Pensieve header: Full testing of the sl_2 portfolio. Continues pensieve://Projects/SL2Portfolio2/PortfolioTesting.nb. Time: 136.744.

Startup

In[∘]:=
```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 1;
<< Engine.m
<< Objects.m
<< KT.m
```

Out[∘]=
```
{2021, 1, 5, 9, 33, 21.0595704}
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
```

```
This version: April 2020. Original version: July 1994.
```

In[∘]:=
```
$k = 2; (*ℏ=γ=1;*)
```

Utilities

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Testing

In[∘]:=
```
Block[{$k = 1}, {
   am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j},
   aS → aS_i, aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k},
   dΔ → dΔ_{i→j,k}, C → C_i, C̄ → C̄_i, Kink → Kink_i, Kink̄ → Kink̄_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

Out[∘]=

$$\text{am} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,(\alpha_i+\alpha_j)+x_k\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right),\ 0\right]$$

$$\text{bm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\,(\beta_i+\beta_j)+y_k\,(\eta_i+\eta_j),\ -y_k\,\beta_i\,\eta_j\right]$$

$$\text{dm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,(\alpha_i+\alpha_j)+b_k\,\beta_i+b_k\,\beta_j+y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\frac{(1-B_k)\,\eta_j\,\xi_i}{\hbar}+x_k\,\xi_j,\right.$$
$$\left.-\frac{y_k\,\beta_i\,\eta_j}{\mathcal{A}_i}-\frac{x_k\,\beta_j\,\xi_i}{\mathcal{A}_j}+a_k\,B_k\,\eta_j\,\xi_i+\frac{\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j}+\frac{(1-3\,B_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i}+\frac{(1-3\,B_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j}+\frac{(1-4\,B_k+3\,B_k^2)\,\eta_j^2\,\xi_i^2}{4\,\hbar}\right]$$

$$R \to \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i+\hbar\,x_j\,y_i,\ -\frac{1}{4}\,\hbar^3\,x_j^2\,y_i^2\right]$$

$$\overline{R} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i-\frac{\hbar\,x_j\,y_i}{B_i},\ -\frac{\hbar^2\,a_j\,x_j\,y_i}{B_i}-\frac{3\,\hbar^3\,x_j^2\,y_i^2}{4\,B_i^2}\right]$$

$$P \to \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar}+\frac{\eta_i\,\xi_j}{\hbar},\ \frac{\eta_i^2\,\xi_j^2}{4\,\hbar}\right]$$

$$\text{aS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ -\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$$

$$\overline{\text{aS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ \hbar\,x_i\,\mathcal{A}_i\,\xi_i-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$$

$$\text{bS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ -\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$$

$$\overline{\text{bS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ \frac{\hbar\,y_i\,\eta_i}{B_i}-\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$$

$$\text{dS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-b_i\,\beta_i-\frac{y_i\,\mathcal{A}_i\,\eta_i}{B_i}-x_i\,\mathcal{A}_i\,\xi_i+\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,B_i},\right.$$
$$\frac{\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{B_i}-\frac{y_i\,\mathcal{A}_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,B_i^2}-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-x_i\,\mathcal{A}_i\,\beta_i\,\xi_i+\frac{a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i}-\frac{\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{B_i}+\frac{(-\mathcal{A}_i+B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{B_i}+$$
$$\left.\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\beta_i\,\eta_i\,\xi_i}{\hbar\,B_i}+\frac{y_i\,(3\,\mathcal{A}_i^2-B_i\,\mathcal{A}_i^2)\,\eta_i^2\,\xi_i}{2\,B_i^2}-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2+\frac{x_i\,(3\,\mathcal{A}_i^2-B_i\,\mathcal{A}_i^2)\,\eta_i\,\xi_i^2}{2\,B_i}+\frac{(-3\,\mathcal{A}_i^2+4\,B_i\,\mathcal{A}_i^2-B_i^2\,\mathcal{A}_i^2)\,\eta_i^2\,\xi_i^2}{4\,\hbar\,B_i^2}\right]$$

$$\text{a}\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+x_j\,\xi_i+x_k\,\xi_i,\ -\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right]$$

$$\text{b}\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[(b_j+b_k)\,\beta_i+B_k\,y_j\,\eta_i+y_k\,\eta_i,\ \frac{1}{2}\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\right]$$

$$\text{d}\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+(b_j+b_k)\,\beta_i+y_j\,\eta_i+B_j\,y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,\right.$$
$$\left.\frac{1}{2}\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2-\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right]$$

$$C \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2},\ -\frac{\hbar\,a_i}{2}\right]$$

$$\overline{C} \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2},\ \frac{\hbar\,a_i}{2}\right]$$

$$\text{Kink} \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2}+\hbar\,a_i\,b_i+\hbar\,x_i\,y_i,\ \frac{\hbar\,a_i}{2}-\frac{1}{4}\,\hbar^3\,x_i^2\,y_i^2\right]$$

$$\overline{\text{Kink}} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2}-\hbar\,a_i\,b_i-\frac{\hbar\,x_i\,y_i}{B_i},\ -\frac{\hbar\,a_i}{2}-\frac{\hbar^2\,a_i\,x_i\,y_i}{B_i}-\frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^2}\right]$$

$$\text{b2t} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i-t_i\,\beta_i+y_i\,\eta_i+x_i\,\xi_i,\ a_i\,\beta_i\right]$$

$$\text{t2b} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i+y_i\,\eta_i+x_i\,\xi_i-b_i\,\tau_i,\ a_i\,\tau_i\right]$$

Check that on the generators this agrees with our conventions in the handout:

In[∘]:=
```
𝔼2Λ[ℰ_] := Module[{k}, Sum[ℰ[k] ϵ^(k-1), {k, 0, ℰ[$]}]];
Timing@Block[{$k = 2}, {
  {
   "[a,x]" → 𝔼2Λ[𝔼_{}→{1,2}[0, a_2 x_1] // am_{1,2→1}] - 𝔼2Λ[𝔼_{}→{1,2}[0, a_1 x_2] // am_{1,2→1}],
   "[b,y]" → 𝔼2Λ[𝔼_{}→{1,2}[0, y_2 b_1, 0] // bm_{1,2→1}] - 𝔼2Λ[𝔼_{}→{1,2}[0, y_1 b_2, 0] // bm_{1,2→1}]
  } /. z_1 → z,
  {
   "Δ[y]" → Last[𝔼_{}→{1}[0, y_1] // bΔ_{1→1,2}],
   "Δ[b]" → Last[𝔼_{}→{1}[0, b_1] // bΔ_{1→1,2}],
   "Δ[a]" → Last[𝔼_{}→{1}[0, a_1] // aΔ_{1→1,2}],
   "Δ[x]" → Last[𝔼_{}→{1}[0, x_1] // aΔ_{1→1,2}]},
  {
   "S(a)" → ((𝔼_{}→{1}[0, a_1] // aS_1)[1]),
   "S(x)" → ((𝔼_{}→{1}[0, x_1] // aS_1)[1]),
   "S(b)" → ((𝔼_{}→{1}[0, b_1] // bS_1)[1]),
   "S(y)" → ((𝔼_{}→{1}[0, y_1] // bS_1)[1])
  } /. z_1 → z
}]
```

Out[∘]=
$$\Big\{3.26563, \big\{\{[a,x] \to -x,\ [b,y] \to -y\,\epsilon\},\ \{\Delta[y] \to B_2\, y_1 + y_2,\ \Delta[b] \to b_1 + b_2,\ \Delta[a] \to a_1 + a_2,\ \Delta[x] \to x_1 + x_2\},\ \{S(a) \to -a,\ S(x) \to -x,\ S(b) \to -b,\ S(y) \to -\frac{y}{B}\}\big\}\Big\}$$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {(am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}
 ]
```

