xi+360 t^2\xi\hbar+120 t^3\xi\hbar^2+30 t^4\xi\hbar^3+6 t^5\xi\hbar^4+t^6\xi\hbar^5\right)} \gamma\eta\xi\hbar, f_{1,1,1,0}[\eta] \to 0,$$

$$f_{2,0,0,0}[\eta] \to \frac{1}{1440} e^{-\frac{1}{720}\eta\left(720 t\xi+360 t^2\xi\hbar+120 t^3\xi\hbar^2+30 t^4\xi\hbar^3+6 t^5\xi\hbar^4+t^6\xi\hbar^5\right)} \left(-480 t\gamma^2\eta^3\xi^3 + 180 t^2\gamma^2\eta^4\xi^4 + 720 t\gamma^2\eta^2\xi^2\hbar - 2160 t^2\gamma^2\eta^3\xi^3\hbar + 720 t^3\gamma^2\eta^4\xi^4\hbar + 1260 t^2\gamma^2\eta^2\xi^2\hbar^2 - 3640 t^3\gamma^2\eta^3\xi^3\hbar^2 + 1320 t^4\gamma^2\eta^4\xi^4\hbar^2 + 1020 t^3\gamma^2\eta^2\xi^2\hbar^3 - 3600 t^4\gamma^2\eta^3\xi^3\hbar^3 + 1530 t^5\gamma^2\eta^4\xi^4\hbar^3 + 555 t^4\gamma^2\eta^2\xi^2\hbar^4 - 2534 t^5\gamma^2\eta^3\xi^3\hbar^4 + 1298 t^6\gamma^2\eta^4\xi^4\hbar^4 + 231 t^5\gamma^2\eta^2\xi^2\hbar^5 - 1396 t^6\gamma^2\eta^3\xi^3\hbar^5 + 873 t^7\gamma^2\eta^4\xi^4\hbar^5\right),$$

$$f_{2,0,0,1}[\eta] \to \frac{1}{1440} e^{-\frac{1}{720}\eta\left(720 t\xi+360 t^2\xi\hbar+120 t^3\xi\hbar^2+30 t^4\xi\hbar^3+6 t^5\xi\hbar^4+t^6\xi\hbar^5\right)} \left(1440\gamma^2\eta^2\xi^3 - 720 t\gamma^2\eta^3\xi^4 - 1440\gamma^2\eta\xi^2\hbar + 5760 t\gamma^2\eta^2\xi^3\hbar - 2520 t^2\gamma^2\eta^3\xi^4\hbar - 1800 t\gamma^2\eta\xi^2\hbar^2 + 7800 t^2\gamma^2\eta^2\xi^3\hbar^2 - 3900 t^3\gamma^2\eta^3\xi^4\hbar^2 - 900 t^2\gamma^2\eta\xi^2\hbar^3 + 5880 t^3\gamma^2\eta^2\xi^3\hbar^3 - 3720 t^4\gamma^2\eta^3\xi^4\hbar^3 - 300 t^3\gamma^2\eta\xi^2\hbar^4 + 3110 t^4\gamma^2\eta^2\xi^3\hbar^4 - 2571 t^5\gamma^2\eta^3\xi^4\hbar^4 - 75 t^4\gamma^2\eta\xi^2\hbar^5 + 1278 t^5\gamma^2\eta^2\xi^3\hbar^5 - 1402 t^6\gamma^2\eta^3\xi^4\hbar^5\right),$$

$$f_{2,0,0,2}[\eta] \to \frac{1}{1440} e^{-\frac{1}{720}\eta\left(720 t\xi+360 t^2\xi\hbar+120 t^3\xi\hbar^2+30 t^4\xi\hbar^3+6 t^5\xi\hbar^4+t^6\xi\hbar^5\right)} \gamma^2\eta\xi^3 \left(720\eta\xi - 1440\hbar + 2160 t\eta\xi\hbar - 1680 t\hbar^2 + 2700 t^2\eta\xi\hbar^2 - 840 t^2\hbar^3 +\right.$$

$1980\,t^3\,\eta\,\xi\,\hbar^3 - 280\,t^3\,\hbar^4 + 1035\,t^4\,\eta\,\xi\,\hbar^4 - 70\,t^4\,\hbar^5 + 423\,t^5\,\eta\,\xi\,\hbar^5\big)$,

$f_{2,0,1,0}[\eta] \to \frac{1}{120}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}\,\gamma\,\eta^2\,\xi^2$
$\big(-120 + 120\,t\,\eta\,\xi - 480\,t\,\hbar + 360\,t^2\,\eta\,\xi\,\hbar - 600\,t^2\,\hbar^2 + 500\,t^3\,\eta\,\xi\,\hbar^2 - 440\,t^3\,\hbar^3 + 450\,t^4\,\eta\,\xi\,\hbar^3 - 230\,t^4\,\hbar^4 + 301\,t^5\,\eta\,\xi\,\hbar^4 - 94\,t^5\,\hbar^5 + 161\,t^6\,\eta\,\xi\,\hbar^5\big)$,

