POLY-TIME KNOT POLYNOMIALS VIA SOLVABLE APPROXIMATIONS

DROR BAR-NATAN AND ROLAND VAN DER VEEN

ABSTRACT. Following Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], we construct the first poly-time-computable knot polynomials since Alexander's [Al, 1928]. We use some new commutator-calculus techniques and a family of Lie algebra $\mathfrak{g}_\nu$ which are solvable yet at the same time they ~~are~~ progressively better approximations of the simple Lie algebra sl_2. The resulting invariants are the strongest genuinely-computable knot invariants presently available and they seem to contain information about some classical topologically-defined knot invariants.

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$\mathfrak{g}_K \to sl_2^{\leq K}$

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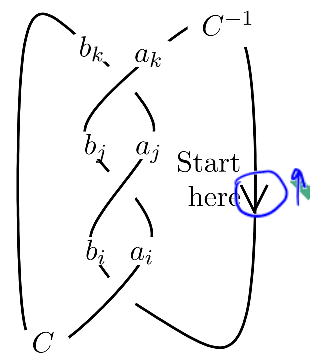

$Z(K) = \sum_{i,j,k} b_i a_j b_k C a_i b_j a_k C^{-1}$

$=\sum a_i \otimes b_i$ under

Put a copy of R on every positive crossing of a knot K with the "a" side on the over-strand and the "b" side on the ~~lower~~ strand, labeling these a's and b's with distinct indices $i, j, k, \ldots$ (similarly put copies of $R^{-1} = \sum a'_i \otimes b'_i$ on the negative crossings; these are absent in our example). Put a copy of $C^{\pm 1}$ on every cuap where the tangent to the knot is pointing to the right (meaning, a C on every such cup and a C^{-1} on every such cap). Now form an expression $Z(K)$ in U by multiplying all the a, b, C letters as they are seen when travelling along K starting from some arbitrary starting point and then summing over all the indices, as shown. If R and C satisfy some conditions dictated by the standard Reidemeister moves of knot theory, the resulting $Z(K)$ is a knot invariant.

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FIGURE 1.1. The standard methodology on an example knot.

Henning? Ohtsuki?

1. Introduction

1.1. From Algebras to Invariants. There is a standard "quantum algebra" methodology that ~~associates~~ a knot invariant to certain triples (U, R, C), where U is a unital algebra, and $R = \sum a_i \otimes b_i \in U \otimes U$ and $C \in U$ are invertible. It is recalled in Figure 1.1.

The ~~best~~ algebras with which to apply this methodology, at least as of 2017, are the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ of semi-simple Lie algebras $\mathfrak{g}$ (or certain completions thereof, or their quantizations). But these algebras are infinite dimensional, and the sums in Figure 1.1 are infinite and not immediately computable.

The dogma solution is to pick a finite dimensional representation of $\mathfrak{g}$ and use it to represent all the elements appearing in Figure 1.1, effectively replacing the algebra by the algebra of endomorphisms of some finite dimensional vector space. This turns the sums finite; yet if the knot K has n crossings, our sum becomes a sum over n indices $i_1, \ldots, i_n$. Thus there are exponentially-many summands to consider and it takes an exponential amount of time to compute $Z(K)$, limiting its computation only to relatively small knots.[1,2] In addition, by choosing a specific representation of $\mathfrak{g}$, one looses the good behaviour of Z under strand-doubling. In Section 6 we explain why such good behaviour is a desirable property.

Alternatively, one may extract finite-type [BN1, CDM] information out of Z by resolving modulo appropriate filtrations of U and its tensor powers. Invariants of type d are computable in time less than or equal to $O(n^d)$ [BN2], and thus for small d, they are ~~easily~~ effectively computable. But there are only very few invariants of sufficiently small type d, they are not very powerful, and there are some no-go theorems that limit the power of any finite number of finite-type invariants to resolve certain ~~topological~~ questions [Ng, St].

1.2. Formulas and Meta-Algebras.

[1] "Divide and conquer" methods often improve the computation time to $O(e^{c\sqrt{n}})$ for some constant c. Utilizing this, the simplest of these "quantum invariants", the Jones, HOMFLY-PT and Kauffman polynomials, corresponding to $sl(2)$, $sl(n)$ and $so(n)$ in their defining representations, can be computed for surprisingly large knots even though ultimately $e^{c\sqrt{n}}$ grows more rapidly than any polynomial.

[2] Note that almost any time the phrases "braided monoidal category" or "TQFT" are used within low dimensional topology, some ~~high~~ tensor powers of some vector spaces need to be considered at some point, and dimensions grow exponentially. Thus our criticism applies in these cases too [BN4].

1.3. Solvable Approximations.

1.4. Section Summaries.

1.5. Acknowledgement.

2. The First Example: Algorithm Only

3. The Zeroth Example in Detail

4. The Lie Algebra $\mathfrak{g}_\beta$ and its Quantization

5. The General $\mathfrak{g}_k$ Invariant

6. Behaviour under strand reversal and strand doubling

7. Experimental Results

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