Out[∘]= {0.3125, {True, True}}

R and *P* are inverses:

In[∘]:=
```
Timing@Block[{$k = 3}, {R_{i,j}, P_{i,k}, HL[(R_{i,j} // P_{i,k}) ≡ aσ_{k→j}]}]
```

Out[∘]=
$$\Big\{0.546875, \Big\{\mathbb{E}_{\{\}\to\{i,j\}}\Big[\hbar\, a_j\, b_i + \hbar\, x_j\, y_i,\ -\frac{1}{4}\hbar^3 x_j^2 y_i^2,\ \frac{1}{9}\hbar^5 x_j^3 y_i^3,\ \frac{1}{48}\left(\hbar^5 x_j^2 y_i^2 - 3\hbar^7 x_j^4 y_i^4\right)\Big],$$
$$\mathbb{E}_{\{i,k\}\to\{\}}\Big[\frac{\alpha_k\,\beta_i}{\hbar} + \frac{\eta_i\,\xi_k}{\hbar},\ \frac{\eta_i^2\,\xi_k^2}{4\hbar},\ \frac{1}{8}\eta_i^2\,\xi_k^2 + \frac{5\,\eta_i^3\,\xi_k^3}{36\hbar},\ \frac{1}{24}\hbar\,\eta_i^2\,\xi_k^2 + \frac{1}{6}\eta_i^3\,\xi_k^3 + \frac{5\,\eta_i^4\,\xi_k^4}{48\hbar}\Big],\ \text{True}\Big\}\Big\}$$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

In[∘]:=
```
Timing[HL /@ {(aS̄_1 // aS_1) ≡ aσ_{1→1}, (bS̄_1 // bS_1) ≡ bσ_{1→1}}]
```