$f_{2,0,1,1}[\eta] \to -\frac{1}{24}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}\,\gamma\,\eta\,\xi^2$
$\big(48\,\eta\,\xi - 72\,\hbar + 120\,t\,\eta\,\xi\,\hbar - 72\,t\,\hbar^2 + 132\,t^2\,\eta\,\xi\,\hbar^2 - 36\,t^2\,\hbar^3 + 92\,t^3\,\eta\,\xi\,\hbar^3 - 12\,t^3\,\hbar^4 + 47\,t^4\,\eta\,\xi\,\hbar^4 - 3\,t^4\,\hbar^5 + 19\,t^5\,\eta\,\xi\,\hbar^5\big)$,

$f_{2,0,1,2}[\eta] \to 0$, $f_{2,0,2,0}[\eta] \to \frac{1}{60}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}$
$\eta\,\xi\,\big(120\,\eta\,\xi - 120\,\hbar + 240\,t\,\eta\,\xi\,\hbar - 120\,t\,\hbar^2 + 240\,t^2\,\eta\,\xi\,\hbar^2 - 60\,t^2\,\hbar^3 + 160\,t^3\,\eta\,\xi\,\hbar^3 - 20\,t^3\,\hbar^4 + 80\,t^4\,\eta\,\xi\,\hbar^4 - 5\,t^4\,\hbar^5 + 32\,t^5\,\eta\,\xi\,\hbar^5\big)$, $f_{2,0,2,1}[\eta] \to 0$,

$f_{2,0,2,2}[\eta] \to 0$, $f_{2,1,0,0}[\eta] \to \frac{1}{1440}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}$
$\big(1440\,\gamma^2\,\eta^3\,\xi^2 - 720\,t\,\gamma^2\,\eta^4\,\xi^3 - 1440\,\gamma^2\,\eta^2\,\xi\,\hbar + 5760\,t\,\gamma^2\,\eta^3\,\xi^2\,\hbar - 2520\,t^2\,\gamma^2\,\eta^4\,\xi^3\,\hbar - 1800\,t\,\gamma^2\,\eta^2\,\xi\,\hbar^2 + 7800\,t^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar^2 - 3900\,t^3\,\gamma^2\,\eta^4\,\xi^3\,\hbar^2 - 900\,t^2\,\gamma^2\,\eta^2\,\xi\,\hbar^3 + 5880\,t^3\,\gamma^2\,\eta^3\,\xi^2\,\hbar^3 - 3720\,t^4\,\gamma^2\,\eta^4\,\xi^3\,\hbar^3 - 300\,t^3\,\gamma^2\,\eta^2\,\xi\,\hbar^4 + 3110\,t^4\,\gamma^2\,\eta^3\,\xi^2\,\hbar^4 - 2571\,t^5\,\gamma^2\,\eta^4\,\xi^3\,\hbar^4 - 75\,t^4\,\gamma^2\,\eta^2\,\xi\,\hbar^5 + 1278\,t^5\,\gamma^2\,\eta^3\,\xi^2\,\hbar^5 - 1402\,t^6\,\gamma^2\,\eta^4\,\xi^3\,\hbar^5\big)$,

$f_{2,1,0,1}[\eta] \to \frac{1}{480}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}\,\gamma^2\,\eta\,\xi$
$\big(480\,\eta^2\,\xi^2 - 1920\,\eta\,\xi\,\hbar + 1680\,t\,\eta^2\,\xi^2\,\hbar + 240\,\hbar^2 - 2520\,t\,\eta\,\xi\,\hbar^2 + 2280\,t^2\,\eta^2\,\xi^2\,\hbar^2 - 1260\,t^2\,\eta\,\xi\,\hbar^3 + 1720\,t^3\,\eta^2\,\xi^2\,\hbar^3 - 420\,t^3\,\eta\,\xi\,\hbar^4 + 910\,t^4\,\eta^2\,\xi^2\,\hbar^4 - 105\,t^4\,\eta\,\xi\,\hbar^5 + 374\,t^5\,\eta^2\,\xi^2\,\hbar^5\big)$,

$f_{2,1,0,2}[\eta] \to -\frac{1}{80}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}\,\gamma^2\,\eta\,\xi^2\,\hbar$
$\big(80\,\eta\,\xi - 40\,\hbar + 120\,t\,\eta\,\xi\,\hbar + 60\,t^2\,\eta\,\xi\,\hbar^2 + 20\,t^3\,\eta\,\xi\,\hbar^3 + 5\,t^4\,\eta\,\xi\,\hbar^4 + t^5\,\eta\,\xi\,\hbar^5\big)$,

