

Pensieve header: Profile with encapsulation of inner Zip2 and Zip3. Time to K31@\$k=3: 7112.92.

Startup

(Alt) In[]:=

```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"..\\Profile\\Profile.m"];
$k = 1;
<< Objects.m
<< KT.m
```

(Alt) Out[]:= {2021, 1, 3, 9, 53, 30.2035381}

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

This version: April 2020. Original version: July 1994.

Engine

Canonical Forms:

(Alt) In[]:=

```
CCF[ $\mathcal{E}$ ] := PPCCF@ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ];
(*Coefficient Canonical Form *)
CF[ $\mathcal{E}$ ] := PPCF@Module[
  {vs = Cases[ $\mathcal{E}$ , (y | a | x |  $\eta$  |  $\beta$  |  $\tau$  |  $\xi$ )_,  $\infty$ ]  $\cup$  {y, a, x,  $\eta$ ,  $\beta$ ,  $\tau$ ,  $\xi$ }},
  Total[(CCF[#][2]] (Times@@vs#[1])) & /@ CoefficientRules[ $\mathcal{E}$ , vs]
];
CF[ $\mathcal{E}_{\mathcal{E}}$ ] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}_{List}$ ] := CF /@  $\mathcal{E}$ ;
CF[ $\mathbb{E}_{sp\_}$ [ $\mathcal{ES}_{\_}$ ]] := CF /@  $\mathbb{E}_{sp}$ [ $\mathcal{ES}$ ];
```

Variables and their duals:

(Alt) In[]:=

```
{t*, b*, y*, a*, x*, z*,  $\tau^*$ ,  $\beta^*$ ,  $\eta^*$ ,  $\alpha^*$ ,  $\xi^*$ ,  $\zeta^*$ } = { $\tau$ ,  $\beta$ ,  $\eta$ ,  $\alpha$ ,  $\xi$ ,  $\zeta$ , t, b, y, a, x, z};
(vs_List)* := (v  $\mapsto$  v*) /@ vs;
(u_i)* := (u*)i;
```

Weights:

(Alt) In[]:=

```
Clear[Wt];
Evaluate[Wt /@ {y, b, t, a, x,  $\eta$ ,  $\beta$ ,  $\tau$ ,  $\alpha$ ,  $\xi$ }] = {1, 0, 0, 2, 1, 1, 2, 2, 0, 1};
Wt[u_i] := Wt[u];
```

The maximal weight \$n, i.e. the n of $gl(n)$. Initially and for a long while this will not be tested beyond $\$n == 2$.

(Alt) In[]:=

```
$n = 2;
```

Upper to lower and lower to Upper:

```
(Alt) In[ ]:=
U21[ $\mathcal{E}_-$ ] :=  $\mathcal{E} / . \{ B_{i_-}^{p_-} \mapsto e^{-p \hbar b_i}, B^{p_-} \mapsto e^{-p \hbar b}, T_{i_-}^{p_-} \mapsto e^{p \hbar t_i}, T^{p_-} \mapsto e^{p \hbar t}, \mathcal{A}_{i_-}^{p_-} \mapsto e^{p \alpha_i}, \mathcal{A}^{p_-} \mapsto e^{p \alpha} \}$ ;
12U[ $\mathcal{E}_-$ ] :=  $\mathcal{E} / . \{ e^{c_- \cdot b_i + d_-} \mapsto B_i^{-c/\hbar} e^d, e^{c_- \cdot b + d_-} \mapsto B^{-c/\hbar} e^d, e^{c_- \cdot t_i + d_-} \mapsto T_i^{c/\hbar} e^d, e^{c_- \cdot t + d_-} \mapsto T^{c/\hbar} e^d, e^{c_- \cdot \alpha_i + d_-} \mapsto \mathcal{A}_i^c e^d, e^{c_- \cdot \alpha + d_-} \mapsto \mathcal{A}^c e^d, e^{\chi_-} \mapsto e^{\text{Expand}@\chi} \}$ ;
12U[ $r\_Rule$ ] := Module[{ $U = r[[1]] / . \{ b \mapsto B, t \mapsto T, \alpha \mapsto \mathcal{A} \}$ },  $U \mapsto 12U[12U[U] / . r]$ ];
AlsoUpper[ $rs\_List$ ] :=  $rs \cup (12U / @ rs)$ ;
```

Derivatives in the presence of exponentiated variables:

```
(Alt) In[ ]:=
Db[ $f_-$ ] :=  $\partial_b f - \hbar B \partial_B f$ ; Dbi[ $f_-$ ] :=  $\partial_{b_i} f - \hbar B_i \partial_{B_i} f$ ;
Dt[ $f_-$ ] :=  $\partial_t f + \hbar T \partial_T f$ ; Dti[ $f_-$ ] :=  $\partial_{t_i} f + \hbar T_i \partial_{T_i} f$ ;
D $\alpha$ [ $f_-$ ] :=  $\partial_\alpha f + \mathcal{A} \partial_{\mathcal{A}} f$ ; D $\alpha_i$ [ $f_-$ ] :=  $\partial_{\alpha_i} f + \mathcal{A}_i \partial_{\mathcal{A}_i} f$ ;
Dv[ $f_-$ ] :=  $\partial_v f$ ;
```

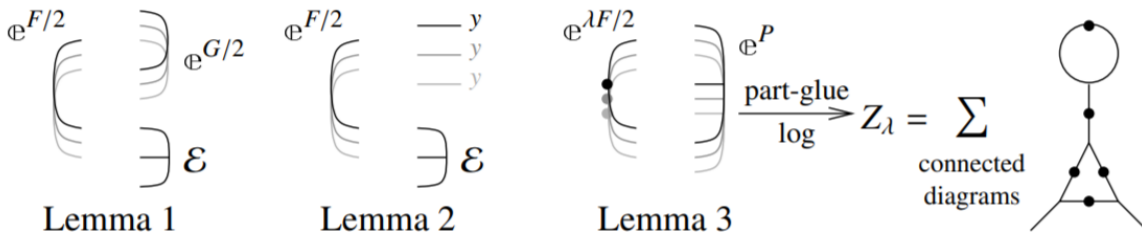
E operations:

```
(Alt) In[ ]:=
 $\mathcal{E}\_E[\$] := \text{Length}[\mathcal{E}] - 1$ ;  $E\_[\mathcal{ES}\_\_\_] [\$] := E[\mathcal{ES}][\$]$ ;
 $\mathcal{E}\_E[k\_Integer] := \mathcal{E}[[k+1]]$ ;  $E\_[\mathcal{ES}\_\_\_] [k\_Integer] := \{\mathcal{ES}\}[[k+1]]$ ;
 $E / : \mathcal{E}1\_E \equiv \mathcal{E}2\_E := \text{Inner}[CF@\#1 == CF@ \#2 \&, \mathcal{E}1, \mathcal{E}2, \text{And}]$ ;
 $E_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_] \equiv E_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_] \wedge := (d1 == d2) \wedge (r1 == r2) \wedge (E[\mathcal{E}1s] \equiv E[\mathcal{E}2s])$ ;
 $E / : \mathcal{E}1\_E * \mathcal{E}2\_E := E @@ \text{Table}[CF[\mathcal{E}1[kk] + \mathcal{E}2[kk]], \{kk, 0, \text{Min}[\mathcal{E}1[\$], \mathcal{E}2[\$]]\}]$ ;
 $E_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_] E_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_] \wedge := E_{(d1 \cup d2) \rightarrow (r1 \cup r2)} @@ (E[\mathcal{E}1s] \times E[\mathcal{E}2s])$ ;
```

```
(Alt) In[ ]:=
 $E_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_] // E_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_] := \text{Module}[\{is = r1 \cap d2, lvs\},$ 
 $lvs = \text{Flatten}@\text{Table}[\{y_{\$ei}, b_{\$ei}, t_{\$ei}, a_{\$ei}, x_{\$ei}\}, \{i, is\}]$ ;
 $E_{(d1 \cup \text{Complement}[d2, is]) \rightarrow (r2 \cup \text{Complement}[r1, is])} @@ (\text{Zip}_{lvs \cup lvs}[\{lvs^*.lvs, \text{Times}[$ 
 $E[\mathcal{E}1s] / . \text{Table}[(v : b | B | t | T | a | x | y)_i \rightarrow v_{\$ei}, \{i, is\}],$ 
 $E[\mathcal{E}2s] / . \text{Table}[(v : \beta | \tau | \alpha | \mathcal{A} | \xi | \eta)_i \rightarrow v_{\$ei}, \{i, is\}]$ 
 $]\})$ 
 $]$ 
```

```
(Alt) In[ ]:=
 $\Lambda 2 E_{d \rightarrow r}[\mathcal{A}_-] := \text{Module}[\{k\}, E_{d \rightarrow r} @@ 12U @ \text{Table}[\text{SeriesCoefficient}[\mathcal{A}, \{\epsilon, 0, k\}], \{k, 0, \$k\}]]$ ;
```

