POLY-TIME KNOT POLYNOMIALS VIA SOLVABLE APPROXIMATIONS

DROR BAR-NATAN AND ROLAND VAN DER VEEN

Abstract. Following Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], we construct the first poly-time-computable knot polynomials since Alexander's [Al, 1928]. We use some new commutator-calculus techniques and a family of Lie algebra $sl_2^{\leqslant k}$ which are solvable yet at the same time they make progressively better approximations of the simple Lie algebra sl_2. The resulting invariants are the strongest genuinely-computable knot invariants presently available and they seem to contain information about some classical topologically-defined knot invariants.

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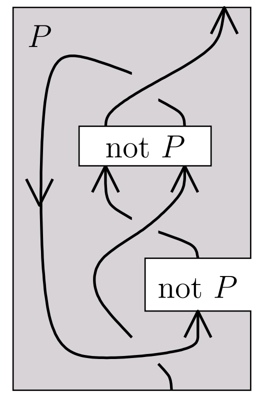
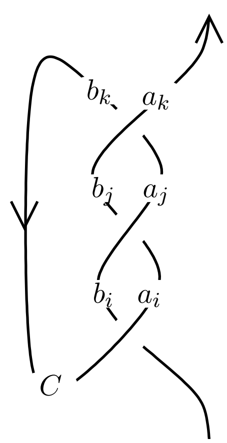
Like elsewhere, for us a "tangle" K is a part of a (multi-component, oriented, framed) knot in a part P of a plane in which an "up" direction is declared. Unlike elsewhere, we do not insist that P would be a disk; it may be a union of disks with a few sub-disks removed. We do insist, however, that the ends of K would lie within the boundary ∂P of P and would be up-going there. We also insist that the components ("strands") of K would be intervals (i.e., not circles), and that they would be placed in a bijection with some finite set S of "strand labels".
In Section 3 we replace this provisional definition with "rotational virtual tangles" in the spirit of [Ka].



Aside 1.1. Provisionally, what we mean by a "tangle".

Draw K in the plane so that at each crossing the two crossing strands are pointed up.
Put a copy of $R = \sum a_i \otimes b_i$ on every positive crossing of K with the "a" side on the over-strand and the "b" side on the under-strand, labeling these a's and b's with distinct indices $i, j, k, \ldots$ (similarly put copies of $R^{-1} = \sum a'_i \otimes b'_i$ on the negative crossings; these are absent in our example). Put a copy of $C^{\pm 1}$ on every cuap where the tangent to the knot is pointing to the right (meaning, a C on every such cup and a C^{-1} on every such cap).
If K is a (long) knot, form an expression $z(K)$ in U by multiplying all the a, b, C letters as they are seen when traveling along K and then summing over all the indices, as shown.



$$z(K) = \sum_{i,j,k} b_i a_j b_k C a_i b_j a_k$$

When K is a tangle with S strands, carry out the multiplications along each strand separately in a different tensor-copy of U, to get $z(K) \in U^{\otimes S}$.
If R and C satisfy some conditions dictated by the standard Reidemeister moves of knot theory, the resulting $z(K)$ is a knot / tangle invariant.

Aside 1.2. The standard methodology on an example knot.

1. Introduction

1.1. **From Algebras to Invariants.** There is a standard "quantum algebra" methodology that associates a framed knot / tangle invariant to certain triples (U, R, C), where U is a unital algebra and $R \in U \otimes U$ and $C \in U$ are invertible (see e.g. [Oh, Section 4.2]). In Aside 1.1 we provisionally explain what we mean by "tangle", and the "quantum algebra" methodology is recalled in Aside 1.2.

The best algebras with which to apply this methodology, at least as of 2017, are the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ of semi-simple Lie algebras $\mathfrak{g}$ (or certain completions

thereof, or their quantizations). But these algebras are infinite dimensional, and the sums in Aside 1.2 are infinite and not immediately computable.