$f_{2,1,1,0}[\eta] \to -\frac{1}{24}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}\,\gamma\,\eta^2\,\xi$
$\big(48\,\eta\,\xi - 72\,\hbar + 120\,t\,\eta\,\xi\,\hbar - 72\,t\,\hbar^2 + 132\,t^2\,\eta\,\xi\,\hbar^2 - 36\,t^2\,\hbar^3 + 92\,t^3\,\eta\,\xi\,\hbar^3 - 12\,t^3\,\hbar^4 + 47\,t^4\,\eta\,\xi\,\hbar^4 - 3\,t^4\,\hbar^5 + 19\,t^5\,\eta\,\xi\,\hbar^5\big)$,

$f_{2,1,1,1}[\eta] \to \frac{1}{60}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}\,\gamma\,\eta^2\,\xi^2\,\hbar$
$\big(120 + 120\,t\,\hbar + 60\,t^2\,\hbar^2 + 20\,t^3\,\hbar^3 + 5\,t^4\,\hbar^4 + t^5\,\hbar^5\big)$,
$f_{2,1,1,2}[\eta] \to 0$, $f_{2,1,2,0}[\eta] \to 0$, $f_{2,1,2,1}[\eta] \to 0$,

$f_{2,2,0,0}[\eta] \to \frac{1}{1440}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}\,\gamma^2$
$\eta^3\,\xi\,\big(720\,\eta\,\xi - 1440\,\hbar + 2160\,t\,\eta\,\xi\,\hbar - 1680\,t\,\hbar^2 + 2700\,t^2\,\eta\,\xi\,\hbar^2 - 840\,t^2\,\hbar^3 + 1980\,t^3\,\eta\,\xi\,\hbar^3 - 280\,t^3\,\hbar^4 + 1035\,t^4\,\eta\,\xi\,\hbar^4 - 70\,t^4\,\hbar^5 + 423\,t^5\,\eta\,\xi\,\hbar^5\big)$,

$f_{2,2,0,1}[\eta] \to -\frac{1}{80}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}\,\gamma^2\,\eta^2\,\xi\,\hbar$
$\big(80\,\eta\,\xi - 40\,\hbar + 120\,t\,\eta\,\xi\,\hbar + 60\,t^2\,\eta\,\xi\,\hbar^2 + 20\,t^3\,\eta\,\xi\,\hbar^3 + 5\,t^4\,\eta\,\xi\,\hbar^4 + t^5\,\eta\,\xi\,\hbar^5\big)$,

$f_{2,2,0,2}[\eta] \to \frac{1}{2}\,e^{-\frac{1}{720}\eta\,(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5)}\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2$,
$f_{2,2,1,0}[\eta] \to 0$, $f_{2,2,1,1}[\eta] \to 0$,
$f_{2,2,2,0}[\eta] \to 0\}\}$

```
Union@Cases[sol, e-, ∞]
```

$$\left\{e^{-\frac{1}{720}\eta\left(720\,t\,\xi+360\,t^2\,\xi\,\hbar+120\,t^3\,\xi\,\hbar^2+30\,t^4\,\xi\,\hbar^3+6\,t^5\,\xi\,\hbar^4+t^6\,\xi\,\hbar^5\right)}\right\}$$

