

Pensieve header: A unified verification notebook for the PPSA project.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

Go;

```
SetOptions[EvaluationNotebook[],
  NotebookEventActions → {
    {"MenuCommand", "Save"} => (
      SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\PPSA"];
      Get["MakeVSnips.m"]
    ),
    PassEventsDown → True
  }];
```

```
HL[ $\mathcal{E}$ _] := Style[ $\mathcal{E}$ , Background → Yellow];
```

Initialization

The “degree carrier / filtration parameter” is \hbar , and all “coupling constants” are proportional to it.

TD

```
$T $\hbar$ D = 5;  $\hbar$  /:  $\hbar^{d-}$  /;  $d > \$T\hbar D$  := 0;
$T $\epsilon$ D = 2;  $\epsilon$  /:  $\epsilon^{d-}$  /;  $d > \$T\epsilon D$  := 0;
SetAttributes[SS, HoldAll];
SS[ $\mathcal{E}$ _] := Block[{ $\hbar$ ,  $\epsilon$ },
  Normal@Series[ $\mathcal{E}$ , { $\hbar$ , 0, $T $\hbar$ D}, { $\epsilon$ , 0, $T $\epsilon$ D}] ]
Simp[ $\mathcal{E}$ _] := Collect[ $\mathcal{E}$ , _CU | _QU, Expand];
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x; 0 ** _ = _ ** 0 = 0;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
```

QLImplementation

```

DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, cp, S, pow,
  gs = Generators /. {opts}, cs = Centrals /. {opts}
},
  gp = Alternatives @@ gs;
  gp = gp | gp; (* generators pattern *)
  sr = Thread[gs → Range@Length@gs]; (* sorting rule *)
  cp = Alternatives @@ cs; (* centrals pattern *)
  S[ε_] := Collect[ε, _U, Expand];
  U_i[ε_] := ε /. {t : cp => t_i, u_U => Replace[u, x_ => x_i, 1]};
  B[U@(x_)_i, U@(y_)_i] := B[U@x_i, U@y_i] = U_i@B[U@x, U@y];
  B[U@(x_)_i, U@(y_)_j] /; i != j := 0;
  B[U@y_, U@x_] := S[-B[U@x, U@y]];
  x_ ** U[] := x; U[] ** x_ := x;
  (a_ * x_U) ** (b_ * y_U) := If[a b == 0, 0, S[a b (x ** y)]];
  (a_ * x_U) ** y_ := S[a (x ** y)]; x_ ** (a_ * y_U) := S[a (x ** y)];
  U[xx___, x_] ** U[y_, yy___] := If[OrderedQ[{x, y} /. sr],
    U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := S[c U@Table[L, {n}] ** U@{r}];
  U@{c_. * l : gp, r___} := S[c U[l] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := S[c U@{r}];
  U@{} = U[];
  U@{L_Plus, r___} := S[U@{#, r} & /@ L];
  U@{L_, r___} := U@{Expand[L], r};
  O_U[poly_, specs___] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, L_List => (l → null), {1}];
    vs = Join@@(First /@ sp);
    us = Join@@(sp /. (l_ → s_) => (l /. x_i_ => x_s));
    S[Total[
      CoefficientRules[poly, vs] /. (p_ → c_) => c U@(us^p)
    ]] /. x_null => x
  ];
  pow[ε_, 0] = U[]; pow[ε_, n_] := pow[ε, n - 1] ** ε;
  S_U[ε_, ss__Rule] := S@Total[
    CoefficientRules[ε, First /@ {ss}] /.
      (p_ → c_) => c NonCommutativeMultiply @@ MapThread[pow, {Last /@ {ss}, p}]
  ]
]

```

DeclareMorphism

QLImplementation

```

DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, (g_ → img_) => (m[U[g]] = img), {1}];
  m[U[]] = V[];
  m[U[g_i]] := V_i[m[U@g]];
  m[U[vs___]] := NonCommutativeMultiply @@ (m /@ U /@ {vs});
  m[ε_] := Simp[ε /. oncs /. u_U => m[u]];
)

