

POLY-TIME KNOT POLYNOMIALS VIA SOLVABLE APPROXIMATIONS

DROR BAR-NATAN AND ROLAND VAN DER VEEN

ABSTRACT. Following Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], we construct the first poly-time-computable knot polynomials since Alexander's [Al, 1928]. We use some new commutator-calculus techniques and a family of Lie algebra $sl_2^{\leq k}$ which are solvable yet at the same time they make progressively better approximations of the simple Lie algebra sl_2 . The resulting invariants are the strongest genuinely-computable knot invariants presently available and they seem to contain information about some classical topologically-defined knot invariants.

Electronic version, links, and related files at $\omega\epsilon\beta := \text{http://drorbn.net/PPSA/}$.

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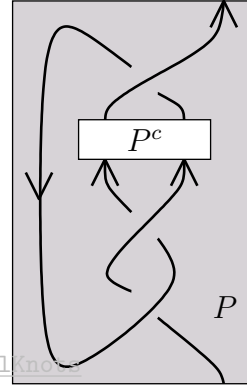
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add framings!

Like elsewhere, for us a “tangle” K is a part of a (multi-component, oriented) knot in a part P of a plane in which an “up” direction is declared. Unlike elsewhere, we do not insist that P would be a disk; it may be a union of disks with a few sub-disks removed. We do insist, however, that the ends of K would lie within the boundary ∂P of P and would be up-going there. We also insist that the components (“strands”) of K would be intervals (i.e., not circles), and that they would be placed in a bijection with some finite set S of “strand labels”. In Section 8.1 we replace this provisional definition with “rotational virtual tangles” in the spirit of [Ka].



make a "Lay"

ASIDE 1.1. Provisionally, what we mean by a “tangle”.

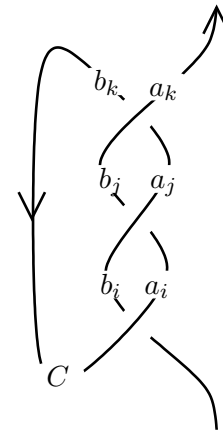
Draw K in the plane so that at each crossing the two crossing strands are pointed up.

Put a copy of $R = \sum a_i \otimes b_i$ on every positive crossing of K with the “ a ” side on the over-strand and the “ b ” side on the under-strand, labeling these a ’s and b ’s with distinct indices i, j, k, \dots (similarly put copies of $R^{-1} = \sum a'_i \otimes b'_i$ on the negative crossings; these are absent in our example). Put a copy of $C^{\pm 1}$ on every cuap where the tangent to the knot is pointing to the right (meaning, a C on every such cup and a C^{-1} on every such cap).

If K is a (long) knot, form an expression $z(K)$ in U by multiplying all the a , b , C letters as they are seen when traveling along K and then summing over all the indices, as shown.

When K is a tangle with S strands, carry out the multiplications along each strand separately in a different tensor-copy of U , to get $z(K) \in U^{\otimes S}$.

If R and C satisfy some conditions dictated by the standard Reidemeister moves of knot theory, the resulting $z(K)$ is a knot / tangle invariant.



$$z(K) = \sum_{i,j,k} b_i a_j b_k C a_i b_j a_k$$

ASIDE 1.2. The standard methodology on an example knot.

1. INTRODUCTION

1.1. From Algebras to Invariants. There is a standard “quantum algebra” methodology that associates a knot / tangle invariant to certain triples (U, R, C) , where U is a unital algebra and $R \in U \otimes U$ and $C \in U$ are invertible (see e.g. [Oh, Section 4.2]). In Aside 1.1 we provisionally explain what we mean by “tangle”, and the “quantum algebra” methodology is recalled in Aside 1.2.

The best algebras with which to apply this methodology, at least as of 2017, are the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ of semi-simple Lie algebras \mathfrak{g} (or certain completions

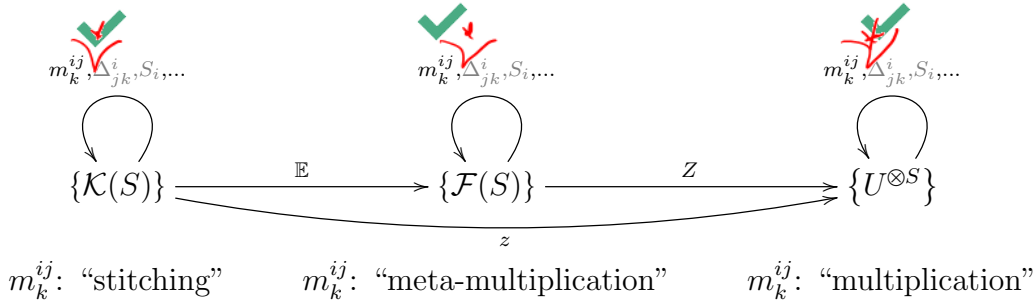
thereof, or their quantizations). But these algebras are infinite dimensional, and the sums in Aside 1.2 are infinite and not immediately computable.