The dogma solution is to pick a finite dimensional representation of $\mathfrak{g}$ and use it to represent all the elements appearing in Aside 1.2, effectively replacing the algebra by the algebra of endomorphisms of some finite dimensional vector space. This turns the sums finite; yet if the knot K has n crossings, our sum becomes a sum over n indices $i_1, \ldots, i_n$. Thus there are exponentially-many summands to consider and it takes an exponential amount of time to compute $z(K)$, limiting its computation only to relatively small knots.[1,2] In addition, by choosing a specific representation of $\mathfrak{g}$, one looses the good behaviour of z under strand-doubling. In Section 8 we explain why such good behaviour is a desirable property.

Alternatively, one may extract finite-type [BN1, CDM] information out of z by reducing modulo appropriate filtrations of U and its tensor powers. Invariants of type d are computable in time less than or equal to $O(n^d)$ [BN2], and thus for small d, they are effectively computable. But there are only very few invariants of sufficiently small type d, they are not very powerful, and there are some no-go theorems that limit the power of any finite number of finite-type invariants to resolve certain topological questions [Ng, St].

1.2. **Formulas and Meta-Algebras.** Our approach to the computation of $z(K)$ is different. Instead of working directly in $U^{\otimes S}$, our invariant $Z(K)$ takes values in spaces $\mathcal{F}(S)$ of "formulas for elements of $U^{\otimes S}$" that have an "value map" $\mathbb{V}\colon \mathcal{F}(S) \to U^{\otimes S}$, taking a formula in $\mathcal{F}(S)$ to its value in $U^{\otimes S}$, for which $z = Z /\!\!/ \mathbb{V}$.[3] We make sure that the following five properties hold:

(1) There are simple and easy to compute (constant time) formulas for the invariants of a crossing and of a cuap.
(2) There are operations on $\mathcal{F}(S)$ that mirror standard operations on the space $U^{\otimes S}$ and on the space $\mathcal{K}(S)$ of S-component tangles, so that a diagram of the following nature commutes:

$$\begin{tikzcd}
\{\mathcal{K}(S)\} \arrow[loop above, "{m^{ij}_k, *, \Delta^i_{jk}, S_i, \ldots}"] \arrow[r, "Z"] \arrow[rr, bend right, "z"'] & \{\mathcal{F}(S)\} \arrow[loop above, "{m^{ij}_k, *, \Delta^i_{jk}, S_i, \ldots}"] \arrow[r, "\mathbb{V}"] & \{U^{\otimes S}\} \arrow[loop above, "{m^{ij}_k, *, \Delta^i_{jk}, S_i, \ldots}"]
\end{tikzcd}$$

m^{ij}_k: "stitching" m^{ij}_k: "meta-multiplication" m^{ij}_k: "multiplication"

The most important of these operations is the operation m^{ij}_k, defined whenever $i \neq j \in S$ and $k \notin S\backslash\{i, j\}$. On tangles, it is "stitching": the operation $\mathcal{K}(S) \to \mathcal{K}((S\backslash\{i, j\}) \cup \{k\})$ that takes the head of component i in a tangle K and stitches it to

[1] "Divide and conquer" methods often improve the computation time to $O(e^{c\sqrt{n}})$ for some constant c. Utilizing this, the simplest of these "quantum invariants", the Jones, HOMFLY-PT and Kauffman polynomials, corresponding to sl_2, sl_N and so_N in their defining representations, can be computed for surprisingly large knots even though ultimately $e^{c\sqrt{n}}$ grows more rapidly than any polynomial.

[2] Note that almost any time the phrases "braided monoidal category" or "TQFT" are used within low dimensional topology, some tensor powers of some vector spaces need to be considered at some point, and dimensions grow exponentially. Thus our criticism applies in these cases too [BN3].

[3] We use properly ordered compositions! $f /\!\!/ g$ means "do f then g", often obfuscated using "$g \circ f$".

