

Pensieve header: Profile with NO encapsulation of Zip3-Inner. Time to K817@\$k=2: 3905.91.

## Startup

```
In[ ]:= Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"..\\Profile\\Profile.m"];
$k = 1;
<< Objects.m
<< KT.m
```

```
Out[ ]:= {2021, 1, 3, 6, 28, 9.5249954}
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

This version: April 2020. Original version: July 1994.

## Engine

Canonical Forms:

```
In[ ]:= CCF[ $\mathcal{E}$ _] := PP_CCF@ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ]; (*Coefficient Canonical Form *)
CF[ $\mathcal{E}$ _] := PP_CF@Module[
  {vs = Cases[ $\mathcal{E}$ , (y | a | x |  $\eta$  |  $\beta$  |  $\tau$  |  $\xi$ )_,  $\infty$ ]  $\cup$  {y, a, x,  $\eta$ ,  $\beta$ ,  $\tau$ ,  $\xi$ }},
  Total[(CCF[#[[2]]] (Times @@ vs^#[[1]])) & /@ CoefficientRules[ $\mathcal{E}$ , vs]]
];
CF[ $\mathcal{E}$ _E] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _List] := CF /@  $\mathcal{E}$ ;
CF[E_sp___[ $\mathcal{ES}$ ___]] := CF /@ E_sp[ $\mathcal{ES}$ ];
```

Variables and their duals:

```
In[ ]:= {t*, b*, y*, a*, x*, z*,  $\tau$ *,  $\beta$ *,  $\eta$ *,  $\alpha$ *,  $\xi$ *,  $\zeta$ *} = { $\tau$ ,  $\beta$ ,  $\eta$ ,  $\alpha$ ,  $\xi$ ,  $\zeta$ , t, b, y, a, x, z};
(vs_List)* := (v  $\mapsto$  v*) /@ vs;
(u_i_)* := (u*)_i;
```

Weights:

```
In[ ]:= Clear[Wt];
Evaluate[Wt /@ {y, b, t, a, x,  $\eta$ ,  $\beta$ ,  $\tau$ ,  $\alpha$ ,  $\xi$ }] = {1, 0, 0, 2, 1, 1, 2, 2, 0, 1};
Wt[u_i_] := Wt[u];
```

The maximal weight \$n\$, i.e. the  $n$  of  $gl(n)$ . Initially and for a long while this will not be tested beyond  $n == 2$ .

```
In[ ]:= $n = 2;
```

Upper to lower and lower to Upper:



```

In[ ]:=
Zip1_{ } = Identity;
Zip1_{vs_} @ {f_, E[Q_, P___]} := PPZip1@Module[{I, F, G, u, v},
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == $n, D[u*, v*, f, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == $n, D[u, v, Q, 0], {u, vs}, {v, vs}];
  {CF[vs*. (F.Inverse[I - G.F]).vs* / 2], E[CF[Q - Log[Det[I - G.F]] / 2 - vs.G.vs / 2], P]}
]

```

Getting rid of linear terms.

**Lemma 2.**  $\left\langle F: \mathcal{E}^{\oplus \sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{C}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F: \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B.$

```

In[ ]:=
Zip2_{ } = Identity;
Zip2_{vs_} @ {f_, E[Q_, P___]} := PPZip2@Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == $n, D[u*, v*, f, 0], {u, vs}, {v, vs}];
  Y = Table[D_v, Q, {v, vs}] /. AlsoUpper@Table[v -> 0, {v, vs}];
  CF /@ ({f_, E[Q - Y.vs + Y.F.Y / 2, P]} /. AlsoUpper@Thread[vs -> vs + F.Y])
]

```

Dealing with Feynman diagrams.

**Lemma 3.** With an extra variable  $\lambda$ ,  $Z_\lambda := \log[\lambda F: \mathbb{C}^P]_B$  satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left( \partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

Note that the power  $m$  of  $\lambda$  is at most  $k - 1 + \frac{2k+2}{2} = 2k$ . We write  $Z_\lambda = \sum Z[m] \lambda^m$ .