Zippping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.



```
(Alt) In[ ]:=
Zipvs[[ $\mathcal{F}_-, \mathcal{E}_-$ ]] :=
 $\{\mathcal{F}, \mathcal{E}\} // \text{Zip1}_{vs} // \text{EZip23}_{\text{Select}[vs, (\theta < \text{Wt}[\#] < \$n) \&]} // \text{Zip2}_{\text{Select}[vs, (\text{Wt}[\#] == \theta \vee \text{Wt}[\#] == \$n) \&]} //$ 
 $\text{Zip3}_{\text{Select}[vs, (\text{Wt}[\#] == \theta \vee \text{Wt}[\#] == \$n) \&]} // \text{Last}$ ;
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} e^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

(Alt) In[]:=

```

Zip1_{ } = Identity;
Zip1_{vs_} @ {f_, E[Q_, P___]} := PPZip1@Module[{I, F, G, u, v},
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == $n, D[u^*, v^*, f, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == $n, D[u, v, Q, 0], {u, vs}, {v, vs}];
  {CF[vs^*.(F.Inverse[I - G.F]).vs^* / 2], E[CF[Q - Log[Det[I - G.F]] / 2 - vs.G.vs / 2], P]}
]

```

Getting rid of linear terms.

Lemma 2. $\left\langle F: \mathcal{E}_{\oplus \sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{E}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F: \mathcal{E}_{|z_B \rightarrow z_B + F y_B} \right\rangle_B$.

(Alt) In[]:=

```

Zip2_{ } = Identity;
Zip2_{vs_} @ {f_, E[Q_, P___]} := PPZip2@Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == $n, D[u^*, v^*, f, 0], {u, vs}, {v, vs}];
  Y = Table[D_v Q, {v, vs}] /. AlsoUpper@Table[v -> 0, {v, vs}];
  CF /@ ({f_, E[Q - Y.vs + Y.F.Y / 2, P]} /. AlsoUpper@Thread[vs -> vs + F.Y])
]

```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F: \oplus^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

```

(Alt) In[ ]:= Zip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := PPZip3@Module[
  {F, u, v, Z, $k, kk, jj, $m = 0, m, n},
  $k = Length[ $\mathcal{E}$ ] - 1;
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0, $k}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == $n,  $\partial_{u^*, v^*} \mathcal{F}$ , 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ $k, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2 (m + 1)}$ 
        (Z[m, kk, u, v] + Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
     $\mathcal{F}$  - Sum[F[u, v] u* v* / 2, {u, vs}, {v, vs}],
    E@@Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, $k}]
  } /. AlsoUpper@Table[v → 0, {v, vs}])
]

```

Encapsulation.

```

(Alt) In[ ]:= EZip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := PPEZip3@Module[
  {nδ, n $\mathcal{F}$ , rc, ps, rr = {(*release rules*)}},
  rc = 0; nδ = Total[
    CoefficientRules[#, vs] /. (ps_ → c_) ⇒ (AppendTo[rr, cδ[++rc] → c]; cδ[rc] (Times@@vsps))
  ] & /@  $\mathcal{E}$ ;
  rc = 0; n $\mathcal{F}$  = Total[CoefficientRules[ $\mathcal{F}$ , vs*] /.
    (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{F}$ [++rc] → c]; c $\mathcal{F}$ [rc] (Times@@(vs*)ps))];
  CF[Expand[{n $\mathcal{F}$ , nδ} // Zip3vs] /. rr]
]

```