sol /. e- → 1

$$\Big\{f_{0,0,0,0}[\eta]\to 1,\ f_{1,0,0,0}[\eta]\to \frac{1}{115200}t\,\gamma\,\eta^2\,\xi^2\left(80+120\,t\,\hbar+60\,t^2\,\hbar^2+20\,t^3\,\hbar^3+5\,t^4\,\hbar^4+t^5\,\hbar^5\right)$$
$$\left(720+360\,t\,\hbar+120\,t^2\,\hbar^2+30\,t^3\,\hbar^3+6\,t^4\,\hbar^4+t^5\,\hbar^5\right),$$
$$f_{1,0,0,1}[\eta]\to -\frac{1}{80}\gamma\,\eta\,\xi^2\left(80+120\,t\,\hbar+60\,t^2\,\hbar^2+20\,t^3\,\hbar^3+5\,t^4\,\hbar^4+t^5\,\hbar^5\right),$$
$$f_{1,0,1,0}[\eta]\to \frac{1}{60}\eta\,\xi\left(120+120\,t\,\hbar+60\,t^2\,\hbar^2+20\,t^3\,\hbar^3+5\,t^4\,\hbar^4+t^5\,\hbar^5\right),\ f_{1,0,1,1}[\eta]\to 0,$$
$$f_{1,1,0,0}[\eta]\to -\frac{1}{80}\gamma\,\eta^2\,\xi\left(80+120\,t\,\hbar+60\,t^2\,\hbar^2+20\,t^3\,\hbar^3+5\,t^4\,\hbar^4+t^5\,\hbar^5\right),$$
$$f_{1,1,0,1}[\eta]\to \gamma\,\eta\,\xi\,\hbar,\ f_{1,1,1,0}[\eta]\to 0,\ f_{2,0,0,0}[\eta]\to$$
$$\frac{1}{1440}\left(-480\,t\,\gamma^2\,\eta^3\,\xi^3+180\,t^2\,\gamma^2\,\eta^4\,\xi^4+720\,t\,\gamma^2\,\eta^2\,\xi^2\,\hbar-2160\,t^2\,\gamma^2\,\eta^3\,\xi^3\,\hbar+720\,t^3\,\gamma^2\,\eta^4\,\xi^4\,\hbar+\right.$$
$$1260\,t^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2-3640\,t^3\,\gamma^2\,\eta^3\,\xi^3\,\hbar^2+1320\,t^4\,\gamma^2\,\eta^4\,\xi^4\,\hbar^2+1020\,t^3\,\gamma^2\,\eta^2\,\xi^2\,\hbar^3-$$
$$3600\,t^4\,\gamma^2\,\eta^3\,\xi^3\,\hbar^3+1530\,t^5\,\gamma^2\,\eta^4\,\xi^4\,\hbar^3+555\,t^4\,\gamma^2\,\eta^2\,\xi^2\,\hbar^4-2534\,t^5\,\gamma^2\,\eta^3\,\xi^3\,\hbar^4+$$
$$\left.1298\,t^6\,\gamma^2\,\eta^4\,\xi^4\,\hbar^4+231\,t^5\,\gamma^2\,\eta^2\,\xi^2\,\hbar^5-1396\,t^6\,\gamma^2\,\eta^3\,\xi^3\,\hbar^5+873\,t^7\,\gamma^2\,\eta^4\,\xi^4\,\hbar^5\right),$$
$$f_{2,0,0,1}[\eta]\to \frac{1}{1440}\left(1440\,\gamma^2\,\eta^2\,\xi^3-720\,t\,\gamma^2\,\eta^3\,\xi^4-1440\,\gamma^2\,\eta\,\xi^2\,\hbar+5760\,t\,\gamma^2\,\eta^2\,\xi^3\,\hbar-\right.$$
$$2520\,t^2\,\gamma^2\,\eta^3\,\xi^4\,\hbar-1800\,t\,\gamma^2\,\eta\,\xi^2\,\hbar^2+7800\,t^2\,\gamma^2\,\eta^2\,\xi^3\,\hbar^2-3900\,t^3\,\gamma^2\,\eta^3\,\xi^4\,\hbar^2-900\,t^2\,\gamma^2\,\eta\,\xi^2\,\hbar^3+$$
$$5880\,t^3\,\gamma^2\,\eta^2\,\xi^3\,\hbar^3-3720\,t^4\,\gamma^2\,\eta^3\,\xi^4\,\hbar^3-300\,t^3\,\gamma^2\,\eta\,\xi^2\,\hbar^4+3110\,t^4\,\gamma^2\,\eta^2\,\xi^3\,\hbar^4-$$
$$\left.2571\,t^5\,\gamma^2\,\eta^3\,\xi^4\,\hbar^4-75\,t^4\,\gamma^2\,\eta\,\xi^2\,\hbar^5+1278\,t^5\,\gamma^2\,\eta^2\,\xi^3\,\hbar^5-1402\,t^6\,\gamma^2\,\eta^3\,\xi^4\,\hbar^5\right),$$
$$f_{2,0,0,2}[\eta]\to \frac{1}{1440}\gamma^2\,\eta\,\xi^3\left(720\,\eta\,\xi-1440\,\hbar+2160\,t\,\eta\,\xi\,\hbar-1680\,t\,\hbar^2+2700\,t^2\,\eta\,\xi\,\hbar^2-\right.$$