```

In[ ]:= Zip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := PPZip3@Module[ {F, u, v, Z, $k, kk, jj, $m = 0, m, n},
  $k = Length[ $\mathcal{E}$ ] - 1;
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0, $k}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == $n,  $\partial_{u^*, v^*} \mathcal{F}$ , 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ $k, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2 (m + 1)}$ 
        (Z[m, kk, u, v] + Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
     $\mathcal{F}$  - Sum[F[u, v] u* v* / 2, {u, vs}, {v, vs}],
    E@@Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, $k}]
  } /. AlsoUpper@Table[v → 0, {v, vs}])
]

```

Encapsulation.

```

In[ ]:= EZip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := PPEZip3@Module[
  {n $\mathcal{E}$ , n $\mathcal{F}$ , j = 0, ps, rr = {(*release rules*)}},
  n $\mathcal{E}$  = Total[
    CoefficientRules[#, vs] /. (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{E}$ [++j] → c]; c $\mathcal{E}$ [j] (Times@@vsps))
  ] & /@  $\mathcal{E}$ ;
  n $\mathcal{F}$  = Total[CoefficientRules[ $\mathcal{F}$ , vs*] /.
    (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{F}$ [++j] → c]; c $\mathcal{F}$ [j] (Times@@(vs*)ps))];
  CF[Expand[{n $\mathcal{F}$ , n $\mathcal{E}$ } // Zip3vs] /. rr]
]

```

## Profiling

```

In[ ]:= BeginProfile[];

```

```

In[ ]:= Timing@Block[{ $k = 1}, Z[Knot[3, 1]]]

```

KnotTheory: Loading precomputed data in PD4Knots`.

$$\text{Out[ ]} = \left\{ 16.1563, \mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[ \frac{1}{2} \left( -4 \mathfrak{t} \hbar - \text{Log} \left[ \left( \frac{1}{\mathfrak{T}^3} - \frac{2}{\mathfrak{T}^2} + \frac{2}{\mathfrak{T}} \right)^2 \right] - \text{Log} \left[ \left( 1 + \frac{\mathfrak{T}}{1 - 2 \mathfrak{T} + 2 \mathfrak{T}^2} - \frac{\mathfrak{T}^2}{1 - 2 \mathfrak{T} + 2 \mathfrak{T}^2} \right)^2 \right] \right), \right. \\
\left. \frac{a \left( -2 \hbar + 2 \mathfrak{T}^2 \hbar \right)}{1 - \mathfrak{T} + \mathfrak{T}^2} + \frac{-2 \hbar + 3 \mathfrak{T} \hbar - 2 \mathfrak{T}^2 \hbar + \mathfrak{T}^3 \hbar}{1 - 2 \mathfrak{T} + 3 \mathfrak{T}^2 - 2 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{x y \left( -2 \hbar^2 - 2 \mathfrak{T} \hbar^2 \right)}{1 - \mathfrak{T} + \mathfrak{T}^2} \right] \right\}$$