Out[∘]= {0.75, {True, True}}

(co)-associativity on both sides

In[∘]:=
$$\text{Timing}[\\ \text{HL} \mathbin{/@} \{(a\Delta_{1\to1,2} \mathbin{//} a\Delta_{2\to2,3}) \equiv (a\Delta_{1\to1,3} \mathbin{//} a\Delta_{1\to1,2}),\ (b\Delta_{1\to1,2} \mathbin{//} b\Delta_{2\to2,3}) \equiv (b\Delta_{1\to1,3} \mathbin{//} b\Delta_{1\to1,2}),\\ (am_{1,2\to1} \mathbin{//} am_{1,3\to1}) \equiv (am_{2,3\to2} \mathbin{//} am_{1,2\to1}),\ (bm_{1,2\to1} \mathbin{//} bm_{1,3\to1}) \equiv (bm_{2,3\to2} \mathbin{//} bm_{1,2\to1})\}]$$

Out[∘]= {0.734375, {True, True, True, True}}

Δ is an algebra morphism

In[∘]:=
$$\text{Timing}[\text{HL} \mathbin{/@} \{(am_{1,2\to1} \mathbin{//} a\Delta_{1\to1,2}) \equiv ((a\Delta_{1\to1,3}\, a\Delta_{2\to2,4}) \mathbin{//} (am_{3,4\to2}\, am_{1,2\to1})),\\ (bm_{1,2\to1} \mathbin{//} b\Delta_{1\to1,2}) \equiv ((b\Delta_{1\to1,3}\, b\Delta_{2\to2,4}) \mathbin{//} (bm_{3,4\to2}\, bm_{1,2\to1}))\}]$$

Out[∘]= {1.39063, {True, True}}

An explicit formula for aS_i

In[∘]:=
$$\text{Timing}@\text{Block}\Big[\{\$k = 4\},\ \text{HL}\Big[\\ aS_i \equiv \Big(\Lambda2\mathbb{E}_{\{i\}\to\{i,j\}}\Big[\\ -\alpha_i\, a_j - \xi_i\, x_i +\\ \text{Log}@\text{Sum}\Big[\text{Expand}\Big[\frac{e^{\xi_i x_i}\,(-\hbar\,\epsilon)^k}{2^k\, k\,!}\ \text{Nest}\Big[\text{Expand}\big[x_i^2\, \partial_{\{x_i,2\}}\#\big]\ \&,\ e^{-\xi_i\, e^{\hbar\epsilon a_i}\, x_i},\ k\Big]\Big],\ \{k, 0, \$k\}\Big]\\ \Big] \mathbin{//} am_{i,j\to i}\Big)\\ \Big]\Big]$$

Out[∘]= {6.75, True}

S is convolution inverse of id

In[∘]:=
$$\text{Timing}[\text{HL}[\# \equiv s\epsilon_1\, s\eta_1]\ \&\ \mathbin{/@} \{\\ (a\Delta_{1\to1,2} \mathbin{//} aS_1) \mathbin{//} am_{1,2\to1},\ (a\Delta_{1\to1,2} \mathbin{//} aS_2) \mathbin{//} am_{1,2\to1},\\ (b\Delta_{1\to1,2} \mathbin{//} bS_1) \mathbin{//} bm_{1,2\to1},\ (b\Delta_{1\to1,2} \mathbin{//} bS_2) \mathbin{//} bm_{1,2\to1}\}]$$

Out[∘]= {1.1875, {True, True, True, True}}

But not with the opposite product:

In[∘]:=
```
Timing[Short[# ≡ sϵ1 sη1] & /@ {
    (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am2,1→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am2,1→1,
    (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm2,1→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm2,1→1}]
```

Out[∘]=

$$\Big\{0.015625,$$

$$\Big\{B_{1,2}\Big[B_1\Big[\mathbb{E}_{\{1\}\to\{1,2\}}\Big[a_1\,\alpha_1+a_2\,\alpha_1+x_1\,\xi_1+x_2\,\xi_1,\ -\hbar\,a_1\ll 1\gg\xi_1+\ll 1\gg,\ \frac{1}{2}\hbar^2\,a_1^2\,x_2\,\xi_1+\ll 5\gg\Big],$$

$$\ll 1\gg\Big],\ \ll 1\gg\Big]\ll 1\gg\ \ll 1\gg,$$

$$B_{1,2}\Big[B_2\Big[\mathbb{E}_{\{1\}\to\{1,2\}}\Big[a_1\,\alpha_1+a_2\,\alpha_1+x_1\,\xi_1+x_2\,\xi_1,\ -\hbar\,a_1\ll 1\gg\xi_1+\ll 1\gg,\ \frac{1}{2}\hbar^2\,a_1^2\,x_2\,\xi_1+\ll 5\gg\Big],$$