```

Implementing $sl_2^{\gamma\epsilon}$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[CU@a, CU@y] = -γ CU@y; B[CU@x, CU@a] = -γ CU@x;
B[CU@x, CU@y] = 2 ε CU@a - t CU[];
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
  Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp,
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  z1 ** (z2 ** z3) - rhs // Simp
}] // Timing
{0.625,
 { (28 t^2 γ^4 + 116 t γ^5 ε + 120 γ^6 ε^2) CU[y, y, y, x, x] + <<23>> + CU[y, y, y, y, <<6>>, x, x, x], 0 }}
```

Implementing $\mathcal{U}_{\gamma\epsilon; \hbar}$

With $q = e^{\hbar\gamma\epsilon}$, $A = e^{-\hbar\epsilon a}$, and $[f, g]_q := fg - qgf$, our algebra is $\mathcal{U}_{\gamma\epsilon; \hbar} = \langle t, y, a, x \rangle / \mathcal{R}$, where $\mathcal{R} = ([t, *] = 0, [a, y] = -\gamma y, [x, y]_q = (1 - T^2 A^2) / \hbar, [x, a] = -\gamma x)$.

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
q := Normal@Series[e^γ ε ħ, {ħ, 0, $TħD}];
B[QU@a, QU@y] = -γ QU@y; B[QU@x, QU@a] = -γ QU@x;
B[QU@x, QU@y] := (q - 1) QU@{y, x} + OQU[SS[(1 - T^2 e^{-2 ħ ε a}) / ħ], {a}];
```

```
With[{bas = QU /@ {y, a, x}}, Table[z1 ** z2 - z2 ** z1 // Simp, {z1, bas}, {z2, bas} ] ]
{{0, γ QU[y], (-1/ħ + T^2/ħ) QU[] - 2 T^2 ε QU[a] + 2 T^2 ε^2 ħ QU[a, a] + (-γ ε ħ - 1/2 γ^2 ε^2 ħ^2) QU[y, x]},
 {-γ QU[y], 0, γ QU[x]},
 {{(1/ħ - T^2/ħ) QU[] + 2 T^2 ε QU[a] - 2 T^2 ε^2 ħ QU[a, a] + (γ ε ħ + 1/2 γ^2 ε^2 ħ^2) QU[y, x], -γ QU[x], 0}}
```

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
  Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp,
    {z1, bas}, {z2, bas}, {z3, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  z1 ** (z2 ** z3) - rhs // Simp
}] // Timing
{49.5938,
 {

$$\left(135 \gamma^6 \epsilon^2 - 730 T^2 \gamma^6 \epsilon^2 + \ll 8 \gg + \frac{198 T^4 \gamma^5 \epsilon}{\hbar}\right) QU[y, y, y, x, x] + \ll 22 \gg + (\ll 1 \gg) \ll 1 \gg, 0 \}$$


```

Verifying that $\lim_{\hbar \rightarrow 0} QU = CU$ using a “random” product (~23 secs):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  Short[lhs = z1 ** (z2 ** z3)],
  Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
  Expand[Limit[rhs /. {QU -> CU, T -> e^{h t/2}}, h -> 0] == lhs]
}] // Timing
{28.0156,
 {

$$2 (8 t^2 \gamma^4 + 16 t \gamma^5 \epsilon) CU[y, y, y, x, x] + \ll 94 \gg + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],$$


$$(-8 T^2 \gamma^6 \epsilon^2 + 8 T^4 \gamma^6 \epsilon^2) QU[y, y, y, x, x] + \ll 422 \gg + \left(\gamma \epsilon \hbar + \frac{15}{2} \gamma^2 \ll 1 \gg \hbar^2\right) \ll 1 \gg, \text{True} \}$$