```
In[ ]:= PrintProfile[]
```

```
Out[ ]:= ProfileRoot is root. Profiled time: 16.141
( 1) 0.063/ 16.140 above Z
( 1) 0/ 0 above RVK
CCF: called 8772 times, time in 6.295/6.295
( 8772) 6.295/ 6.295 under CF
CF: called 12930 times, time in 5.342/11.637
( 84) 0.187/ 0.359 under Z
( 76) 0.078/ 0.109 under Boot
( 90) 0.109/ 0.267 under Zip1
( 270) 1.623/ 5.968 under Zip2
( 12410) 3.345/ 4.934 under Zip3
( 8772) 6.295/ 6.295 above CCF
Zip3: called 90 times, time in 2.488/7.422
( 44) 1.252/ 4.296 under Z
( 46) 1.236/ 3.126 under Boot
( 12410) 3.345/ 4.934 above CF
Zip1: called 45 times, time in 1.021/1.288
( 22) 0.345/ 0.471 under Z
( 23) 0.676/ 0.817 under Boot
( 90) 0.109/ 0.267 above CF
Zip2: called 90 times, time in 0.761/6.729
( 44) 0.327/ 5.889 under Z
( 46) 0.434/ 0.840 under Boot
( 270) 1.623/ 5.968 above CF
Boot: called 23 times, time in 0.171/14.11
( 5) 0.046/ 5.063 under Z
( 18) 0.125/ 9.047 under Boot
( 18) 0.125/ 9.047 above Boot
( 76) 0.078/ 0.109 above CF
( 23) 0.676/ 0.817 above Zip1
( 46) 0.434/ 0.840 above Zip2
( 46) 1.236/ 3.126 above Zip3
Z: called 1 times, time in 0.063/16.141
( 1) 0.063/ 16.140 under ProfileRoot
( 5) 0.046/ 5.063 above Boot
( 84) 0.187/ 0.359 above CF
( 22) 0.345/ 0.471 above Zip1
( 44) 0.327/ 5.889 above Zip2
( 44) 1.252/ 4.296 above Zip3
RVK: called 1 times, time in 0./0.
( 1) 0/ 0 under ProfileRoot
```

In[#:]= **Timing@Block**[{**\$k** = 1}, **Z[Knot**[8, 17]]]

$$\text{Out[#:]= } \left\{ 61.6875, \mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[ \frac{1}{2} \left( -2 \mathfrak{t} \hbar - \text{Log} \left[ \left( -1 - \frac{1}{\mathfrak{T}^4} + \frac{4}{\mathfrak{T}^3} - \frac{6}{\mathfrak{T}^2} + \frac{5}{\mathfrak{T}} \right)^2 \right] - \right. \right. \\ \left. \text{Log} \left[ \left( 1 + \frac{\mathfrak{T}}{1 - 4 \mathfrak{T} + 6 \mathfrak{T}^2 - 5 \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{2 \mathfrak{T}^2}{1 - 4 \mathfrak{T} + 6 \mathfrak{T}^2 - 5 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{\mathfrak{T}^3}{1 - 4 \mathfrak{T} + 6 \mathfrak{T}^2 - 5 \mathfrak{T}^3 + \mathfrak{T}^4} \right)^2 \right] - \right. \\ \left. \text{Log} \left[ \left( 1 - \frac{\mathfrak{T}}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{4 \mathfrak{T}^2}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{7 \mathfrak{T}^3}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} + \right. \right. \\ \left. \left. \frac{7 \mathfrak{T}^4}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{4 \mathfrak{T}^5}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{\mathfrak{T}^6}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} \right)^2 \right] \right] \right), \\ \frac{-3 \hbar + 8 \mathfrak{T} \hbar - 8 \mathfrak{T}^2 \hbar + 8 \mathfrak{T}^4 \hbar - 8 \mathfrak{T}^5 \hbar + 3 \mathfrak{T}^6 \hbar}{1 - 4 \mathfrak{T} + 8 \mathfrak{T}^2 - 11 \mathfrak{T}^3 + 8 \mathfrak{T}^4 - 4 \mathfrak{T}^5 + \mathfrak{T}^6} + \frac{\mathfrak{a} \left( -6 \hbar + 16 \mathfrak{T} \hbar - 16 \mathfrak{T}^2 \hbar + 16 \mathfrak{T}^4 \hbar - 16 \mathfrak{T}^5 \hbar + 6 \mathfrak{T}^6 \hbar \right)}{1 - 4 \mathfrak{T} + 8 \mathfrak{T}^2 - 11 \mathfrak{T}^3 + 8 \mathfrak{T}^4 - 4 \mathfrak{T}^5 + \mathfrak{T}^6} + \\ \frac{\mathfrak{x} \mathfrak{y} \left( -6 \hbar^2 + 10 \mathfrak{T} \hbar^2 - 6 \mathfrak{T}^2 \hbar^2 - 6 \mathfrak{T}^3 \hbar^2 + 10 \mathfrak{T}^4 \hbar^2 - 6 \mathfrak{T}^5 \hbar^2 \right)}{1 - 4 \mathfrak{T} + 8 \mathfrak{T}^2 - 11 \mathfrak{T}^3 + 8 \mathfrak{T}^4 - 4 \mathfrak{T}^5 + \mathfrak{T}^6} \left. \right] \right\}$$

In[ ]:= **PrintProfile**[ ]

Out[ ]:= ProfileRoot is root. Profiled time: 77.812  
 ( 2) 0.188/ 77.812 above Z  
 ( 2) 0/ 0 above RVK  
 CCF: called 25936 times, time in 36.892/36.892  
 ( 25936) 36.892/ 36.892 under CF  
 CF: called 26978 times, time in 29.935/66.827  
 ( 298) 1.034/ 2.286 under Z  
 ( 88) 0.108/ 0.170 under Boot  
 ( 212) 0.344/ 0.721 under Zip1  
 ( 636) 15.436/ 40.996 under Zip2  
 ( 25744) 13.013/ 22.654 under Zip3  
 ( 25936) 36.892/ 36.892 above CCF  
 Zip3: called 212 times, time in 6.586/29.24  
 ( 158) 5.086/ 25.458 under Z  
 ( 54) 1.500/ 3.782 under Boot  
 ( 25744) 13.013/ 22.654 above CF  
 Zip1: called 106 times, time in 2.092/2.813  
 ( 79) 1.339/ 1.887 under Z  
 ( 27) 0.753/ 0.926 under Boot  
 ( 212) 0.344/ 0.721 above CF  
 Zip2: called 212 times, time in 1.932/42.928  
 ( 158) 1.388/ 41.868 under Z  
 ( 54) 0.544/ 1.060 under Boot  
 ( 636) 15.436/ 40.996 above CF  
 Z: called 2 times, time in 0.188/77.812  
 ( 2) 0.188/ 77.812 under ProfileRoot  
 ( 7) 0.046/ 6.125 above Boot  
 ( 298) 1.034/ 2.286 above CF  
 ( 79) 1.339/ 1.887 above Zip1  
 ( 158) 1.388/ 41.868 above Zip2  
 ( 158) 5.086/ 25.458 above Zip3  
 Boot: called 27 times, time in 0.187/15.735  
 ( 7) 0.046/ 6.125 under Z  
 ( 20) 0.141/ 9.610 under Boot  
 ( 20) 0.141/ 9.610 above Boot  
 ( 88) 0.108/ 0.170 above CF  
 ( 27) 0.753/ 0.926 above Zip1  