```

(Alt) In[ ]:= EZip23vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := PPEZip3@Module[
  {nδ, n $\mathcal{F}$ , rc, ps, rr = {(*release rules*)}},
  rc = 0; nδ = Total[
    CoefficientRules[#, vs] /. (ps_ → c_) ⇒ (AppendTo[rr, cδ[++rc] → c]; cδ[rc] (Times@@vsps))
  ] & /@  $\mathcal{E}$ ;
  rc = 0; n $\mathcal{F}$  = Total[CoefficientRules[ $\mathcal{F}$ , vs*] /.
    (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{F}$ [++rc] → c]; c $\mathcal{F}$ [rc] (Times@@(vs*)ps))];
  CF[Expand[{n $\mathcal{F}$ , nδ} // Zip2vs // Zip3vs] /. rr]
]

```

Profiling

(Alt) In[]:= **BeginProfile**[];

(Alt) In[]:= **cm**_{1,2→1} // **cm**_{1,3→1}

(Alt) Out[]:= $\mathbb{E}_{\{1,2,3\} \rightarrow \{1\}} \left[\begin{aligned} & a_1 (\alpha_1 + \alpha_2 + \alpha_3) + b_1 \beta_1 + b_1 \beta_2 + b_1 \beta_3 + y_1 \eta_1 + \frac{y_1 \eta_2}{\mathcal{A}_1} + \frac{y_1 \eta_3}{\mathcal{A}_1 \mathcal{A}_2} + \frac{x_1 \xi_1}{\mathcal{A}_2 \mathcal{A}_3} + b_1 \eta_2 \xi_1 + \frac{b_1 \eta_3 \xi_1}{\mathcal{A}_2} + \frac{x_1 \xi_2}{\mathcal{A}_3} + b_1 \eta_3 \xi_2 + x_1 \xi_3, \\ & - \frac{y_1 \beta_1 \eta_2}{\mathcal{A}_1} - \frac{y_1 \beta_1 \eta_3}{\mathcal{A}_1 \mathcal{A}_2} - \frac{y_1 \beta_2 \eta_3}{\mathcal{A}_1 \mathcal{A}_2} - \frac{x_1 \beta_2 \xi_1}{\mathcal{A}_2 \mathcal{A}_3} - \frac{x_1 \beta_3 \xi_1}{\mathcal{A}_2 \mathcal{A}_3} + a_1 \eta_2 \xi_1 - \frac{y_1 \eta_2^2 \xi_1}{\mathcal{A}_1} + \frac{a_1 \eta_3 \xi_1}{\mathcal{A}_2} - \\ & \frac{b_1 \beta_2 \eta_3 \xi_1}{\mathcal{A}_2} - \frac{2 y_1 \eta_2 \eta_3 \xi_1}{\mathcal{A}_1 \mathcal{A}_2} - \frac{y_1 \eta_3^2 \xi_1}{\mathcal{A}_1 \mathcal{A}_2^2} - \frac{x_1 \eta_2 \xi_1^2}{\mathcal{A}_2 \mathcal{A}_3} - \frac{1}{2} b_1 \eta_2^2 \xi_1^2 - \frac{x_1 \eta_3 \xi_1^2}{\mathcal{A}_2^2 \mathcal{A}_3} - \frac{b_1 \eta_2 \eta_3 \xi_1^2}{\mathcal{A}_2} - \\ & \frac{b_1 \eta_3^2 \xi_1^2}{2 \mathcal{A}_2^2} - \frac{x_1 \beta_3 \xi_2}{\mathcal{A}_3} + a_1 \eta_3 \xi_2 - \frac{y_1 \eta_3^2 \xi_2}{\mathcal{A}_1 \mathcal{A}_2} - \frac{2 x_1 \eta_3 \xi_1 \xi_2}{\mathcal{A}_2 \mathcal{A}_3} - \frac{b_1 \eta_3^2 \xi_1 \xi_2}{\mathcal{A}_2} - \frac{x_1 \eta_3 \xi_2^2}{\mathcal{A}_3} - \frac{1}{2} b_1 \eta_3^2 \xi_2^2 \end{aligned} \right]$

(Alt) In[]:= **Timing@Block**[{**\$k** = 1}, **Z**[**Knot**[3, 1]]]

KnotTheory: Loading precomputed data in PD4Knots`.

(Alt) Out[]:= $\left\{ 16.0313, \mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[\frac{1}{2} \left(-4 t \hbar - \text{Log} \left[\left(\frac{1}{T^3} - \frac{2}{T^2} + \frac{2}{T} \right)^2 \right] - \text{Log} \left[\left(1 + \frac{T}{1 - 2T + 2T^2} - \frac{T^2}{1 - 2T + 2T^2} \right)^2 \right] \right), \right. \right. \\ \left. \left. \frac{a (-2 \hbar + 2 T^2 \hbar)}{1 - T + T^2} + \frac{-2 \hbar + 3 T \hbar - 2 T^2 \hbar + T^3 \hbar}{1 - 2T + 3 T^2 - 2 T^3 + T^4} + \frac{x y (-2 \hbar^2 - 2 T \hbar^2)}{1 - T + T^2} \right] \right\}$

(Alt) In[]:= **PrintProfile**[]

(Alt) Out[]:= ProfileRoot is root. Profiled time: 16.36

- (1) 0.079/ 15.990 above Z
- (1) 0.219/ 0.219 above Boot
- (2) 0.015/ 0.015 above CF
- (1) 0/ 0.047 above EZip3
- (1) 0/ 0 above RVK
- (1) 0/ 0.016 above Zip1
- (1) 0/ 0.031 above Zip2
- (1) 0.016/ 0.047 above Zip3

CF: called 13144 times, time in 5.962/10.51

- (84) 0.265/ 0.469 under Z
- (76) 0.094/ 0.202 under Boot
- (138) 0.610/ 2.153 under EZip3
- (2) 0.015/ 0.015 under ProfileRoot
- (92) 0.172/ 0.393 under Zip1
- (276) 0.577/ 1.809 under Zip2
- (12476) 4.229/ 5.469 under Zip3
- (7621) 4.548/ 4.548 above CCF

CCF: called 7621 times, time in 4.548/4.548

- (7621) 4.548/ 4.548 under CF

Zip3: called 92 times, time in 2.894/8.363

- (22) 0.879/ 2.999 under Z
- (23) 1.078/ 2.674 under Boot
- (46) 0.921/ 2.643 under EZip3
- (1) 0.016/ 0.047 under ProfileRoot
- (12476) 4.229/ 5.469 above CF

Zip1: called 46 times, time in 1.248/1.641

```

( 22) 0.499/ 0.686 under Z
( 23) 0.749/ 0.939 under Boot
( 1) 0/ 0.016 under ProfileRoot
( 92) 0.172/ 0.393 above CF
Zip2: called 92 times, time in 1.015/2.824
( 22) 0.300/ 1.050 under Z
( 23) 0.357/ 0.574 under Boot
( 46) 0.358/ 1.169 under EZip3
( 1) 0/ 0.031 under ProfileRoot
( 276) 0.577/ 1.809 above CF
Boot: called 24 times, time in 0.314/17.655
( 5) 0.016/ 6.594 under Z
( 18) 0.079/ 10.840 under Boot
( 1) 0.219/ 0.219 under ProfileRoot
( 18) 0.079/ 10.840 above Boot
( 76) 0.094/ 0.202 above CF
( 23) 0.109/ 2.110 above EZip3
( 23) 0.749/ 0.939 above Zip1
( 23) 0.357/ 0.574 above Zip2
( 23) 1.078/ 2.674 above Zip3
EZip3: called 46 times, time in 0.3/6.265
( 22) 0.191/ 4.108 under Z
( 23) 0.109/ 2.110 under Boot
( 1) 0/ 0.047 under ProfileRoot
( 138) 0.610/ 2.153 above CF
( 46) 0.358/ 1.169 above Zip2
( 46) 0.921/ 2.643 above Zip3
Z: called 1 times, time in 0.079/15.985
( 1) 0.079/ 15.990 under ProfileRoot
( 5) 0.016/ 6.594 above Boot
( 84) 0.265/ 0.469 above CF
( 22) 0.191/ 4.108 above EZip3
( 22) 0.499/ 0.686 above Zip1
( 22) 0.300/ 1.050 above Zip2
( 22) 0.879/ 2.999 above Zip3
RVK: called 1 times, time in 0./0.
( 1) 0/ 0 under ProfileRoot

```

(Alt) In[]:= **Timing@Block**[{**\$k** = 1}, **Z[Knot**[8, 17]]]

$$\begin{aligned}
\text{(Alt) Out[]} = & \left\{ 52.9531, \mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[\frac{1}{2} \left(-2 \, \mathfrak{t} \, \hbar - \text{Log} \left[\left(-1 - \frac{1}{\mathfrak{T}^4} + \frac{4}{\mathfrak{T}^3} - \frac{6}{\mathfrak{T}^2} + \frac{5}{\mathfrak{T}} \right)^2 \right] - \right. \right. \\
& \text{Log} \left[\left(1 + \frac{\mathfrak{T}}{1 - 4 \, \mathfrak{T} + 6 \, \mathfrak{T}^2 - 5 \, \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{2 \, \mathfrak{T}^2}{1 - 4 \, \mathfrak{T} + 6 \, \mathfrak{T}^2 - 5 \, \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{\mathfrak{T}^3}{1 - 4 \, \mathfrak{T} + 6 \, \mathfrak{T}^2 - 5 \, \mathfrak{T}^3 + \mathfrak{T}^4} \right)^2 \right] - \\
& \text{Log} \left[\left(1 - \frac{\mathfrak{T}}{1 - 3 \, \mathfrak{T} + 4 \, \mathfrak{T}^2 - 4 \, \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{4 \, \mathfrak{T}^2}{1 - 3 \, \mathfrak{T} + 4 \, \mathfrak{T}^2 - 4 \, \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{7 \, \mathfrak{T}^3}{1 - 3 \, \mathfrak{T} + 4 \, \mathfrak{T}^2 - 4 \, \mathfrak{T}^3 + \mathfrak{T}^4} + \right. \\
& \left. \left. \frac{7 \, \mathfrak{T}^4}{1 - 3 \, \mathfrak{T} + 4 \, \mathfrak{T}^2 - 4 \, \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{4 \, \mathfrak{T}^5}{1 - 3 \, \mathfrak{T} + 4 \, \mathfrak{T}^2 - 4 \, \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{\mathfrak{T}^6}{1 - 3 \, \mathfrak{T} + 4 \, \mathfrak{T}^2 - 4 \, \mathfrak{T}^3 + \mathfrak{T}^4} \right)^2 \right] \right], \\
& -3 \, \hbar + 8 \, \mathfrak{T} \, \hbar - 8 \, \mathfrak{T}^2 \, \hbar + 8 \, \mathfrak{T}^4 \, \hbar - 8 \, \mathfrak{T}^5 \, \hbar + 3 \, \mathfrak{T}^6 \, \hbar \Bigg] + \frac{a \left(-6 \, \hbar + 16 \, \mathfrak{T} \, \hbar - 16 \, \mathfrak{T}^2 \, \hbar + 16 \, \mathfrak{T}^4 \, \hbar - 16 \, \mathfrak{T}^5 \, \hbar + 6 \, \mathfrak{T}^6 \, \hbar \right)}{1 - 4 \, \mathfrak{T} + 8 \, \mathfrak{T}^2 - 11 \, \mathfrak{T}^3 + 8 \, \mathfrak{T}^4 - 4 \, \mathfrak{T}^5 + \mathfrak{T}^6} + \\
& \frac{x \, y \left(-6 \, \hbar^2 + 10 \, \mathfrak{T} \, \hbar^2 - 6 \, \mathfrak{T}^2 \, \hbar^2 - 6 \, \mathfrak{T}^3 \, \hbar^2 + 10 \, \mathfrak{T}^4 \, \hbar^2 - 6 \, \mathfrak{T}^5 \, \hbar^2 \right)}{1 - 4 \, \mathfrak{T} + 8 \, \mathfrak{T}^2 - 11 \, \mathfrak{T}^3 + 8 \, \mathfrak{T}^4 - 4 \, \mathfrak{T}^5 + \mathfrak{T}^6} \Bigg] \Bigg\}
\end{aligned}$$