$$\ll 1\gg\Big],\ \ll 1\gg\Big]\ll 1\gg\ \ll 1\gg,$$

$$B_{1,2}\Big[B_1\Big[\ll 1\gg,\ \mathbb{E}_{\{1\}\to\{1\}}\Big[-b_1\,\beta_1-\frac{y_1\,\eta_1}{B_1},\ \ll 1\gg,\ -\frac{y_1\,\beta_1^2\,\eta_1}{2\,B_1}+\frac{\ll 1\gg}{\ll 1\gg}-\ll 1\gg-\frac{\hbar^2\ll 1\gg\ \ll 1\gg}{2\ll 1\gg}\Big]\Big],$$

$$\ll 1\gg\Big]\equiv\ll 1\gg,$$

$$B_{1,2}\Big[B_2\Big[\ll 1\gg,\ \mathbb{E}_{\{2\}\to\{2\}}\Big[-b_2\,\beta_2-\frac{y_2\,\eta_2}{B_2},\ \ll 1\gg,\ -\frac{y_2\,\beta_2^2\,\eta_2}{2\,B_2}+\frac{\ll 1\gg}{\ll 1\gg}-\ll 1\gg-\frac{\hbar^2\ll 1\gg\ \ll 1\gg}{2\ll 1\gg}\Big]\Big],$$

$$\ll 1\gg\Big]\equiv\ll 1\gg\Big\}\Big\}$$

S is an algebra anti-(co)morphism

In[∘]:=
```
Timing[HL /@ {(am1,2→1 // aS1) ≡ ((aS1 aS2) // am2,1→1), (bm1,2→1 // bS1) ≡ ((bS1 bS2) // bm2,1→1),
    (aS1 // aΔ1→1,2) ≡ (aΔ1→2,1 // (aS1 aS2)), (bS1 // bΔ1→1,2) ≡ (bΔ1→2,1 // (bS1 bS2))}]
```

Out[∘]= {1.45313, {True, True, True, True}}

Pairing axioms

In[∘]:=
```
Timing[HL /@ {((bm1,2→1 sϒ3→0,0,3,3 // sϵ0) // P1,3) ≡
     (( (sϒ1→1,1,0,0 // sϵ0) (sϒ2→2,2,0,0 // sϵ0) aΔ3→4,5) // P1,4 // P2,5),
    ((bΔ1→1,2 (sϒ3→0,0,3,3 // sϵ0) (sϒ4→0,0,4,4 // sϵ0)) // P1,3 // P2,4) ≡
     (((sϒ1→1,1,0,0 // sϵ0) am3,4→3) // P1,3 )}]
```

Out[∘]= {1.92188, {True, True}}

In[∘]:=
```
Timing[HL /@ {((bS1 aσ2→2) // P1,2) ≡ ((bσ1→1 aS2) // P1,2),
    ((bS̄1 aσ2→2) // P1,2) ≡ ((bσ1→1 aS̄2) // P1,2)}]
```

Out[∘]= {0.921875, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
In[◦]:= (*Timing@{{
        "[a,y]"→ ((𝔼{}→{1,2}[0,0,y2a1]~B1,2~dm1,2→1)[3]-(𝔼{}→{1,2}[0,0,y1a2]~B1,2~dm1,2→1)[3]),
        "[b,x]"   →
         ((𝔼{}→{1,2}[0,0,x2b1]~B1,2~dm1,2→1)[3]-(𝔼{}→{1,2}[0,0,x1b2]~B1,2~dm1,2→1)[3]),"xy-qyx"→
         ((𝔼{}→{1,2}[0,0,x1y2]~B1,2~dm1,2→1)[3]-(1+ϵ)(𝔼{}→{1,2}[0,0,y1x2]~B1,2~dm1,2→1)[3])
       }/.{z_1→z}//Expand//Factor,
   {
     "Δ(a)"→((𝔼{}→{1}[0,0,a1]~B1~dΔ1→1,2)[3]),
     "Δ(x)"→((𝔼{}→{1}[0,0,x1]~B1~dΔ1→1,2)[3]),
     "Δ(b)"→((𝔼{}→{1}[0,0,b1]~B1~dΔ1→1,2)[3]),
     "Δ(y)"→((𝔼{}→{1}[0,0,y1]~B1~dΔ1→1,2)[3])
    }//Simplify,
   {
      "S(a)"→((𝔼{}→{1}[0,0,a1]~B1~dS1)[3]),
      "S(x)"→((𝔼{}→{1}[0,0,x1]~B1~dS1)[3]),
      "S(b)"→((𝔼{}→{1}[0,0,b1]~B1~dS1)[3]),
      "S(y)"→((𝔼{}→{1}[0,0,y1]~B1~dS1)[3])
     }/.{z_1→z}//Simplify
  }*)
```