 ( 54) 0.544/ 1.060 above Zip2  
 ( 54) 1.500/ 3.782 above Zip3  
 RVK: called 2 times, time in 0./0.  
 ( 2) 0/ 0 under ProfileRoot

In[ ]:= **Timing@Block**[{**\$k** = 2}, **Z[Knot**[3, 1]]]

Out[ ]:= 
$$\left\{ 320.688, \mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[ \frac{1}{2} \left( -4 \mathfrak{t} \hbar - \text{Log} \left[ \left( \frac{1}{\mathfrak{T}^3} - \frac{2}{\mathfrak{T}^2} + \frac{2}{\mathfrak{T}} \right)^2 \right] - \text{Log} \left[ \left( 1 + \frac{\mathfrak{T}}{1 - 2 \mathfrak{T} + 2 \mathfrak{T}^2} - \frac{\mathfrak{T}^2}{1 - 2 \mathfrak{T} + 2 \mathfrak{T}^2} \right)^2 \right] \right), \right.$$
  

$$\frac{a \left( -2 \hbar + 2 \mathfrak{T}^2 \hbar \right)}{1 - \mathfrak{T} + \mathfrak{T}^2} + \frac{-2 \hbar + 3 \mathfrak{T} \hbar - 2 \mathfrak{T}^2 \hbar + \mathfrak{T}^3 \hbar}{1 - 2 \mathfrak{T} + 3 \mathfrak{T}^2 - 2 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{x y \left( -2 \hbar^2 - 2 \mathfrak{T} \hbar^2 \right)}{1 - \mathfrak{T} + \mathfrak{T}^2}, \frac{a^2 \left( 2 \mathfrak{T} \hbar^2 - 8 \mathfrak{T}^2 \hbar^2 + 2 \mathfrak{T}^3 \hbar^2 \right)}{1 - 2 \mathfrak{T} + 3 \mathfrak{T}^2 - 2 \mathfrak{T}^3 + \mathfrak{T}^4} +$$
  

$$\frac{a \left( 2 \mathfrak{T} \hbar^2 - 14 \mathfrak{T}^2 \hbar^2 + 12 \mathfrak{T}^3 \hbar^2 - 6 \mathfrak{T}^4 \hbar^2 + 2 \mathfrak{T}^5 \hbar^2 \right)}{1 - 3 \mathfrak{T} + 6 \mathfrak{T}^2 - 7 \mathfrak{T}^3 + 6 \mathfrak{T}^4 - 3 \mathfrak{T}^5 + \mathfrak{T}^6} + \frac{\mathfrak{T} \hbar^2 - 11 \mathfrak{T}^2 \hbar^2 + 16 \mathfrak{T}^3 \hbar^2 - 12 \mathfrak{T}^4 \hbar^2 + 8 \mathfrak{T}^5 \hbar^2 - 3 \mathfrak{T}^6 \hbar^2 + \mathfrak{T}^7 \hbar^2}{2 - 8 \mathfrak{T} + 20 \mathfrak{T}^2 - 32 \mathfrak{T}^3 + 38 \mathfrak{T}^4 - 32 \mathfrak{T}^5 + 20 \mathfrak{T}^6 - 8 \mathfrak{T}^7 + 2 \mathfrak{T}^8} +$$
  

$$\left. \frac{a x y \left( 8 \mathfrak{T} \hbar^3 - 8 \mathfrak{T}^2 \hbar^3 - 4 \mathfrak{T}^3 \hbar^3 \right)}{1 - 2 \mathfrak{T} + 3 \mathfrak{T}^2 - 2 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{x y \left( -2 \hbar^3 - 2 \mathfrak{T}^2 \hbar^3 - 6 \mathfrak{T}^3 \hbar^3 + 2 \mathfrak{T}^5 \hbar^3 \right)}{1 - 3 \mathfrak{T} + 6 \mathfrak{T}^2 - 7 \mathfrak{T}^3 + 6 \mathfrak{T}^4 - 3 \mathfrak{T}^5 + \mathfrak{T}^6} + \frac{x^2 y^2 \left( \hbar^4 + 5 \mathfrak{T} \hbar^4 + \mathfrak{T}^2 \hbar^4 \right)}{1 - 2 \mathfrak{T} + 3 \mathfrak{T}^2 - 2 \mathfrak{T}^3 + \mathfrak{T}^4} \right\}$$