```
(Alt) In[ ]:= PrintProfile[]
```

```
(Alt) Out[ ]:= ProfileRoot is root. Profiled time: 69.313
```

```
( 2) 0.358/ 68.938 above Z
( 1) 0.219/ 0.219 above Boot
( 2) 0.015/ 0.015 above CF
( 1) 0/ 0.047 above EZip3
( 2) 0/ 0 above RVK
( 1) 0/ 0.016 above Zip1
( 1) 0/ 0.031 above Zip2
( 1) 0.016/ 0.047 above Zip3
CCF: called 21248 times, time in 28.875/28.875
( 21248) 28.875/ 28.875 under CF
CF: called 27375 times, time in 26.763/55.638
( 298) 1.436/ 3.190 under Z
( 88) 0.094/ 0.249 under Boot
( 321) 7.910/ 20.043 under EZip3
( 2) 0.015/ 0.015 under ProfileRoot
( 214) 0.422/ 1.191 under Zip1
( 642) 3.898/ 10.281 under Zip2
( 25810) 12.988/ 20.669 under Zip3
( 21248) 28.875/ 28.875 above CCF
Zip3: called 214 times, time in 6.973/27.642
( 79) 3.535/ 18.683 under Z
( 27) 1.204/ 2.956 under Boot
( 107) 2.218/ 5.956 under EZip3
( 1) 0.016/ 0.047 under ProfileRoot
( 25810) 12.988/ 20.669 above CF
Zip1: called 107 times, time in 2.465/3.656
( 79) 1.654/ 2.592 under Z
( 27) 0.811/ 1.048 under Boot
( 1) 0/ 0.016 under ProfileRoot
( 214) 0.422/ 1.191 above CF
Zip2: called 214 times, time in 2.342/12.623
( 79) 1.127/ 8.661 under Z
( 27) 0.403/ 0.683 under Boot
( 107) 0.812/ 3.248 under EZip3
( 1) 0/ 0.031 under ProfileRoot
( 642) 3.898/ 10.281 above CF
EZip3: called 107 times, time in 1.193/30.44
( 79) 1.084/ 28.001 under Z
( 27) 0.109/ 2.392 under Boot
( 1) 0/ 0.047 under ProfileRoot
( 321) 7.910/ 20.043 above CF
( 107) 0.812/ 3.248 above Zip2
( 107) 2.218/ 5.956 above Zip3
Z: called 2 times, time in 0.358/68.938
( 2) 0.358/ 68.938 under ProfileRoot
( 7) 0.031/ 7.453 above Boot
( 298) 1.436/ 3.190 above CF
( 79) 1.084/ 28.001 above EZip3
( 79) 1.654/ 2.592 above Zip1
( 79) 1.127/ 8.661 above Zip2
( 79) 3.535/ 18.683 above Zip3
Boot: called 28 times, time in 0.344/19.014
( 7) 0.031/ 7.453 under Z
( 20) 0.094/ 11.342 under Boot
```

```
( 1) 0.219/ 0.219 under ProfileRoot
( 20) 0.094/ 11.342 above Boot
( 88) 0.094/ 0.249 above CF
( 27) 0.109/ 2.392 above EZip3
( 27) 0.811/ 1.048 above Zip1
( 27) 0.403/ 0.683 above Zip2
( 27) 1.204/ 2.956 above Zip3
RVK: called 2 times, time in 0./0.
( 2) 0/ 0 under ProfileRoot
```

```
(Alt) In[ ]:= Timing@Block[{ $k = 2}, Z[Knot[3, 1]]]
```

$$\begin{aligned} \text{(Alt) Out[]} = & \left\{ 164.172, \mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[\frac{1}{2} \left(-4 t \hbar - \text{Log} \left[\left(\frac{1}{T^3} - \frac{2}{T^2} + \frac{2}{T} \right)^2 \right] - \text{Log} \left[\left(1 + \frac{T}{1 - 2T + 2T^2} - \frac{T^2}{1 - 2T + 2T^2} \right)^2 \right] \right) \right], \right. \\ & \frac{a \left(-2 \hbar + 2 T^2 \hbar \right)}{1 - T + T^2} + \frac{-2 \hbar + 3 T \hbar - 2 T^2 \hbar + T^3 \hbar}{1 - 2T + 3 T^2 - 2 T^3 + T^4} + \frac{x y \left(-2 \hbar^2 - 2 T \hbar^2 \right)}{1 - T + T^2}, \frac{a^2 \left(2 T \hbar^2 - 8 T^2 \hbar^2 + 2 T^3 \hbar^2 \right)}{1 - 2T + 3 T^2 - 2 T^3 + T^4} + \\ & \frac{a \left(2 T \hbar^2 - 14 T^2 \hbar^2 + 12 T^3 \hbar^2 - 6 T^4 \hbar^2 + 2 T^5 \hbar^2 \right)}{1 - 3T + 6 T^2 - 7 T^3 + 6 T^4 - 3 T^5 + T^6} + \frac{T \hbar^2 - 11 T^2 \hbar^2 + 16 T^3 \hbar^2 - 12 T^4 \hbar^2 + 8 T^5 \hbar^2 - 3 T^6 \hbar^2 + T^7 \hbar^2}{2 - 8T + 20 T^2 - 32 T^3 + 38 T^4 - 32 T^5 + 20 T^6 - 8 T^7 + 2 T^8} + \\ & \left. \frac{a x y \left(8 T \hbar^3 - 8 T^2 \hbar^3 - 4 T^3 \hbar^3 \right)}{1 - 2T + 3 T^2 - 2 T^3 + T^4} + \frac{x y \left(-2 \hbar^3 - 2 T^2 \hbar^3 - 6 T^3 \hbar^3 + 2 T^5 \hbar^3 \right)}{1 - 3T + 6 T^2 - 7 T^3 + 6 T^4 - 3 T^5 + T^6} + \frac{x^2 y^2 \left(\hbar^4 + 5 T \hbar^4 + T^2 \hbar^4 \right)}{1 - 2T + 3 T^2 - 2 T^3 + T^4} \right\} \end{aligned}$$