```
In[◦]:= {HL[((sϒ1→0,0,1,1 // sϵ0) (sϒ2→0,0,2,2 // sϵ0) // dm1,2→1) ≡ am1,2→1],
  HL[((sϒ1→1,1,0,0 // sϵ0) (sϒ2→2,2,0,0 // sϵ0) // dm1,2→1) ≡ bm1,2→1]}
Out[◦]= {True, True}
```

(co)-associativity

```
In[◦]:= Timing[Block[{$k = 1},
   HL /@ {(dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→1,3 // dΔ1→1,2), (dm1,2→1 // dm1,3→1) ≡ (dm2,3→2 // dm1,2→1)}]
 ]
Out[◦]= {0.4375, {True, True}}
```

Δ is an algebra morphism

```
In[◦]:= Timing@HL[(dm1,2→1 // dΔ1→1,2) ≡ ((dΔ1→1,3 dΔ2→2,4) // (dm3,4→2 dm1,2→1))]
Out[◦]= {2.5, True}
```

dS and $\overline{\text{dS}}$ are inverses:

```
In[◦]:= Timing@HL[(d̅S̅1 // dS1) ≡ dσ1→1]
Out[◦]= {2.125, True}
```

S_2 inverts R, but not S_1:

```
In[∘]:= Timing@{(R1,2 // dS1) ≡ R̄1,2, HL[(R1,2 // dS2) ≡ R̄1,2]}
```

$$\text{Out}=\left\{0.390625,\ \left\{\frac{\hbar^2 x_2 y_1}{B_1}-\frac{\hbar^2 a_2 x_2 y_1}{B_1}-\frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2}==-\frac{\hbar^2 a_2 x_2 y_1}{B_1}-\frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2}\ \&\&\ -\frac{\hbar^3 x_2 y_1}{2 B_1}+\frac{\hbar^3 a_2 x_2 y_1}{B_1}-\frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1}+\frac{2\hbar^4 x_2^2 y_1^2}{B_1^2}-\frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2}-\frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3}==-\frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1}+\frac{\hbar^4 x_2^2 y_1^2}{2 B_1^2}-\frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2}-\frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3},\ \text{True}\right\}\right\}$$

dS is convolution inverse of id

```
In[∘]:= Timing[HL[# ≡ dϵ1 dη1] & /@ {(dΔ1→1,2 // dS1) // dm1,2→1, (dΔ1→1,2 // dS2) // dm1,2→1}]
Out[∘]= {2.76563, {True, True}}
```

dS is a (co)-algebra anti-morphism

```
In[∘]:= Timing[HL /@
          Expand /@ {(dm1,2→1 // dS1) ≡ ((dS1 dS2) // dm2,1→1), (dS1 // dΔ1→1,2) ≡ (dΔ1→2,1 // (dS1 dS2))}]
Out[∘]= {5.04688, {True, True}}
```

Quasi-triangular axiom 1:

```
In[∘]:= Timing[
         HL /@ {(R1,3 // dΔ1→1,2) ≡ ((R1,4 R2,3) // dm3,4→3), (R1,2 // dΔ2→2,3) ≡ ((R1,2 R4,3) // dm1,4→1)}]
Out[∘]= {0.640625, {True, True}}
```

Quasi-triangular axiom 2:

```
In[∘]:= Timing@HL[((dΔ1->1,2 R3,4) // (dm1,3→1 dm2,4→2)) ≡ ((R1,2 dΔ1→3,4) // (dm1,4→1 dm2,3→2))]
Out[∘]= {2.10938, True}
```

The Drinfel'd element inverse property, $(u_1\,\bar{u}_2)$ // $\mathrm{dm}_{1,2\to1} \equiv \mathrm{d}\epsilon_i$:

```
In[∘]:= Timing@HL[(((R1,2 // dS1 // dm2,1→i) (R1,2 // dS2 // dS2 // dm2,1→j)) // dmi,j→i) ≡ dηi]
```

$$\text{Out}=\left\{3.4375,\ -\text{Log}\left[\frac{1}{B_i}\right]-\text{Log}[B_i]==0\right\}$$

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