```
In[ ]:= PrintProfile[]
```

```
Out[ ]:= ProfileRoot is root. Profiled time: 398.499
( 3) 0.329/ 398.499 above Z
( 3) 0/ 0 above RVK
CCF: called 51756 times, time in 209.633/209.633
( 51756) 209.633/ 209.633 under CF
CF: called 40772 times, time in 165.855/375.488
( 424) 1.544/ 3.361 under Z
( 202) 0.265/ 0.453 under Boot
( 302) 0.456/ 1.022 under Zip1
( 996) 34.351/ 99.917 under Zip2
( 38848) 129.239/ 270.735 under Zip3
( 51756) 209.633/ 209.633 above CCF
Zip3: called 302 times, time in 16.504/287.239
( 202) 13.428/ 279.492 under Z
( 100) 3.076/ 7.747 under Boot
( 38848) 129.239/ 270.735 above CF
Zip1: called 151 times, time in 3.043/4.065
( 101) 1.697/ 2.388 under Z
( 50) 1.346/ 1.677 under Boot
( 302) 0.456/ 1.022 above CF
Zip2: called 302 times, time in 2.793/102.71
( 202) 1.889/ 100.398 under Z
( 100) 0.904/ 2.312 under Boot
( 996) 34.351/ 99.917 above CF
Boot: called 47 times, time in 0.342/32.388
( 12) 0.062/ 12.531 under Z
( 35) 0.280/ 19.857 under Boot
( 35) 0.280/ 19.857 above Boot
( 202) 0.265/ 0.453 above CF
( 50) 1.346/ 1.677 above Zip1
( 100) 0.904/ 2.312 above Zip2
( 100) 3.076/ 7.747 above Zip3
Z: called 3 times, time in 0.329/398.499
( 3) 0.329/ 398.499 under ProfileRoot
( 12) 0.062/ 12.531 above Boot
( 424) 1.544/ 3.361 above CF
( 101) 1.697/ 2.388 above Zip1
( 202) 1.889/ 100.398 above Zip2
( 202) 13.428/ 279.492 above Zip3
RVK: called 3 times, time in 0./0.
( 3) 0/ 0 under ProfileRoot
```



In[ ]:= **Timing@Block**[{**\$k** = 2}, **Z[Knot**[8, 17]]]

$$\begin{aligned}
 \text{Out[ ]} = & \left\{ 3507.41, \mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[ \frac{1}{2} \left( -2 \mathfrak{t} \hbar - \text{Log} \left[ \left( -1 - \frac{1}{\mathfrak{T}^4} + \frac{4}{\mathfrak{T}^3} - \frac{6}{\mathfrak{T}^2} + \frac{5}{\mathfrak{T}} \right)^2 \right] - \right. \right. \\
 & \text{Log} \left[ \left( 1 + \frac{\mathfrak{T}}{1 - 4 \mathfrak{T} + 6 \mathfrak{T}^2 - 5 \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{2 \mathfrak{T}^2}{1 - 4 \mathfrak{T} + 6 \mathfrak{T}^2 - 5 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{\mathfrak{T}^3}{1 - 4 \mathfrak{T} + 6 \mathfrak{T}^2 - 5 \mathfrak{T}^3 + \mathfrak{T}^4} \right)^2 \right] - \\
 & \text{Log} \left[ \left( 1 - \frac{\mathfrak{T}}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{4 \mathfrak{T}^2}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{7 \mathfrak{T}^3}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} + \right. \right. \\
 & \left. \left. \frac{7 \mathfrak{T}^4}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} - \frac{4 \mathfrak{T}^5}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} + \frac{\mathfrak{T}^6}{1 - 3 \mathfrak{T} + 4 \mathfrak{T}^2 - 4 \mathfrak{T}^3 + \mathfrak{T}^4} \right)^2 \right] \right], \\
 & \frac{-3 \hbar + 8 \mathfrak{T} \hbar - 8 \mathfrak{T}^2 \hbar + 8 \mathfrak{T}^4 \hbar - 8 \mathfrak{T}^5 \hbar + 3 \mathfrak{T}^6 \hbar}{1 - 4 \mathfrak{T} + 8 \mathfrak{T}^2 - 11 \mathfrak{T}^3 + 8 \mathfrak{T}^4 - 4 \mathfrak{T}^5 + \mathfrak{T}^6} + \frac{a \left( -6 \hbar + 16 \mathfrak{T} \hbar - 16 \mathfrak{T}^2 \hbar + 16 \mathfrak{T}^4 \hbar - 16 \mathfrak{T}^5 \hbar + 6 \mathfrak{T}^6 \hbar \right)}{1 - 4 \mathfrak{T} + 8 \mathfrak{T}^2 - 11 \mathfrak{T}^3 + 8 \mathfrak{T}^4 - 4 \mathfrak{T}^5 + \mathfrak{T}^6} + \\
 & \frac{x y \left( -6 \hbar^2 + 10 \mathfrak{T} \hbar^2 - 6 \mathfrak{T}^2 \hbar^2 - 6 \mathfrak{T}^3 \hbar^2 + 10 \mathfrak{T}^4 \hbar^2 - 6 \mathfrak{T}^5 \hbar^2 \right)}{1 - 4 \mathfrak{T} + 8 \mathfrak{T}^2 - 11 \mathfrak{T}^3 + 8 \mathfrak{T}^4 - 4 \mathfrak{T}^5 + \mathfrak{T}^6}, \\