```
(Alt) In[ ]:= PrintProfile[]
```

```
(Alt) Out[ ]:= ProfileRoot is root. Profiled time: 233.484
```

```
( 3) 0.578/ 233.110 above Z
( 1) 0.219/ 0.219 above Boot
( 2) 0.015/ 0.015 above CF
( 1) 0/ 0.047 above EZip3
( 3) 0/ 0 above RVK
( 1) 0/ 0.016 above Zip1
( 1) 0/ 0.031 above Zip2
( 1) 0.016/ 0.047 above Zip3
CCF: called 36901 times, time in 122.686/122.686
( 36901) 122.690/ 122.690 under CF
CF: called 41349 times, time in 83.328/206.014
( 424) 2.124/ 4.908 under Z
( 202) 0.202/ 0.685 under Boot
( 501) 25.033/ 64.418 under EZip3
( 2) 0.015/ 0.015 under ProfileRoot
( 304) 0.640/ 1.690 under Zip1
( 1002) 8.161/ 21.563 under Zip2
( 38914) 47.153/ 112.740 under Zip3
( 36901) 122.690/ 122.690 above CCF
Zip3: called 304 times, time in 15.672/128.407
( 101) 6.083/ 77.808 under Z
( 50) 2.547/ 6.208 under Boot
( 152) 7.026/ 44.344 under EZip3
( 1) 0.016/ 0.047 under ProfileRoot
( 38914) 47.153/ 112.740 above CF
Zip1: called 152 times, time in 3.922/5.612
( 101) 2.110/ 3.328 under Z
( 50) 1.812/ 2.268 under Boot
( 1) 0/ 0.016 under ProfileRoot
( 304) 0.640/ 1.690 above CF
Zip2: called 304 times, time in 3.654/25.217
```



```

( 101) 1.534/ 16.161 under Z
( 50) 0.716/ 1.448 under Boot
( 152) 1.404/ 7.577 under EZip3
( 1) 0/ 0.031 under ProfileRoot
( 1002) 8.161/ 21.563 above CF
EZip3: called 152 times, time in 3.177/119.516
( 101) 2.944/ 113.910 under Z
( 50) 0.233/ 5.564 under Boot
( 1) 0/ 0.047 under ProfileRoot
( 501) 25.033/ 64.418 above CF
( 152) 1.404/ 7.577 above Zip2
( 152) 7.026/ 44.344 above Zip3
Z: called 3 times, time in 0.578/233.109
( 3) 0.578/ 233.110 under ProfileRoot
( 12) 0.061/ 16.421 above Boot
( 424) 2.124/ 4.908 above CF
( 101) 2.944/ 113.910 above EZip3
( 101) 2.110/ 3.328 above Zip1
( 101) 1.534/ 16.161 above Zip2
( 101) 6.083/ 77.808 above Zip3
Boot: called 48 times, time in 0.467/41.996
( 12) 0.061/ 16.421 under Z
( 35) 0.187/ 25.356 under Boot
( 1) 0.219/ 0.219 under ProfileRoot
( 35) 0.187/ 25.356 above Boot
( 202) 0.202/ 0.685 above CF
( 50) 0.233/ 5.564 above EZip3
( 50) 1.812/ 2.268 above Zip1
( 50) 0.716/ 1.448 above Zip2
( 50) 2.547/ 6.208 above Zip3
RVK: called 3 times, time in 0./0.
( 3) 0/ 0 under ProfileRoot

```

(Alt) In[]:= **Timing@Block**[{**\$k** = 2}, **Z[Knot**[8, 17]]]

$$\begin{aligned}
& \{1716.23, \mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[\frac{1}{2} \left(-2 \mathfrak{t} \hbar - \text{Log} \left[\left(-1 - \frac{1}{\mathfrak{T}^4} + \frac{4}{\mathfrak{T}^3} - \frac{6}{\mathfrak{T}^2} + \frac{5}{\mathfrak{T}} \right)^2 \right] - \right. \right. \\
& \quad \text{Log} \left[\left(1 + \frac{\mathfrak{T}}{1 - 4 \mathfrak{T} + 6 \mathfrak{T}^2 - 5 \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{2 \mathfrak{T}^2}{1 - 4 \mathfrak{T} + 6 \mathfrak{T}^2 - 5 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{\mathfrak{T}^3}{1 - 4 \mathfrak{T} + 6 \mathfrak{T}^2 - 5 \mathfrak{T}^3 + \mathfrak{T}^4} \right)^2 \right] - \\
& \quad \text{Log} \left[\left(1 - \frac{\mathfrak{T}}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{4 \mathfrak{T}^2}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{7 \mathfrak{T}^3}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} + \right. \right. \\
& \quad \left. \left. \frac{7 \mathfrak{T}^4}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{4 \mathfrak{T}^5}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{\mathfrak{T}^6}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} \right)^2 \right] \Bigg], \\
& \quad \frac{-3 \hbar + 8 \mathfrak{T} \hbar - 8 \mathfrak{T}^2 \hbar + 8 \mathfrak{T}^4 \hbar - 8 \mathfrak{T}^5 \hbar + 3 \mathfrak{T}^6 \hbar}{1 - 4 \mathfrak{T} + 8 \mathfrak{T}^2 - 11 \mathfrak{T}^3 + 8 \mathfrak{T}^4 - 4 \mathfrak{T}^5 + \mathfrak{T}^6} + \frac{a \left(-6 \hbar + 16 \mathfrak{T} \hbar - 16 \mathfrak{T}^2 \hbar + 16 \mathfrak{T}^4 \hbar - 16 \mathfrak{T}^5 \hbar + 6 \mathfrak{T}^6 \hbar \right)}{1 - 4 \mathfrak{T} + 8 \mathfrak{T}^2 - 11 \mathfrak{T}^3 + 8 \mathfrak{T}^4 - 4 \mathfrak{T}^5 + \mathfrak{T}^6} + \\
& \quad \frac{x y \left(-6 \hbar^2 + 10 \mathfrak{T} \hbar^2 - 6 \mathfrak{T}^2 \hbar^2 - 6 \mathfrak{T}^3 \hbar^2 + 10 \mathfrak{T}^4 \hbar^2 - 6 \mathfrak{T}^5 \hbar^2 \right)}{1 - 4 \mathfrak{T} + 8 \mathfrak{T}^2 - 11 \mathfrak{T}^3 + 8 \mathfrak{T}^4 - 4 \mathfrak{T}^5 + \mathfrak{T}^6}, \\
& \quad \left(a \left(8 \mathfrak{T} \hbar^2 - 64 \mathfrak{T}^2 \hbar^2 + 262 \mathfrak{T}^3 \hbar^2 - 608 \mathfrak{T}^4 \hbar^2 + 952 \mathfrak{T}^5 \hbar^2 - 1096 \mathfrak{T}^6 \hbar^2 + 952 \mathfrak{T}^7 \hbar^2 - 608 \mathfrak{T}^8 \hbar^2 + 262 \mathfrak{T}^9 \hbar^2 - 64 \mathfrak{T}^{10} \hbar^2 + \right. \right. \\
& \quad \left. \left. 8 \mathfrak{T}^{11} \hbar^2 \right) \right) / \left(1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) + \\
& \quad \left(a^2 \left(8 \mathfrak{T} \hbar^2 - 64 \mathfrak{T}^2 \hbar^2 + 262 \mathfrak{T}^3 \hbar^2 - 608 \mathfrak{T}^4 \hbar^2 + 952 \mathfrak{T}^5 \hbar^2 - 1096 \mathfrak{T}^6 \hbar^2 + 952 \mathfrak{T}^7 \hbar^2 - 608 \mathfrak{T}^8 \hbar^2 + 262 \mathfrak{T}^9 \hbar^2 - 64 \mathfrak{T}^{10} \hbar^2 + \right. \right. \\
& \quad \left. \left. 8 \mathfrak{T}^{11} \hbar^2 \right) \right) / \left(1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) + \\
& \quad \left(4 \mathfrak{T} \hbar^2 - 50 \mathfrak{T}^2 \hbar^2 + 307 \mathfrak{T}^3 \hbar^2 - 1160 \mathfrak{T}^4 \hbar^2 + 3062 \mathfrak{T}^5 \hbar^2 - 6127 \mathfrak{T}^6 \hbar^2 + 9760 \mathfrak{T}^7 \hbar^2 - 12754 \mathfrak{T}^8 \hbar^2 + 13916 \mathfrak{T}^9 \hbar^2 - \right. \\
& \quad \left. 12754 \mathfrak{T}^{10} \hbar^2 + 9760 \mathfrak{T}^{11} \hbar^2 - 6127 \mathfrak{T}^{12} \hbar^2 + 3062 \mathfrak{T}^{13} \hbar^2 - 1160 \mathfrak{T}^{14} \hbar^2 + 307 \mathfrak{T}^{15} \hbar^2 - 50 \mathfrak{T}^{16} \hbar^2 + 4 \mathfrak{T}^{17} \hbar^2 \right) / \\
& \quad \left(2 - 24 \mathfrak{T} + 144 \mathfrak{T}^2 - 578 \mathfrak{T}^3 + 1728 \mathfrak{T}^4 - 4056 \mathfrak{T}^5 + 7708 \mathfrak{T}^6 - 12072 \mathfrak{T}^7 + 15744 \mathfrak{T}^8 - 17194 \mathfrak{T}^9 + \right. \\
& \quad \left. 15744 \mathfrak{T}^{10} - 12072 \mathfrak{T}^{11} + 7708 \mathfrak{T}^{12} - 4056 \mathfrak{T}^{13} + 1728 \mathfrak{T}^{14} - 578 \mathfrak{T}^{15} + 144 \mathfrak{T}^{16} - 24 \mathfrak{T}^{17} + 2 \mathfrak{T}^{18} \right) + \\
& \quad \left(a x y \left(28 \mathfrak{T} \hbar^3 - 168 \mathfrak{T}^2 \hbar^3 + 544 \mathfrak{T}^3 \hbar^3 - 1000 \mathfrak{T}^4 \hbar^3 + 1248 \mathfrak{T}^5 \hbar^3 - 1096 \mathfrak{T}^6 \hbar^3 + \right. \right. \\
& \quad \left. \left. 656 \mathfrak{T}^7 \hbar^3 - 216 \mathfrak{T}^8 \hbar^3 - 20 \mathfrak{T}^9 \hbar^3 + 40 \mathfrak{T}^{10} \hbar^3 - 12 \mathfrak{T}^{11} \hbar^3 \right) \right) / \\
& \quad \left(1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) + \\
& \quad \left(x y \left(-18 \hbar^3 + 78 \mathfrak{T} \hbar^3 - 146 \mathfrak{T}^2 \hbar^3 + 110 \mathfrak{T}^3 \hbar^3 + 78 \mathfrak{T}^4 \hbar^3 - 274 \mathfrak{T}^5 \hbar^3 + \right. \right. \\
& \quad \left. \left. 274 \mathfrak{T}^6 \hbar^3 - 78 \mathfrak{T}^7 \hbar^3 - 110 \mathfrak{T}^8 \hbar^3 + 146 \mathfrak{T}^9 \hbar^3 - 78 \mathfrak{T}^{10} \hbar^3 + 18 \mathfrak{T}^{11} \hbar^3 \right) \right) / \\
& \quad \left(1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) + \\
& \quad \left(x^2 y^2 \left(3 \hbar^4 - 37 \mathfrak{T}^2 \hbar^4 + 153 \mathfrak{T}^3 \hbar^4 - 261 \mathfrak{T}^4 \hbar^4 + 325 \mathfrak{T}^5 \hbar^4 - 261 \mathfrak{T}^6 \hbar^4 + 153 \mathfrak{T}^7 \hbar^4 - 37 \mathfrak{T}^8 \hbar^4 + 3 \mathfrak{T}^{10} \hbar^4 \right) \right) / \\
& \quad \left(1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) \Bigg] \Bigg\}
\end{aligned}$$

(Alt) In[*]:= **PrintProfile[]**

(Alt) Out[*]:= ProfileRoot is root. Profiled time: 1949.72