```
In[∘]:= Timing@
         Block[{$k = 2}, (((R1,2 // dS1 // dm2,1→i) // dSi) (R1,2 // dS2 // dS2 // dm2,1→j)) // dmi,j→i]
```

$$\text{Out}=\left\{5.79688,\ \mathbb{E}_{\{\}\to\{i\}}\left[-\text{Log}\left[\frac{1}{B_i}\right]-2\,\text{Log}[B_i],\ \hbar\, a_i,\ 0\right]\right\}$$

In[∘]:= $\text{Timing@Block}\left[\{\$k = 2\}, \text{HL} \mathbin{/@} \left\{\left(\left(C_i\, \overline{C}_j\right) \mathbin{//} dm_{i,j\to i}\right) \equiv d\eta_i,\ \left(\left(\overline{C}_i\, \overline{C}_j\right) \mathbin{//} dm_{i,j\to i}\right) \equiv \left(\left(\left(R_{1,2} \mathbin{//} dS_1 \mathbin{//} dm_{2,1\to i}\right) \mathbin{//} dS_i\right) \left(R_{1,2} \mathbin{//} dS_2 \mathbin{//} dS_2 \mathbin{//} dm_{2,1\to j}\right) \mathbin{//} dm_{i,j\to i}\right)\right\}\right]$

Out[∘]= $\left\{6.04688, \left\{\text{True}, \hbar\, b_i == -\text{Log}\left[\frac{1}{B_i}\right] - 2\,\text{Log}[B_i]\right\}\right\}$

In[∘]:= $\text{Timing@Block}\left[\{\$k = 2\}, \text{HL} \mathbin{/@} \left\{\left(\left(C_i\, \overline{C}_j\right) \mathbin{//} dm_{i,j\to i}\right) \equiv d\eta_i,\ \left(\left(\overline{C}_i\, \overline{C}_j\right) \mathbin{//} dm_{i,j\to i}\right) \equiv \left(\left(\left(\left(R_{1,2} \mathbin{//} dS_1 \mathbin{//} dm_{2,1\to i}\right) \mathbin{//} dS_i\right) \left(R_{1,2} \mathbin{//} dS_2 \mathbin{//} dS_2 \mathbin{//} dm_{2,1\to j}\right)\right) \mathbin{//} dm_{i,j\to i}\right)\right\}\right]$

Out[∘]= $\left\{6.42188, \left\{\text{True}, \hbar\, b_i == -\text{Log}\left[\frac{1}{B_i}\right] - 2\,\text{Log}[B_i]\right\}\right\}$

Reidemeister 2:

In[∘]:= $\text{Timing}\left[\text{HL}[\# \equiv d\eta_1\, d\eta_2]\ \&\ \mathbin{/@} \left\{\left(\overline{R}_{1,2}\, R_{3,4}\right) \mathbin{//} (dm_{1,3\to 1}\, dm_{2,4\to 2}),\ \left(R_{1,2}\, \overline{R}_{3,4}\right) \mathbin{//} (dm_{1,3\to 1}\, dm_{2,4\to 2})\right\}\right]$

Out[∘]= {1.75, {True, True}}

Cyclic Reidemeister 2:

In[∘]:= $\text{Timing@HL}\left[\left(\left(R_{1,4}\, \overline{R}_{5,2}\, \overline{C}_3\right) \mathbin{//} dm_{2,4\to 2} \mathbin{//} dm_{1,3\to 1} \mathbin{//} dm_{1,5\to 1}\right) \equiv \overline{C}_1\, d\eta_2\right]$

Out[∘]= {0.859375, True}

Reidemeister 3:

In[∘]:= $\text{Timing@HL}[(R_{1,2}\, R_{6,3}\, R_{4,5} \mathbin{//} dm_{1,6\to 1}\, dm_{2,4\to 2}\, dm_{3,5\to 3}) \equiv (R_{2,3}\, R_{1,4}\, R_{5,6} \mathbin{//} dm_{1,5\to 1}\, dm_{2,6\to 2}\, dm_{3,4\to 3})]$

Out[∘]= {3.64063, True}

Relations between the four kinks:

In[∘]:= $\text{Timing}\left[\text{HL} \mathbin{/@} \left\{\text{Kink}_i \equiv ((R_{3,1}\, C_2) \mathbin{//} dm_{1,2\to 1} \mathbin{//} dm_{1,3\to i}),\ \overline{\text{Kink}}_j \equiv \left(\left(\overline{R}_{3,1}\, \overline{C}_2\right) \mathbin{//} dm_{1,2\to 1} \mathbin{//} dm_{1,3\to j}\right),\ \left(\left(\text{Kink}_i\, \overline{\text{Kink}}_j\right) \mathbin{//} dm_{i,j\to 1}\right) \equiv d\eta_1\right\}\right]$

Out[∘]= $\left\{4.59375, \left\{\frac{\hbar\, b_i}{2} + \hbar\, a_i\, b_i + \hbar\, x_i\, y_i == \hbar\, a_i\, b_i + \frac{1}{2}\left(-2\,\text{Log}[B_i] - \hbar\, b_i\right) + \hbar\, x_i\, y_i,\ -\frac{\hbar\, b_j}{2} - \hbar\, a_j\, b_j - \frac{\hbar\, x_j\, y_j}{B_j} == -\hbar\, a_j\, b_j + \frac{1}{2}\left(-2\,\text{Log}\left[\frac{1}{B_j}\right] + \hbar\, b_j\right) - \frac{\hbar\, x_j\, y_j}{B_j}, \text{True}\right\}\right\}$