$$\left.840\,t^2\,\hbar^3+1980\,t^3\,\eta\,\xi\,\hbar^3-280\,t^3\,\hbar^4+1035\,t^4\,\eta\,\xi\,\hbar^4-70\,t^4\,\hbar^5+423\,t^5\,\eta\,\xi\,\hbar^5\right),$$
$$f_{2,0,1,0}[\eta]\to \frac{1}{120}\gamma\,\eta^2\,\xi^2\left(-120+120\,t\,\eta\,\xi-480\,t\,\hbar+360\,t^2\,\eta\,\xi\,\hbar-600\,t^2\,\hbar^2+500\,t^3\,\eta\,\xi\,\hbar^2-\right.$$
$$\left.440\,t^3\,\hbar^3+450\,t^4\,\eta\,\xi\,\hbar^3-230\,t^4\,\hbar^4+301\,t^5\,\eta\,\xi\,\hbar^4-94\,t^5\,\hbar^5+161\,t^6\,\eta\,\xi\,\hbar^5\right),$$
$$f_{2,0,1,1}[\eta]\to -\frac{1}{24}\gamma\,\eta\,\xi^2\left(48\,\eta\,\xi-72\,\hbar+120\,t\,\eta\,\xi\,\hbar-72\,t\,\hbar^2+132\,t^2\,\eta\,\xi\,\hbar^2-36\,t^2\,\hbar^3+\right.$$
$$\left.92\,t^3\,\eta\,\xi\,\hbar^3-12\,t^3\,\hbar^4+47\,t^4\,\eta\,\xi\,\hbar^4-3\,t^4\,\hbar^5+19\,t^5\,\eta\,\xi\,\hbar^5\right),\ f_{2,0,1,2}[\eta]\to 0,$$
$$f_{2,0,2,0}[\eta]\to \frac{1}{60}\eta\,\xi\left(120\,\eta\,\xi-120\,\hbar+240\,t\,\eta\,\xi\,\hbar-120\,t\,\hbar^2+240\,t^2\,\eta\,\xi\,\hbar^2-60\,t^2\,\hbar^3+\right.$$
$$\left.160\,t^3\,\eta\,\xi\,\hbar^3-20\,t^3\,\hbar^4+80\,t^4\,\eta\,\xi\,\hbar^4-5\,t^4\,\hbar^5+32\,t^5\,\eta\,\xi\,\hbar^5\right),\ f_{2,0,2,1}[\eta]\to 0,\ f_{2,0,2,2}[\eta]\to 0,$$
$$f_{2,1,0,0}[\eta]\to \frac{1}{1440}\left(1440\,\gamma^2\,\eta^3\,\xi^2-720\,t\,\gamma^2\,\eta^4\,\xi^3-1440\,\gamma^2\,\eta^2\,\xi\,\hbar+5760\,t\,\gamma^2\,\eta^3\,\xi^2\,\hbar-\right.$$
$$2520\,t^2\,\gamma^2\,\eta^4\,\xi^3\,\hbar-1800\,t\,\gamma^2\,\eta^2\,\xi\,\hbar^2+7800\,t^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar^2-3900\,t^3\,\gamma^2\,\eta^4\,\xi^3\,\hbar^2-900\,t^2\,\gamma^2\,\eta^2\,\xi\,\hbar^3+$$
$$5880\,t^3\,\gamma^2\,\eta^3\,\xi^2\,\hbar^3-3720\,t^4\,\gamma^2\,\eta^4\,\xi^3\,\hbar^3-300\,t^3\,\gamma^2\,\eta^2\,\xi\,\hbar^4+3110\,t^4\,\gamma^2\,\eta^3\,\xi^2\,\hbar^4-$$
$$\left.2571\,t^5\,\gamma^2\,\eta^4\,\xi^3\,\hbar^4-75\,t^4\,\gamma^2\,\eta^2\,\xi\,\hbar^5+1278\,t^5\,\gamma^2\,\eta^3\,\xi^2\,\hbar^5-1402\,t^6\,\gamma^2\,\eta^4\,\xi^3\,\hbar^5\right),$$
$$f_{2,1,0,1}[\eta]\to \frac{1}{480}\gamma^2\,\eta\,\xi\left(480\,\eta^2\,\xi^2-1920\,\eta\,\xi\,\hbar+1680\,t\,\eta^2\,\xi^2\,\hbar+240\,\hbar^2-2520\,t\,\eta\,\xi\,\hbar^2+\right.$$
$$2280\,t^2\,\eta^2\,\xi^2\,\hbar^2-1260\,t^2\,\eta\,\xi\,\hbar^3+1720\,t^3\,\eta^2\,\xi^2\,\hbar^3-420\,t^3\,\eta\,\xi\,\hbar^4+$$
$$\left.910\,t^4\,\eta^2\,\xi^2\,\hbar^4-105\,t^4\,\eta\,\xi\,\hbar^5+374\,t^5\,\eta^2\,\xi^2\,\hbar^5\right),\ f_{2,1,0,2}[\eta]\to$$
$$-\frac{1}{80}\gamma^2\,\eta\,\xi^2\,\hbar\left(80\,\eta\,\xi-40\,\hbar+120\,t\,\eta\,\xi\,\hbar+60\,t^2\,\eta\,\xi\,\hbar^2+20\,t^3\,\eta\,\xi\,\hbar^3+5\,t^4\,\eta\,\xi\,\hbar^4+t^5\,\eta\,\xi\,\hbar^5\right),$$