```

( 4) 1.439/ 1949.340 above Z
( 1) 0.219/ 0.219 above Boot
( 2) 0.015/ 0.015 above CF
( 1) 0/ 0.047 above EZip3
( 4) 0/ 0 above RVK
( 1) 0/ 0.016 above Zip1
( 1) 0/ 0.031 above Zip2
( 1) 0.016/ 0.047 above Zip3

```

CCF: called 75053 times, time in 1015.35/1015.35

(75053) 1015.350/ 1015.350 under CF

CF: called 57038 times, time in 871.023/1886.37

```

( 745) 10.701/ 23.216 under Z
( 220) 0.233/ 0.780 under Boot
( 745) 405.395/ 878.573 under EZip3
( 2) 0.015/ 0.015 under ProfileRoot

```

```

( 426) 0.842/ 2.267 under Zip1
( 1490) 51.850/ 118.873 under Zip2
( 53410) 401.987/ 862.646 under Zip3
( 75053) 1015.350/ 1015.350 above CCF
Zip3: called 426 times, time in 36.966/899.612
( 158) 18.219/ 763.903 under Z
( 54) 2.689/ 6.536 under Boot
( 213) 16.042/ 129.126 under EZip3
( 1) 0.016/ 0.047 under ProfileRoot
( 53410) 401.987/ 862.646 above CF
EZip3: called 213 times, time in 13.35/1037.16
( 158) 13.102/ 1031.220 under Z
( 54) 0.248/ 5.892 under Boot
( 1) 0/ 0.047 under ProfileRoot
( 745) 405.395/ 878.573 above CF
( 213) 1.946/ 16.107 above Zip2
( 213) 16.042/ 129.126 above Zip3
Zip2: called 426 times, time in 5.905/124.778
( 158) 3.196/ 107.098 under Z
( 54) 0.763/ 1.542 under Boot
( 213) 1.946/ 16.107 under EZip3
( 1) 0/ 0.031 under ProfileRoot
( 1490) 51.850/ 118.873 above CF
Zip1: called 213 times, time in 5.221/7.488
( 158) 3.348/ 5.096 under Z
( 54) 1.873/ 2.376 under Boot
( 1) 0/ 0.016 under ProfileRoot
( 426) 0.842/ 2.267 above CF
Z: called 4 times, time in 1.439/1949.34
( 4) 1.439/ 1949.340 under ProfileRoot
( 14) 0.061/ 17.374 above Boot
( 745) 10.701/ 23.216 above CF
( 158) 13.102/ 1031.220 above EZip3
( 158) 3.348/ 5.096 above Zip1
( 158) 3.196/ 107.098 above Zip2
( 158) 18.219/ 763.903 above Zip3
Boot: called 52 times, time in 0.467/43.512
( 14) 0.061/ 17.374 under Z
( 37) 0.187/ 25.919 under Boot
( 1) 0.219/ 0.219 under ProfileRoot
( 37) 0.187/ 25.919 above Boot
( 220) 0.233/ 0.780 above CF
( 54) 0.248/ 5.892 above EZip3
( 54) 1.873/ 2.376 above Zip1
( 54) 0.763/ 1.542 above Zip2
( 54) 2.689/ 6.536 above Zip3
RVK: called 4 times, time in 0./0.
( 4) 0/ 0 under ProfileRoot

```

(Alt) In[]:= **Timing@Block**[{**\$k** = 3}, **Z[Knot**[3, 1]]]

$$\begin{aligned}
(\text{Alt}) \text{ Out}[*]:= & \left\{ 5163.2, \mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[\frac{1}{2} \left(-4 \mathfrak{t} \mathfrak{h} - \text{Log} \left[\left(\frac{1}{\mathfrak{T}^3} - \frac{2}{\mathfrak{T}^2} + \frac{2}{\mathfrak{T}} \right)^2 \right] - \text{Log} \left[\left(1 + \frac{\mathfrak{T}}{1 - 2 \mathfrak{T} + 2 \mathfrak{T}^2} - \frac{\mathfrak{T}^2}{1 - 2 \mathfrak{T} + 2 \mathfrak{T}^2} \right)^2 \right] \right) \right], \right. \\
& \frac{a \left(-2 \mathfrak{h} + 2 \mathfrak{T}^2 \mathfrak{h} \right)}{1 - \mathfrak{T} + \mathfrak{T}^2} + \frac{-2 \mathfrak{h} + 3 \mathfrak{T} \mathfrak{h} - 2 \mathfrak{T}^2 \mathfrak{h} + \mathfrak{T}^3 \mathfrak{h}}{1 - 2 \mathfrak{T} + 3 \mathfrak{T}^2 - 2 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{x y \left(-2 \mathfrak{h}^2 - 2 \mathfrak{T} \mathfrak{h}^2 \right)}{1 - \mathfrak{T} + \mathfrak{T}^2}, \frac{a^2 \left(2 \mathfrak{T} \mathfrak{h}^2 - 8 \mathfrak{T}^2 \mathfrak{h}^2 + 2 \mathfrak{T}^3 \mathfrak{h}^2 \right)}{1 - 2 \mathfrak{T} + 3 \mathfrak{T}^2 - 2 \mathfrak{T}^3 + \mathfrak{T}^4} + \\
& \frac{a \left(2 \mathfrak{T} \mathfrak{h}^2 - 14 \mathfrak{T}^2 \mathfrak{h}^2 + 12 \mathfrak{T}^3 \mathfrak{h}^2 - 6 \mathfrak{T}^4 \mathfrak{h}^2 + 2 \mathfrak{T}^5 \mathfrak{h}^2 \right)}{1 - 3 \mathfrak{T} + 6 \mathfrak{T}^2 - 7 \mathfrak{T}^3 + 6 \mathfrak{T}^4 - 3 \mathfrak{T}^5 + \mathfrak{T}^6} + \frac{\mathfrak{T} \mathfrak{h}^2 - 11 \mathfrak{T}^2 \mathfrak{h}^2 + 16 \mathfrak{T}^3 \mathfrak{h}^2 - 12 \mathfrak{T}^4 \mathfrak{h}^2 + 8 \mathfrak{T}^5 \mathfrak{h}^2 - 3 \mathfrak{T}^6 \mathfrak{h}^2 + \mathfrak{T}^7 \mathfrak{h}^2}{2 - 8 \mathfrak{T} + 20 \mathfrak{T}^2 - 32 \mathfrak{T}^3 + 38 \mathfrak{T}^4 - 32 \mathfrak{T}^5 + 20 \mathfrak{T}^6 - 8 \mathfrak{T}^7 + 2 \mathfrak{T}^8} + \\
& \frac{a x y \left(8 \mathfrak{T} \mathfrak{h}^3 - 8 \mathfrak{T}^2 \mathfrak{h}^3 - 4 \mathfrak{T}^3 \mathfrak{h}^3 \right)}{1 - 2 \mathfrak{T} + 3 \mathfrak{T}^2 - 2 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{x y \left(-2 \mathfrak{h}^3 - 2 \mathfrak{T}^2 \mathfrak{h}^3 - 6 \mathfrak{T}^3 \mathfrak{h}^3 + 2 \mathfrak{T}^5 \mathfrak{h}^3 \right)}{1 - 3 \mathfrak{T} + 6 \mathfrak{T}^2 - 7 \mathfrak{T}^3 + 6 \mathfrak{T}^4 - 3 \mathfrak{T}^5 + \mathfrak{T}^6} + \frac{x^2 y^2 \left(\mathfrak{h}^4 + 5 \mathfrak{T} \mathfrak{h}^4 + \mathfrak{T}^2 \mathfrak{h}^4 \right)}{1 - 2 \mathfrak{T} + 3 \mathfrak{T}^2 - 2 \mathfrak{T}^3 + \mathfrak{T}^4}, \\