```

Implementing θ

theta

```
DeclareMorphism[Cθ, CU -> CU, {y -> -CU@x, a -> -CU@a, x -> -CU@y}, {t -> -t, T -> T^{-1}}];
DeclareMorphism[Qθ, QU -> QU, {y -> Q_{QU}[SS[-T^{-1} e^{h \epsilon^a} x], {a, x}],
  a -> -QU@a, x -> Q_{QU}[SS[-T^{-1} e^{h \epsilon^a} y], {a, y}], {t -> -t, T -> T^{-1}}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[z -> Cθ[z] -> Cθ[Cθ[z]], {z, bas} ] ]
{CU[y] -> -CU[x] -> CU[y], CU[a] -> -CU[a] -> CU[a], CU[x] -> -CU[y] -> CU[x]}
```

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
  Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2], {z1, bas}, {z2, bas} ] ]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[z → QΘ[z] → HL[QΘ[QΘ[z]]], {z, bas}] ]
{QU[y] → - $\frac{QU[x]}{\tau}$  -  $\frac{\epsilon \hbar QU[a, x]}{\tau}$  -  $\frac{\epsilon^2 \hbar^2 QU[a, a, x]}{2 \tau}$  → QU[y], QU[a] → -QU[a] → QU[a],
  QU[x] →  $\left(-\frac{1}{\tau} + \frac{\gamma \epsilon \hbar}{\tau} - \frac{\gamma^2 \epsilon^2 \hbar^2}{2 \tau}\right) QU[y] + \left(-\frac{\epsilon \hbar}{\tau} + \frac{\gamma \epsilon^2 \hbar^2}{\tau}\right) QU[y, a] - \frac{\epsilon^2 \hbar^2 QU[y, a, a]}{2 \tau}$  → QU[x]}
```

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → Simp[QΘ[z1 ** z2] - QΘ[z1] ** QΘ[z2]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\text{ADeq}\$f = e^{\hbar (t/2 - \epsilon a)} \left(\left(\cosh \left[\hbar \left(\frac{\epsilon - t}{2} + \epsilon a \right) \right] - \cosh \left[\hbar \sqrt{\left(\frac{t - \epsilon}{2} \right)^2 + \epsilon \omega} \right] \right) / \right. \\ \left. \left(\hbar \sinh \left[\frac{-\epsilon \hbar}{2} \right] (\omega - \epsilon a^2 + (t - \epsilon) a) \right) \right) y;$$

ADeq

```
DeclareMorphism[ADeq, QU → CU, {a → CU@a, x → CU@x,
  y → SCU[SS[ADeq$f], a → CU[a], ω → CU[y, x] + ε CU[a, a] - (t - ε) CU[a], y → CU[y]] }]]
{Short[tt = ADeq[QU[y, y, a, a, x, x]]], tt /. ħ → 0}
{<<1>>, CU[y, y, a, a, x, x]}