$$f_{2,1,1,0}[\eta] \to -\frac{1}{24}\gamma\,\eta^2\,\xi\,\big(48\,\eta\,\xi - 72\,\hbar + 120\,t\,\eta\,\xi\,\hbar - 72\,t\,\hbar^2 + 132\,t^2\,\eta\,\xi\,\hbar^2 - 36\,t^2\,\hbar^3 + 92\,t^3\,\eta\,\xi\,\hbar^3 - 12\,t^3\,\hbar^4 + 47\,t^4\,\eta\,\xi\,\hbar^4 - 3\,t^4\,\hbar^5 + 19\,t^5\,\eta\,\xi\,\hbar^5\big),$$

$$f_{2,1,1,1}[\eta] \to \frac{1}{60}\gamma\,\eta^2\,\xi^2\,\hbar\,\big(120 + 120\,t\,\hbar + 60\,t^2\,\hbar^2 + 20\,t^3\,\hbar^3 + 5\,t^4\,\hbar^4 + t^5\,\hbar^5\big),$$

$$f_{2,1,1,2}[\eta] \to 0,$$
$$f_{2,1,2,0}[\eta] \to 0,$$
$$f_{2,1,2,1}[\eta] \to 0,$$

$$f_{2,2,0,0}[\eta] \to \frac{1}{1440}\gamma^2\,\eta^3\,\xi\,\big(720\,\eta\,\xi - 1440\,\hbar + 2160\,t\,\eta\,\xi\,\hbar - 1680\,t\,\hbar^2 + 2700\,t^2\,\eta\,\xi\,\hbar^2 - 840\,t^2\,\hbar^3 + 1980\,t^3\,\eta\,\xi\,\hbar^3 - 280\,t^3\,\hbar^4 + 1035\,t^4\,\eta\,\xi\,\hbar^4 - 70\,t^4\,\hbar^5 + 423\,t^5\,\eta\,\xi\,\hbar^5\big),\ f_{2,2,0,1}[\eta] \to$$
$$-\frac{1}{80}\gamma^2\,\eta^2\,\xi\,\hbar\,\big(80\,\eta\,\xi - 40\,\hbar + 120\,t\,\eta\,\xi\,\hbar + 60\,t^2\,\eta\,\xi\,\hbar^2 + 20\,t^3\,\eta\,\xi\,\hbar^3 + 5\,t^4\,\eta\,\xi\,\hbar^4 + t^5\,\eta\,\xi\,\hbar^5\big),$$

$$f_{2,2,0,2}[\eta] \to \frac{1}{2}\gamma^2\,\eta^2\,\xi^2\,\hbar^2,\ f_{2,2,1,0}[\eta] \to 0,$$
$$f_{2,2,1,1}[\eta] \to 0,\ f_{2,2,2,0}[\eta] \to 0\}$$

Stitching Direct

```
MatrixExp[η₁ ρ[CU@y]].MatrixExp[α₁ ρ[CU@a]].MatrixExp[ξ₁ ρ[CU@x]].MatrixExp[η₂ ρ[CU@y]].
  MatrixExp[α₂ ρ[CU@a]].MatrixExp[ξ₂ ρ[CU@x]] // Simplify // MatrixForm
```

$$\begin{pmatrix} e^{\gamma(\alpha_1+\alpha_2)}\,(1+\gamma\,\epsilon\,\eta_2\,\xi_1) & e^{\gamma\,\alpha_1}\,\gamma\,\big(e^{\gamma\,\alpha_2}\,\xi_2 + \xi_1\,(1 + e^{\gamma\,\alpha_2}\,\gamma\,\epsilon\,\eta_2\,\xi_2)\big) \\ e^{\gamma\,\alpha_2}\,\epsilon\,\big(\eta_2 + e^{\gamma\,\alpha_1}\,\eta_1\,(1+\gamma\,\epsilon\,\eta_2\,\xi_1)\big) & 1 + e^{\gamma\,\alpha_1}\,\gamma\,\epsilon\,\eta_1\,\xi_1 + e^{\gamma\,\alpha_2}\,\gamma\,\epsilon\,\big(\eta_2 + e^{\gamma\,\alpha_1}\,\eta_1\,(1+\gamma\,\epsilon\,\eta_2\,\xi_1)\big)\,\xi_2 \end{pmatrix}$$

```
eqn = MatrixExp[η₁ ρ[CU@y]].MatrixExp[α₁ ρ[CU@a]].MatrixExp[ξ₁ ρ[CU@x]].
  MatrixExp[η₂ ρ[CU@y]].MatrixExp[α₂ ρ[CU@a]].MatrixExp[ξ₂ ρ[CU@x]] ==
 e^(τ0 ϵ γ) MatrixExp[η0 ρ[CU@y]].MatrixExp[α0 ρ[CU@a]].MatrixExp[ξ0 ρ[CU@x]]
```