& \frac{a^3 \left(-4 \mathfrak{T} \mathfrak{h}^3 + 28 \mathfrak{T}^2 \mathfrak{h}^3 - 28 \mathfrak{T}^4 \mathfrak{h}^3 + 4 \mathfrak{T}^5 \mathfrak{h}^3 \right)}{3 - 9 \mathfrak{T} + 18 \mathfrak{T}^2 - 21 \mathfrak{T}^3 + 18 \mathfrak{T}^4 - 9 \mathfrak{T}^5 + 3 \mathfrak{T}^6} + \frac{a^2 \left(-2 \mathfrak{T} \mathfrak{h}^3 + 24 \mathfrak{T}^2 \mathfrak{h}^3 - 12 \mathfrak{T}^3 \mathfrak{h}^3 - 32 \mathfrak{T}^4 \mathfrak{h}^3 + 20 \mathfrak{T}^5 \mathfrak{h}^3 - 8 \mathfrak{T}^6 \mathfrak{h}^3 + 2 \mathfrak{T}^7 \mathfrak{h}^3 \right)}{1 - 4 \mathfrak{T} + 10 \mathfrak{T}^2 - 16 \mathfrak{T}^3 + 19 \mathfrak{T}^4 - 16 \mathfrak{T}^5 + 10 \mathfrak{T}^6 - 4 \mathfrak{T}^7 + \mathfrak{T}^8} + \\
& \frac{a \left(-\mathfrak{T} \mathfrak{h}^3 + 19 \mathfrak{T}^2 \mathfrak{h}^3 - 19 \mathfrak{T}^3 \mathfrak{h}^3 - 34 \mathfrak{T}^4 \mathfrak{h}^3 + 40 \mathfrak{T}^5 \mathfrak{h}^3 - 22 \mathfrak{T}^6 \mathfrak{h}^3 + 11 \mathfrak{T}^7 \mathfrak{h}^3 - 3 \mathfrak{T}^8 \mathfrak{h}^3 + \mathfrak{T}^9 \mathfrak{h}^3 \right)}{1 - 5 \mathfrak{T} + 15 \mathfrak{T}^2 - 30 \mathfrak{T}^3 + 45 \mathfrak{T}^4 - 51 \mathfrak{T}^5 + 45 \mathfrak{T}^6 - 30 \mathfrak{T}^7 + 15 \mathfrak{T}^8 - 5 \mathfrak{T}^9 + \mathfrak{T}^{10}} + \\
& \frac{\left(-\mathfrak{T} \mathfrak{h}^3 + 29 \mathfrak{T}^2 \mathfrak{h}^3 - 43 \mathfrak{T}^3 \mathfrak{h}^3 - 71 \mathfrak{T}^4 \mathfrak{h}^3 + 131 \mathfrak{T}^5 \mathfrak{h}^3 - 84 \mathfrak{T}^6 \mathfrak{h}^3 + 53 \mathfrak{T}^7 \mathfrak{h}^3 - 23 \mathfrak{T}^8 \mathfrak{h}^3 + 11 \mathfrak{T}^9 \mathfrak{h}^3 - 3 \mathfrak{T}^{10} \mathfrak{h}^3 + \mathfrak{T}^{11} \mathfrak{h}^3 \right)}{\left(6 - 36 \mathfrak{T} + 126 \mathfrak{T}^2 - 300 \mathfrak{T}^3 + 540 \mathfrak{T}^4 - 756 \mathfrak{T}^5 + 846 \mathfrak{T}^6 - 756 \mathfrak{T}^7 + 540 \mathfrak{T}^8 - 300 \mathfrak{T}^9 + 126 \mathfrak{T}^{10} - 36 \mathfrak{T}^{11} + 6 \mathfrak{T}^{12} \right)} / \\
& \frac{a^2 x y \left(-8 \mathfrak{T} \mathfrak{h}^4 + 8 \mathfrak{T}^2 \mathfrak{h}^4 + 36 \mathfrak{T}^3 \mathfrak{h}^4 - 20 \mathfrak{T}^4 \mathfrak{h}^4 - 4 \mathfrak{T}^5 \mathfrak{h}^4 \right)}{1 - 3 \mathfrak{T} + 6 \mathfrak{T}^2 - 7 \mathfrak{T}^3 + 6 \mathfrak{T}^4 - 3 \mathfrak{T}^5 + \mathfrak{T}^6} + \\
& \frac{a x y \left(12 \mathfrak{T} \mathfrak{h}^4 - 16 \mathfrak{T}^2 \mathfrak{h}^4 + 40 \mathfrak{T}^3 \mathfrak{h}^4 - 16 \mathfrak{T}^4 \mathfrak{h}^4 - 56 \mathfrak{T}^5 \mathfrak{h}^4 + 8 \mathfrak{T}^6 \mathfrak{h}^4 + 4 \mathfrak{T}^7 \mathfrak{h}^4 \right)}{1 - 4 \mathfrak{T} + 10 \mathfrak{T}^2 - 16 \mathfrak{T}^3 + 19 \mathfrak{T}^4 - 16 \mathfrak{T}^5 + 10 \mathfrak{T}^6 - 4 \mathfrak{T}^7 + \mathfrak{T}^8} + \\
& \frac{x y \left(-4 \mathfrak{h}^4 + 3 \mathfrak{T} \mathfrak{h}^4 - 6 \mathfrak{T}^2 \mathfrak{h}^4 - 9 \mathfrak{T}^3 \mathfrak{h}^4 - 15 \mathfrak{T}^4 \mathfrak{h}^4 - 63 \mathfrak{T}^5 \mathfrak{h}^4 - 9 \mathfrak{T}^6 \mathfrak{h}^4 + 42 \mathfrak{T}^7 \mathfrak{h}^4 + 3 \mathfrak{T}^8 \mathfrak{h}^4 - 4 \mathfrak{T}^9 \mathfrak{h}^4 \right)}{3 - 15 \mathfrak{T} + 45 \mathfrak{T}^2 - 90 \mathfrak{T}^3 + 135 \mathfrak{T}^4 - 153 \mathfrak{T}^5 + 135 \mathfrak{T}^6 - 90 \mathfrak{T}^7 + 45 \mathfrak{T}^8 - 15 \mathfrak{T}^9 + 3 \mathfrak{T}^{10}} + \\
& \frac{a x^2 y^2 \left(-14 \mathfrak{T} \mathfrak{h}^5 - 6 \mathfrak{T}^2 \mathfrak{h}^5 + 30 \mathfrak{T}^3 \mathfrak{h}^5 + 4 \mathfrak{T}^4 \mathfrak{h}^5 \right)}{1 - 3 \mathfrak{T} + 6 \mathfrak{T}^2 - 7 \mathfrak{T}^3 + 6 \mathfrak{T}^4 - 3 \mathfrak{T}^5 + \mathfrak{T}^6} + \\
& \frac{x^2 y^2 \left(2 \mathfrak{h}^5 + 23 \mathfrak{T} \mathfrak{h}^5 - 10 \mathfrak{T}^2 \mathfrak{h}^5 + 11 \mathfrak{T}^3 \mathfrak{h}^5 + 42 \mathfrak{T}^4 \mathfrak{h}^5 - 29 \mathfrak{T}^5 \mathfrak{h}^5 - 8 \mathfrak{T}^6 \mathfrak{h}^5 \right)}{1 - 4 \mathfrak{T} + 10 \mathfrak{T}^2 - 16 \mathfrak{T}^3 + 19 \mathfrak{T}^4 - 16 \mathfrak{T}^5 + 10 \mathfrak{T}^6 - 4 \mathfrak{T}^7 + \mathfrak{T}^8} + \\
& \left. \frac{x^3 y^3 \left(-2 \mathfrak{h}^6 - 24 \mathfrak{T} \mathfrak{h}^6 - 24 \mathfrak{T}^2 \mathfrak{h}^6 - 2 \mathfrak{T}^3 \mathfrak{h}^6 \right)}{3 - 9 \mathfrak{T} + 18 \mathfrak{T}^2 - 21 \mathfrak{T}^3 + 18 \mathfrak{T}^4 - 9 \mathfrak{T}^5 + 3 \mathfrak{T}^6} \right\}
\end{aligned}$$

(Alt) In[*]:= **PrintProfile[]**

(Alt) Out[*]:= ProfileRoot is root. Profiled time: 7112.92