The Trefoil

In[∘]:=
$$\begin{aligned}&\text{Timing@Block}\big[\{\$k = 1\},\\&\quad Z31 = R_{1,5}\,R_{6,2}\,R_{3,7}\,\overline{C}_4\,\overline{\text{Kink}}_8\,\overline{\text{Kink}}_9\,\overline{\text{Kink}}_{10};\\&\quad \text{Do}[Z31 = Z31\ //\ dm_{1,r\to 1},\ \{r, 2, 10\}];\\&\quad \{\text{Simplify}\ /@\ Z31,\ \text{Simplify}\ /@\ (Z31\ //\ b2t_1\ /.\ T_1 \to T)\}\big]\end{aligned}$$

Out[∘]=
$$\Big\{3.,\ \Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[-\text{Log}\big[1 - B_1 + B_1^2\big] - \hbar\, b_1,\ -\frac{\hbar\,\big(B_1 - 2B_1^2 - 2B_1^4 - a_1\,(-1 + B_1 - B_1^3 + B_1^4) + 2\hbar\, x_1 y_1 + B_1^3\,(3 + 2\hbar\, x_1 y_1)\big)}{(1 - B_1 + B_1^2)^2}\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[-\text{Log}\big[1 - T_1 + T_1^2\big] + \hbar\, t_1,\ -\frac{\hbar\,\big(T_1 - 2T_1^2 - 2T_1^4 - 2a_1\,(-1 + T_1 - T_1^3 + T_1^4) + 2\hbar\, x_1 y_1 + T_1^3\,(3 + 2\hbar\, x_1 y_1)\big)}{(1 - T_1 + T_1^2)^2}\Big]\Big\}\Big\}$$

b2t, t2b, knot tensors.

In[∘]:= $\text{HL}[(b2t_i\ //\ t2b_i) \equiv d\sigma_{i\to i}]$

Out[∘]= True

In[∘]:= $t2b_i\ //\ b2t_i$

Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}[a_i\,\alpha_i + y_i\,\eta_i + x_i\,\xi_i + t_i\,\tau_i,\ 0,\ 0]$

Reidemeister 2:

In[∘]:= $\text{Timing}\big[\text{HL}[\# \equiv d\eta_1\, d\eta_2]\ \&\ /@\ \{(\overline{kR}_{1,2}\, kR_{3,4})\ //\ (km_{1,3\to 1}\, km_{2,4\to 2}),\ (kR_{1,2}\,\overline{kR}_{3,4})\ //\ (km_{1,3\to 1}\, km_{2,4\to 2})\}\big]$

Out[∘]= $\{3.03125,\ \{\text{True},\ \text{True}\}\}$

Cyclic Reidemeister 2:

In[∘]:= $\text{Timing@HL}\big[\big((kR_{1,4}\,\overline{kR}_{5,2}\,\overline{kC}_3)\ //\ km_{2,4\to 2}\ //\ km_{1,3\to 1}\ //\ km_{1,5\to 1}\big) \equiv \overline{kC}_1\, d\eta_2\big]$

Out[∘]= $\{0.828125,\ \text{True}\}$

Reidemeister 3:

In[∘]:=
$$\begin{aligned}&\text{Timing@HL}[(kR_{1,2}\, kR_{4,3}\, kR_{5,6}\ //\ km_{1,4\to 1}\ //\ km_{2,5\to 2}\ //\ km_{3,6\to 3}) \equiv\\&\quad (kR_{1,6}\, kR_{2,3}\, kR_{4,5}\ //\ km_{1,4\to 1}\ //\ km_{2,5\to 2}\ //\ km_{3,6\to 3})]\end{aligned}$$

Out[∘]= $\{1.59375,\ \text{True}\}$

Relations between the four kinks:

In[∘]:=
$$\begin{aligned}&\text{Timing}\big[\text{HL}\ /@\ \{kKink_i \equiv ((kR_{3,1}\, kC_2)\ //\ km_{1,2\to 1}\ //\ km_{1,3\to i}),\\&\quad \overline{kKink}_j \equiv \big((\overline{kR}_{3,1}\,\overline{kC}_2)\ //\ km_{1,2\to 1}\ //\ km_{1,3\to j}\big),\ \big((kKink_i\,\overline{kKink}_j)\ //\ km_{i,j\to 1}\big) \equiv d\eta_1\}\big]\end{aligned}$$

Out[∘]=
$$\Big\{2.82813,\ \Big\{-\frac{t\hbar}{2} - t\hbar\, a_i + \hbar\, x_i y_i == \frac{1}{2}\,(t\hbar - 2\,\text{Log}[T]) - t\hbar\, a_i + \hbar\, x_i y_i,$$
$$\frac{t\hbar}{2} + t\hbar\, a_j - \frac{\hbar\, x_j y_j}{T} == \frac{1}{2}\,(-t\hbar + 2\,\text{Log}[T]) + t\hbar\, a_j - \frac{\hbar\, x_j y_j}{T},\ \text{True}\Big\}\Big\}$$

The Trefoil

In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = kR1,5 kR6,2 kR3,7 kC4 kKink8 kKink9 kKink10;
  Do[Z31 = Z31 // km1,r→1, {r, 2, 10}];
  Simplify /@ Z31]
```