$$\big\{\big\{e^{\gamma\,\alpha_2}\,\big(e^{\gamma\,\alpha_1} + e^{\gamma\,\alpha_1}\,\gamma\,\epsilon\,\eta_2\,\xi_1\big),\ e^{\gamma\,\alpha_1}\,\gamma\,\xi_1 + e^{\gamma\,\alpha_2}\,\gamma\,\big(e^{\gamma\,\alpha_1} + e^{\gamma\,\alpha_1}\,\gamma\,\epsilon\,\eta_2\,\xi_1\big)\,\xi_2\big\},$$
$$\big\{e^{\gamma\,\alpha_2}\,\big(e^{\gamma\,\alpha_1}\,\epsilon\,\eta_1 + \epsilon\,\eta_2\,(1 + e^{\gamma\,\alpha_1}\,\gamma\,\epsilon\,\eta_1\,\xi_1)\big),$$
$$1 + e^{\gamma\,\alpha_1}\,\gamma\,\epsilon\,\eta_1\,\xi_1 + e^{\gamma\,\alpha_2}\,\gamma\,\big(e^{\gamma\,\alpha_1}\,\epsilon\,\eta_1 + \epsilon\,\eta_2\,(1 + e^{\gamma\,\alpha_1}\,\gamma\,\epsilon\,\eta_1\,\xi_1)\big)\,\xi_2\big\}\big\} ==$$
$$\big\{\big\{e^{\alpha 0\,\gamma+\gamma\,\epsilon\,\tau 0},\ e^{\alpha 0\,\gamma+\gamma\,\epsilon\,\tau 0}\,\gamma\,\xi 0\big\},\ \big\{e^{\alpha 0\,\gamma+\gamma\,\epsilon\,\tau 0}\,\epsilon\,\eta 0,\ e^{\gamma\,\epsilon\,\tau 0}\,\big(1 + e^{\alpha 0\,\gamma}\,\gamma\,\epsilon\,\eta 0\,\xi 0\big)\big\}\big\}$$

```
sol = Block[{ϵ}, Solve[Thread[Flatten /@ eqn], {τ0, η0, α0, ξ0}]][[1]]
```

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is

$\mathrm{Log}[e^{\gamma(\alpha 0+\epsilon\,\tau 0)}] - \mathrm{Log}[e^{\gamma\,\alpha_2}\,(e^{\gamma\,\mathrm{Subscript}[\ll 2\gg]} + e^{\mathrm{Times}[\ll 2\gg]}\,\gamma\,\epsilon\,\eta_2\,\xi_1)] == 0.$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Solve: Equations may not give solutions for all "solve" variables.

$$\Big\{\tau 0 \to \frac{1}{\gamma\epsilon}\left(-\mathrm{Log}\left[e^{\alpha 0\,\gamma}\right] + \mathrm{Log}\left[e^{\gamma\alpha_1+\gamma\alpha_2} + e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_2\xi_1\right]\right),\ \eta 0 \to \frac{1}{\gamma\epsilon\left(\xi_1 + e^{\gamma\alpha_2}\xi_2 + e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_1\xi_2\right)}$$
$$e^{-\gamma\alpha_1}\Bigg(\frac{1}{2} + \frac{1}{2}e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\xi_2 + \frac{1}{2}e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2\xi_1\xi_2 -$$
$$\frac{1}{2}\sqrt{\Big(\left(-1 - e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\xi_2 - e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2\xi_1\xi_2\right)^2 +}$$
$$4\,e^{-\alpha 0\,\gamma+\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\left(-e^{\gamma\alpha_1}\eta_1\xi_1 - \eta_2\xi_1 - e^{\gamma\alpha_1}\gamma\epsilon\eta_1\eta_2\xi_1^2 - e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\xi_2 - e^{\gamma\alpha_2}\eta_2\xi_2 -\right.$$
$$\left.2\,e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\eta_2\xi_1\xi_2 - e^{\gamma\alpha_2}\gamma\epsilon\eta_2^2\xi_1\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2^2\xi_1^2\xi_2\right)\Big)\Bigg),$$
$$\xi 0 \to \frac{1}{e^{\gamma\alpha_1}\eta_1 + \eta_2 + e^{\gamma\alpha_1}\gamma\epsilon\eta_1\eta_2\xi_1}\,e^{-\gamma\alpha_2}\Bigg(\frac{1}{2\gamma\epsilon} + \frac{1}{2}e^{\gamma\alpha_1}\eta_1\xi_1 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\xi_2 +$$
$$\frac{1}{2}e^{\gamma\alpha_2}\eta_2\xi_2 + \frac{1}{2}e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\eta_2\xi_1\xi_2 -$$
$$\frac{1}{2\gamma\epsilon}\Bigg(\sqrt{\Big(\left(-1 - e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\xi_2 - e^{\gamma\alpha_2}\gamma\epsilon\eta_2\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2\xi_1\xi_2\right)^2 +}$$
$$4\,e^{-\alpha 0\,\gamma+\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\left(-e^{\gamma\alpha_1}\eta_1\xi_1 - \eta_2\xi_1 - e^{\gamma\alpha_1}\gamma\epsilon\eta_1\eta_2\xi_1^2 - e^{\gamma\alpha_1+\gamma\alpha_2}\eta_1\xi_2 - e^{\gamma\alpha_2}\eta_2\xi_2 -\right.$$
$$\left.2\,e^{\gamma\alpha_1+\gamma\alpha_2}\gamma\epsilon\eta_1\eta_2\xi_1\xi_2 - e^{\gamma\alpha_2}\gamma\epsilon\eta_2^2\xi_1\xi_2 - e^{\gamma\alpha_1+\gamma\alpha_2}\gamma^2\epsilon^2\eta_1\eta_2^2\xi_1^2\xi_2\right)\Big)\Bigg)\Bigg)\Big\}$$

```
eqn = MatrixExp[η1 ρ[CU@y]].MatrixExp[α1 ρ[CU@a]].MatrixExp[ξ1 ρ[CU@x]].
    MatrixExp[η2 ρ[CU@y]].MatrixExp[α2 ρ[CU@a]].MatrixExp[ξ2 ρ[CU@x]] ==
   T0 MatrixExp[η0 ρ[CU@y]].MatrixExp[α0 ρ[CU@a]].MatrixExp[ξ0 ρ[CU@x]]
```