 & \left( a \left( 8 \mathfrak{T} \hbar^2 - 64 \mathfrak{T}^2 \hbar^2 + 262 \mathfrak{T}^3 \hbar^2 - 608 \mathfrak{T}^4 \hbar^2 + 952 \mathfrak{T}^5 \hbar^2 - 1096 \mathfrak{T}^6 \hbar^2 + 952 \mathfrak{T}^7 \hbar^2 - 608 \mathfrak{T}^8 \hbar^2 + 262 \mathfrak{T}^9 \hbar^2 - 64 \mathfrak{T}^{10} \hbar^2 + \right. \right. \\
 & \left. \left. 8 \mathfrak{T}^{11} \hbar^2 \right) \right) / \left( 1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) + \\
 & \left( a^2 \left( 8 \mathfrak{T} \hbar^2 - 64 \mathfrak{T}^2 \hbar^2 + 262 \mathfrak{T}^3 \hbar^2 - 608 \mathfrak{T}^4 \hbar^2 + 952 \mathfrak{T}^5 \hbar^2 - 1096 \mathfrak{T}^6 \hbar^2 + 952 \mathfrak{T}^7 \hbar^2 - 608 \mathfrak{T}^8 \hbar^2 + 262 \mathfrak{T}^9 \hbar^2 - 64 \mathfrak{T}^{10} \hbar^2 + \right. \right. \\
 & \left. \left. 8 \mathfrak{T}^{11} \hbar^2 \right) \right) / \left( 1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) + \\
 & \left( 4 \mathfrak{T} \hbar^2 - 50 \mathfrak{T}^2 \hbar^2 + 307 \mathfrak{T}^3 \hbar^2 - 1160 \mathfrak{T}^4 \hbar^2 + 3062 \mathfrak{T}^5 \hbar^2 - 6127 \mathfrak{T}^6 \hbar^2 + 9760 \mathfrak{T}^7 \hbar^2 - 12754 \mathfrak{T}^8 \hbar^2 + 13916 \mathfrak{T}^9 \hbar^2 - \right. \\
 & \left. 12754 \mathfrak{T}^{10} \hbar^2 + 9760 \mathfrak{T}^{11} \hbar^2 - 6127 \mathfrak{T}^{12} \hbar^2 + 3062 \mathfrak{T}^{13} \hbar^2 - 1160 \mathfrak{T}^{14} \hbar^2 + 307 \mathfrak{T}^{15} \hbar^2 - 50 \mathfrak{T}^{16} \hbar^2 + 4 \mathfrak{T}^{17} \hbar^2 \right) / \\
 & \left( 2 - 24 \mathfrak{T} + 144 \mathfrak{T}^2 - 578 \mathfrak{T}^3 + 1728 \mathfrak{T}^4 - 4056 \mathfrak{T}^5 + 7708 \mathfrak{T}^6 - 12072 \mathfrak{T}^7 + 15744 \mathfrak{T}^8 - 17194 \mathfrak{T}^9 + \right. \\
 & \left. 15744 \mathfrak{T}^{10} - 12072 \mathfrak{T}^{11} + 7708 \mathfrak{T}^{12} - 4056 \mathfrak{T}^{13} + 1728 \mathfrak{T}^{14} - 578 \mathfrak{T}^{15} + 144 \mathfrak{T}^{16} - 24 \mathfrak{T}^{17} + 2 \mathfrak{T}^{18} \right) + \\
 & \left( a x y \left( 28 \mathfrak{T} \hbar^3 - 168 \mathfrak{T}^2 \hbar^3 + 544 \mathfrak{T}^3 \hbar^3 - 1000 \mathfrak{T}^4 \hbar^3 + 1248 \mathfrak{T}^5 \hbar^3 - 1096 \mathfrak{T}^6 \hbar^3 + \right. \right. \\
 & \left. \left. 656 \mathfrak{T}^7 \hbar^3 - 216 \mathfrak{T}^8 \hbar^3 - 20 \mathfrak{T}^9 \hbar^3 + 40 \mathfrak{T}^{10} \hbar^3 - 12 \mathfrak{T}^{11} \hbar^3 \right) \right) / \\
 & \left( 1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) + \\
 & \left( x y \left( -18 \hbar^3 + 78 \mathfrak{T} \hbar^3 - 146 \mathfrak{T}^2 \hbar^3 + 110 \mathfrak{T}^3 \hbar^3 + 78 \mathfrak{T}^4 \hbar^3 - 274 \mathfrak{T}^5 \hbar^3 + \right. \right. \\
 & \left. \left. 274 \mathfrak{T}^6 \hbar^3 - 78 \mathfrak{T}^7 \hbar^3 - 110 \mathfrak{T}^8 \hbar^3 + 146 \mathfrak{T}^9 \hbar^3 - 78 \mathfrak{T}^{10} \hbar^3 + 18 \mathfrak{T}^{11} \hbar^3 \right) \right) / \\
 & \left( 1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) + \\
 & \left( x^2 y^2 \left( 3 \hbar^4 - 37 \mathfrak{T}^2 \hbar^4 + 153 \mathfrak{T}^3 \hbar^4 - 261 \mathfrak{T}^4 \hbar^4 + 325 \mathfrak{T}^5 \hbar^4 - 261 \mathfrak{T}^6 \hbar^4 + 153 \mathfrak{T}^7 \hbar^4 - 37 \mathfrak{T}^8 \hbar^4 + 3 \mathfrak{T}^{10} \hbar^4 \right) \right) / \\
 & \left. \left( 1 - 8 \mathfrak{T} + 32 \mathfrak{T}^2 - 86 \mathfrak{T}^3 + 168 \mathfrak{T}^4 - 248 \mathfrak{T}^5 + 283 \mathfrak{T}^6 - 248 \mathfrak{T}^7 + 168 \mathfrak{T}^8 - 86 \mathfrak{T}^9 + 32 \mathfrak{T}^{10} - 8 \mathfrak{T}^{11} + \mathfrak{T}^{12} \right) \right] \}
 \end{aligned}$$