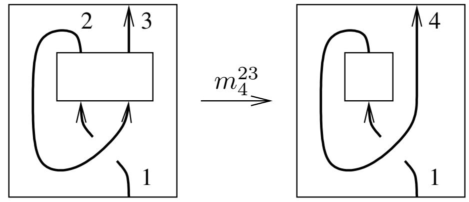


Figure 1.1. Stitching.

the tail of component j, renaming the resulting single component k, as in Figure 1.1[4]. Clearly from the construction in Aside 1.2, the corresponding operation on $\{U^{\otimes S}\}$ is "multiply tensor factor i with tensor factor j, storing the result in tensor factor k. We have a "meta-multiplication" operation $m_k^{ij}\colon \mathcal{F}(S) \to \mathcal{F}((S\backslash\{i,j\}) \cup \{k\})$ which takes "the formula for an element ζ in $U^{\otimes S}$" to "the formula for $m_k^{ij}(\zeta)$", and which likewise intertwines Z. Namely, we have $\mathbb{V}/\!\!/m_k^{ij} = m^{ij}/\!\!/\mathbb{V}$ and $m_k^{ij}/\!\!/Z = Z/\!\!/m_k^{ij}$.

(3) Similarly, if $S_1 \cap S_2 = \varnothing$, there is a "disjoint union" operation $*\colon \mathcal{K}(S_1) \times \mathcal{K}(S_2) \to \mathcal{K}(S_1 \cup S_2)$.[5] The corresponding operation on $\{\mathcal{U}^S\}$ is the tensor product operation $* = \otimes\colon U^{\otimes S_1} \times U^{\otimes S_2} \to U^{\otimes(S_1\cup S_2)}$. We ensure that there is a compatible $*\colon \mathcal{F}(S_1) \times \mathcal{F}(S_2) \to \mathcal{F}(S_1 \cup S_2)$.

(4) $\mathbb{V}$ is injective. A formula is determined its value.

(5) The rank of $\mathcal{F}(S)$ (over some ring $\mathcal{R}$ of Laurent polynomials which we will specify later) grows polynomially in the size $|S|$ of S, and all the operations on $\mathcal{F}(S)$ are computable using a polynomial number of ring operations.

These five properties taken together are almost enough for what we want. If K is an n-crossing tangle, it be presented as some stitching of a disjoint union of n individual crossings floating indepedently. Hence by using (1)–(3), a formula $Z(K)$ for the invariant $z(K)$ can be computed using $O(n)$ stitchings and unions. By (4), that formula is in itself an invariant. Finally, by (5), $Z(K)$ can be computed using a polynomial number of ring operations (and some combinatorial overhead which amounts to much less).

To show that the computation of Z is poly-time it remains to bound the complexity of the ring elements that we encounter, and hence the complexity of ring operations among them. This is done in Section **??**.

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1.3. Solvable Approximations. MORE.

1.4. Section Summaries. MORE.

1.5. Acknowledgement. MORE.

2. The First Example: Only the Algorithm, Only for Knots

MORE.

[4]The careful reader will notice that stitching is only partially defined, for the head of i must lie next to the tail of j for m_k^{ij} to make sense, and that it is sometimes ill-defined, for there may be more than one path connecting the head of i with the tail of j. Please accept our assurances that these issues do not lead to any difficulties, and that they are fully resolved in Section 3.

[5]As in footnote 4, there is a minor placement issue here. It is resolved in Section 3.

3. Rotational Virtual Tangles

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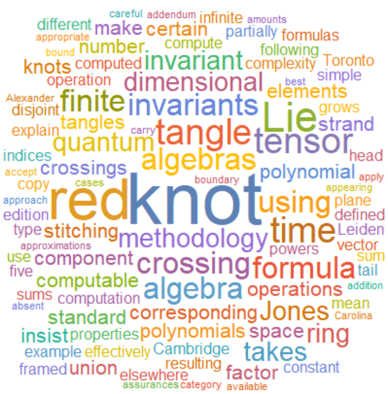
[Wo] *Wolfram Language & System Documentation Center,* ωεβ/Wolf. See pp. .



Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada

E-mail address: `drorbn@math.toronto.edu`

URL: `http://www.math.toronto.edu/drorbn`

Mathematisch Instituut, Universiteit Leiden, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands

E-mail address: `roland.mathematics@gmail.com`

URL: `http://www.rolandvdv.nl/`