$$\Big\{\Big\{e^{\gamma\alpha_2}\left(e^{\gamma\alpha_1} + e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1\right),\ e^{\gamma\alpha_1}\gamma\xi_1 + e^{\gamma\alpha_2}\gamma\left(e^{\gamma\alpha_1} + e^{\gamma\alpha_1}\gamma\epsilon\eta_2\xi_1\right)\xi_2\Big\},$$
$$\Big\{e^{\gamma\alpha_2}\left(e^{\gamma\alpha_1}\epsilon\eta_1 + \epsilon\eta_2\left(1 + e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1\right)\right),$$
$$1 + e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1 + e^{\gamma\alpha_2}\gamma\left(e^{\gamma\alpha_1}\epsilon\eta_1 + \epsilon\eta_2\left(1 + e^{\gamma\alpha_1}\gamma\epsilon\eta_1\xi_1\right)\right)\xi_2\Big\}\Big\} ==$$
$$\Big\{\Big\{e^{\alpha 0\,\gamma}\,\mathrm{T0},\ e^{\alpha 0\,\gamma}\,\mathrm{T0}\,\gamma\,\xi 0\Big\},\ \Big\{e^{\alpha 0\,\gamma}\,\mathrm{T0}\,\epsilon\,\eta 0,\ \mathrm{T0}\left(1 + e^{\alpha 0\,\gamma}\gamma\epsilon\,\eta 0\,\xi 0\right)\Big\}\Big\}$$

```
sol = Block[{ϵ}, Solve[Thread[Flatten /@ eqn], {T0, η0, α0, ξ0}]][[1]]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\Big\{\mathrm{T0} \to \frac{1}{1 + \gamma\epsilon\eta_2\xi_1},\ \eta 0 \to \frac{\eta_1 + e^{-\gamma\alpha_1}\eta_2 + \gamma\epsilon\eta_1\eta_2\xi_1}{1 + \gamma\epsilon\eta_2\xi_1},$$
$$\alpha 0 \to \frac{\mathrm{Log}\left[e^{\gamma\alpha_1+\gamma\alpha_2}\left(1 + \gamma\epsilon\eta_2\xi_1\right)^2\right]}{\gamma},\ \xi 0 \to \frac{e^{-\gamma\alpha_2}\xi_1 + \xi_2 + \gamma\epsilon\eta_2\xi_1\xi_2}{1 + \gamma\epsilon\eta_2\xi_1}\Big\}$$

$\mathbb{E}$

$\mathbb{E}[L, Q, P]$ means $e^{\hbar(L+Q)}\,P$, where L is linear in the a's, Q is a combination of $x_i\,y_j$, and P is a perturbation polynomial. It should be interpreted via $\mathbb{CO}[\mathbb{E}[\,...], \{x_1, a_1, y_1\}_i, ...]$ (with some default for direct interpretation), or likewise via $\mathbb{QO}[\mathbb{E}[\,...], \{x_1, a_1, y_1\}_i, ...]$. In themselves, $\mathbb{CO}$ and $\mathbb{QO}$ should have an interpretation in CU/QU by casting.