With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → Simp[
    ADeq[z1 ** z2] - ADeq[z1] ** ADeq[z2] /. {γ → 1, τ → SS[eħ t/2]}
  ], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} →  $\frac{t^6 \hbar^5 CU[]}{23 \ 040}$ , {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

```

With[{bas = QU /@ {y, a, x}},
Table[{z1, z2} → Simp[
  ADeq[z1 ** z2] - ADeq[z1] ** ADeq[z2] /. {T → SS[eht/2]}
], {z1, bas}, {z2, bas}] ]
{ {QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0,
  {QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0,
  {QU[x], QU[y]} →  $\left( -t \in \hbar + t \gamma \in \hbar - \frac{2}{3} t^2 \in \hbar^2 + \frac{2}{3} t^2 \gamma \in \hbar^2 + \frac{1}{2} t \in^2 \hbar^2 - \frac{7}{6} t \gamma \in^2 \hbar^2 + \right.$ 
 $\frac{2}{3} t \gamma^2 \in^2 \hbar^2 - \frac{1}{4} t^3 \in \hbar^3 + \frac{1}{4} t^3 \gamma \in \hbar^3 + \frac{5}{12} t^2 \in^2 \hbar^3 - \frac{11}{12} t^2 \gamma \in^2 \hbar^3 + \frac{1}{2} t^2 \gamma^2 \in^2 \hbar^3 -$ 
 $\frac{1}{15} t^4 \in \hbar^4 + \frac{1}{15} t^4 \gamma \in \hbar^4 + \frac{7}{40} t^3 \in^2 \hbar^4 - \frac{3}{8} t^3 \gamma \in^2 \hbar^4 + \frac{1}{5} t^3 \gamma^2 \in^2 \hbar^4 + \frac{t^6 \hbar^5}{23040} -$ 
 $\left. \frac{1}{72} t^5 \in \hbar^5 + \frac{1}{72} t^5 \gamma \in \hbar^5 + \frac{1}{20} t^4 \in^2 \hbar^5 - \frac{19}{180} t^4 \gamma \in^2 \hbar^5 + \frac{1}{18} t^4 \gamma^2 \in^2 \hbar^5 \right) \text{CU}[] +$ 
 $\left( 2 \in^2 \hbar - 2 \gamma \in^2 \hbar + \frac{8}{3} t \in^2 \hbar^2 - \frac{8}{3} t \gamma \in^2 \hbar^2 + \frac{3}{2} t^2 \in^2 \hbar^3 - \frac{3}{2} t^2 \gamma \in^2 \hbar^3 + \frac{8}{15} t^3 \in^2 \hbar^4 - \right.$ 
 $\left. \frac{8}{15} t^3 \gamma \in^2 \hbar^4 + \frac{5}{36} t^4 \in^2 \hbar^5 - \frac{5}{36} t^4 \gamma \in^2 \hbar^5 \right) \text{CU}[a] +$ 
 $\left( \frac{1}{6} t \in \hbar^2 - \frac{1}{6} t \gamma \in \hbar^2 + \frac{1}{2} \gamma \in^2 \hbar^2 - \frac{1}{2} \gamma^2 \in^2 \hbar^2 + \frac{1}{12} t^2 \in \hbar^3 - \frac{1}{12} t^2 \gamma \in \hbar^3 - \frac{1}{6} t \in^2 \hbar^3 + \frac{3}{4} t \gamma \in^2 \hbar^3 - \right.$ 
 $\frac{7}{12} t \gamma^2 \in^2 \hbar^3 + \frac{1}{40} t^3 \in \hbar^4 - \frac{1}{40} t^3 \gamma \in \hbar^4 - \frac{13}{120} t^2 \in^2 \hbar^4 + \frac{3}{8} t^2 \gamma \in^2 \hbar^4 - \frac{4}{15} t^2 \gamma^2 \in^2 \hbar^4 +$ 
 $\left. \frac{1}{180} t^4 \in \hbar^5 - \frac{1}{180} t^4 \gamma \in \hbar^5 - \frac{3}{80} t^3 \in^2 \hbar^5 + \frac{7}{60} t^3 \gamma \in^2 \hbar^5 - \frac{19}{240} t^3 \gamma^2 \in^2 \hbar^5 \right) \text{CU}[y, x] +$ 
 $\left( -\frac{1}{3} \in^2 \hbar^2 + \frac{1}{3} \gamma \in^2 \hbar^2 - \frac{1}{3} t \in^2 \hbar^3 + \frac{1}{3} t \gamma \in^2 \hbar^3 - \frac{3}{20} t^2 \in^2 \hbar^4 + \frac{3}{20} t^2 \gamma \in^2 \hbar^4 - \right.$ 
 $\left. \frac{2}{45} t^3 \in^2 \hbar^5 + \frac{2}{45} t^3 \gamma \in^2 \hbar^5 \right) \text{CU}[y, a, x] +$ 
 $\left( \frac{1}{120} t \in^2 \hbar^4 - \frac{1}{120} t \gamma \in^2 \hbar^4 + \frac{1}{240} t^2 \in^2 \hbar^5 - \frac{1}{240} t^2 \gamma \in^2 \hbar^5 \right) \text{CU}[y, y, x, x],$ 
  {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0 } }

```