```

( 5)      2.001/ 7112.546 above Z
( 1)      0.219/   0.219 above Boot
( 2)      0.015/   0.015 above CF
( 1)      0/      0.047 above EZip3
( 5)      0/      0 above RVK
( 1)      0/      0.016 above Zip1
( 1)      0/      0.031 above Zip2
( 1)      0.016/   0.047 above Zip3

```

CCF: called 119154 times, time in 4179.14/4179.14

```
( 119154) 4179.141/ 4179.141 under CF
```

CF: called 71965 times, time in 2558.99/6738.14

```

( 913)    13.217/   28.779 under Z
( 372)     0.514/    1.589 under Boot
( 970)    876.916/ 1771.275 under EZip3
( 2)      0.015/   0.015 under ProfileRoot
( 516)    1.031/    2.722 under Zip1
( 1940)   80.550/   195.897 under Zip2

```

```

( 67252) 1586.751/ 4737.858 under Zip3
( 119154) 4179.141/ 4179.141 above CCF
Zip3: called 516 times, time in 249.015/4986.87
( 180) 31.949/ 2177.216 under Z
( 77) 4.762/ 11.017 under Boot
( 258) 212.288/ 2798.593 under EZip3
( 1) 0.016/ 0.047 under ProfileRoot
( 67252) 1586.751/ 4737.858 above CF
EZip3: called 258 times, time in 108.725/4718.62
( 180) 108.258/ 4707.606 under Z
( 77) 0.467/ 10.968 under Boot
( 1) 0/ 0.047 under ProfileRoot
( 970) 876.916/ 1771.275 above CF
( 258) 2.474/ 40.028 above Zip2
( 258) 212.288/ 2798.593 above Zip3
Zip2: called 516 times, time in 7.7/203.597
( 180) 4.119/ 160.709 under Z
( 77) 1.107/ 2.829 under Boot
( 258) 2.474/ 40.028 under EZip3
( 1) 0/ 0.031 under ProfileRoot
( 1940) 80.550/ 195.897 above CF
Zip1: called 258 times, time in 6.61/9.332
( 180) 3.941/ 5.940 under Z
( 77) 2.669/ 3.376 under Boot
( 1) 0/ 0.016 under ProfileRoot
( 516) 1.031/ 2.722 above CF
Z: called 5 times, time in 2.001/7112.55
( 5) 2.001/ 7112.546 under ProfileRoot
( 19) 0.109/ 30.295 above Boot
( 913) 13.217/ 28.779 above CF
( 180) 108.258/ 4707.606 above EZip3
( 180) 3.941/ 5.940 above Zip1
( 180) 4.119/ 160.709 above Zip2
( 180) 31.949/ 2177.216 above Zip3
Boot: called 72 times, time in 0.735/75.604
( 19) 0.109/ 30.295 under Z
( 52) 0.407/ 45.090 under Boot
( 1) 0.219/ 0.219 under ProfileRoot
( 52) 0.407/ 45.090 above Boot
( 372) 0.514/ 1.589 above CF
( 77) 0.467/ 10.968 above EZip3
( 77) 2.669/ 3.376 above Zip1
( 77) 1.107/ 2.829 above Zip2
( 77) 4.762/ 11.017 above Zip3
RVK: called 5 times, time in 0./0.
( 5) 0/ 0 under ProfileRoot

```

```

(Alt) In[ ]:= Timing@Block[{$k = 3}, Z[Knot[8, 17]]]

```

```

In[ ]:= PrintProfile[]

```