The dogma solution is to pick a finite dimensional representation of \mathfrak{g} and use it to represent all the elements appearing in Aside 1.2, effectively replacing the algebra by the algebra of endomorphisms of some finite dimensional vector space. This turns the sums finite; yet if the knot K has n crossings, our sum becomes a sum over n indices i_1, \dots, i_n . Thus there are exponentially-many summands to consider and it takes an exponential amount of time to compute $z(K)$, limiting its computation only to relatively small knots.^{1,2} In addition, by choosing a specific representation of \mathfrak{g} , one loses the good behaviour of z under strand-doubling. In Section 6 we explain why such good behaviour is a desirable property.

Alternatively, one may extract finite-type [BN1, CDM] information out of z by reducing modulo appropriate filtrations of U and its tensor powers. Invariants of type d are computable in time less than or equal to $O(n^d)$ [BN2], and thus for small d , they are effectively computable. But there are only very few invariants of sufficiently small type d , they are not very powerful, and there are some no-go theorems that limit the power of any finite number of finite-type invariants to resolve certain topological questions [Ng, St].

1.2. Formulas and Meta-Algebras. Our approach to the computation of $z(K)$ is different. Instead of working directly in $U^{\otimes S}$, our invariant $Z(K)$ takes values in a space $\mathcal{F}(S)$ of “formulas for elements of $U^{\otimes S}$ ” that have an “interpretation map” $\mathbb{E}: \mathcal{F}(S) \rightarrow U^{\otimes S}$ for which $z = Z//\mathbb{E}$.³ We make sure that the following two properties hold:

- (1) There are operations on $\mathcal{F}(S)$ that mirror standard operations on the space $U^{\otimes S}$ and on the space $\mathcal{K}(S)$ of S -component tangles, so that a diagram of the following nature commutes:



The most important of these operations is the operation m_k^{ij} . On tangles, it is “stitching”: the operation that takes the head of component i in a tangle K and stitches it to the tail of component j , renaming the resulting single component k , as in Figure 1.1⁴.

¹“Divide and conquer” methods often improve the computation time to $O(e^{c\sqrt{n}})$ for some constant c . Utilizing this, the simplest of these “quantum invariants”, the Jones, HOMFLY-PT and Kauffman polynomials, corresponding to sl_2 , sl_N and so_N in their defining representations, can be computed for surprisingly large knots even though ultimately $e^{c\sqrt{n}}$ grows more rapidly than any polynomial.

²Note that almost any time the phrases “braided monoidal category” or “TQFT” are used within low dimensional topology, some tensor powers of some vector spaces need to be considered at some point, and dimensions grow exponentially. Thus our criticism applies in these cases too [BN3].

³We use properly ordered compositions! $f//g$ means “do f then g ”, often obfuscated using “ $g \circ f$ ”.

⁴The careful reader will notice that stitching is only partially defined, for the head of i must lie next to the tail of j for m_k^{ij} to make sense, and that it is sometimes ill-defined, for there may be more than one path

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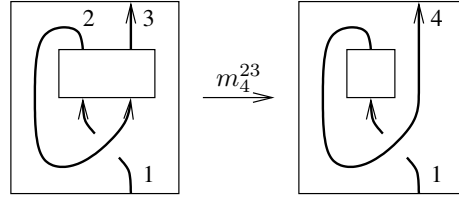


FIGURE 1.1. Stitching.

- (2) The rank of $\mathcal{F}(S)$ (over some ring of Laurent polynomials which we will specify later) grows polynomially in the size $|S|$ of S , and all the operations on $\mathcal{F}(S)$ are computable using a polynomial number of ring operations.

MORE.

1.3. Solvable Approximations.

1.4. Section Summaries.

1.5. Acknowledgement.

2. THE FIRST EXAMPLE: ALGORITHM ONLY

3. THE ZEROth EXAMPLE IN DETAIL

4. THE LIE ALGEBRA $sl_2^{\alpha,\beta}$ AND ITS QUANTIZATION5. THE GENERAL $sl_2^{\leq k}$ INVARIANT

6. BEHAVIOUR UNDER STRAND REVERSAL AND STRAND DOUBLING

7. EXPERIMENTAL RESULTS

8. ODDS AND ENDS

8.1. Rotational Virtual Tangles.

9. TABLES

connecting the head of i with the tail of j . Please accept our assurances that these issues do not lead to any difficulties, and that they are fully resolved in Section 8.1.

Invariants
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Ng:Ribbon
Invariants
bay:Thesis
nsky:Flat1
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