```
In[ ]:= PrintProfile[]
```

```
Out[ ]:= ProfileRoot is root. Profiled time: 3905.91
(      4)      1.391/ 3905.905 above Z
(      4)      0/      0 above RVK
CF: called 56217 times, time in 1927.4/3828.7
(   745)      7.394/   15.875 under Z
(   220)      0.281/    0.501 under Boot
(   424)      0.660/    1.526 under Zip1
(  1484)     315.690/   775.411 under Zip2
( 53344)    1603.374/  3035.388 under Zip3
( 126292)   1901.302/  1901.302 above CCF
CCF: called 126292 times, time in 1901.3/1901.3
( 126292)   1901.302/  1901.302 under CF
Zip3: called 424 times, time in 66.68/3102.07
(   316)     63.463/  3094.009 under Z
(   108)      3.217/    8.059 under Boot
( 53344)    1603.374/  3035.388 above CF
Zip2: called 424 times, time in 4.767/780.178
(   316)      3.800/   777.631 under Z
(   108)      0.967/    2.547 under Boot
(  1484)     315.690/   775.411 above CF
Zip1: called 212 times, time in 3.978/5.504
(   158)      2.554/    3.749 under Z
(    54)      1.424/    1.755 under Boot
(   424)      0.660/    1.526 above CF
Z: called 4 times, time in 1.391/3905.91
(      4)      1.391/ 3905.905 under ProfileRoot
(     14)      0.077/   13.250 above Boot
(   745)      7.394/   15.875 above CF
(   158)      2.554/    3.749 above Zip1
(   316)      3.800/   777.631 above Zip2
(   316)     63.463/  3094.009 above Zip3
Boot: called 51 times, time in 0.388/33.544
(    14)      0.077/   13.250 under Z
(    37)      0.311/   20.294 under Boot
(    37)      0.311/   20.294 above Boot
(   220)      0.281/    0.501 above CF
(    54)      1.424/    1.755 above Zip1
(   108)      0.967/    2.547 above Zip2
(   108)      3.217/    8.059 above Zip3
RVK: called 4 times, time in 0./0.
(      4)      0/      0 under ProfileRoot
```

```
In[ ]:= Timing@Block[{$k = 3}, Z[Knot[3, 1]]]
```

```
Out[ ]:= $Aborted
```

```
In[ ]:= PrintProfile[]
```

```
Timing@Block[{$k = 3}, Z[Knot[8, 17]]]
```

```
In[ ]:= PrintProfile[]
```