Out[∘]=
$$\left\{3.23438, \mathbb{E}_{\{\}\to\{1\}}\left[t\hbar - \mathrm{Log}\left[1 - T + T^2\right], -\frac{\hbar\left(T - 2T^2 + 3T^3 - 2T^4 - 2\left(-1 + T - T^3 + T^4\right)a_1 + 2\left(1 + T^3\right)\hbar x_1 y_1\right)}{\left(1 - T + T^2\right)^2}\right]\right\}$$

In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = kR1,5 kR6,2 kR3,7 kC4 kKink8 kKink9 kKink10;
  Do[Z31 = Z31 // km1,r→1, {r, 2, 10}];
  Simplify /@ Z31]
```

Out[∘]=
$$\left\{2.53125, \mathbb{E}_{\{\}\to\{1\}}\left[t\hbar - \mathrm{Log}\left[1 - T + T^2\right], -\frac{\hbar\left(T - 2T^2 + 3T^3 - 2T^4 - 2\left(-1 + T - T^3 + T^4\right)a_1 + 2\left(1 + T^3\right)\hbar x_1 y_1\right)}{\left(1 - T + T^2\right)^2}\right]\right\}$$

In[∘]:=
```
Timing@Block[{$k = 1}, Z[Knot[8, 17]]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
$$\left\{32.7188, \mathbb{E}_{\{\}\to\{0\}}\left[-t\hbar - \mathrm{Log}\left[-1 - \frac{1}{T^4} + \frac{4}{T^3} - \frac{6}{T^2} + \frac{5}{T}\right] - \mathrm{Log}\left[1 + \frac{T}{1 - 4T + 6T^2 - 5T^3 + T^4} - \frac{2T^2}{1 - 4T + 6T^2 - 5T^3 + T^4} + \frac{T^3}{1 - 4T + 6T^2 - 5T^3 + T^4}\right] - \mathrm{Log}\left[1 - \frac{T}{1 - 3T + 4T^2 - 4T^3 + T^4} + \frac{4T^2}{1 - 3T + 4T^2 - 4T^3 + T^4} - \frac{7T^3}{1 - 3T + 4T^2 - 4T^3 + T^4} + \frac{7T^4}{1 - 3T + 4T^2 - 4T^3 + T^4} - \frac{4T^5}{1 - 3T + 4T^2 - 4T^3 + T^4} + \frac{T^6}{1 - 3T + 4T^2 - 4T^3 + T^4}\right], \frac{-3\hbar + 8T\hbar - 8T^2\hbar + 8T^4\hbar - 8T^5\hbar + 3T^6\hbar}{1 - 4T + 8T^2 - 11T^3 + 8T^4 - 4T^5 + T^6} + \frac{a\left(-6\hbar + 16T\hbar - 16T^2\hbar + 16T^4\hbar - 16T^5\hbar + 6T^6\hbar\right)}{1 - 4T + 8T^2 - 11T^3 + 8T^4 - 4T^5 + T^6} + \frac{x\,y\left(-6\hbar^2 + 10T\hbar^2 - 6T^2\hbar^2 - 6T^3\hbar^2 + 10T^4\hbar^2 - 6T^5\hbar^2\right)}{1 - 4T + 8T^2 - 11T^3 + 8T^4 - 4T^5 + T^6}\right]\right\}$$

CU

Associativity of CU:

In[∘]:=
```
Timing@Block[{$k = 3}, HL[(cm1,2→1 // cm1,3→1) ≡ (cm2,3→2 // cm1,2→1)]]
```

Out[∘]= {1.1875, True}

Associativity, co-associativity, and Δ is an algebra morphism:

In[ ]:=
```
Timing@Block[{$k = 3}, HL /@ {(cm_{1,2→1} // cm_{1,3→1}) ≡ (cm_{2,3→2} // cm_{1,2→1}),
    (cΔ_{1→1,2} // cΔ_{2→2,3}) ≡ (cΔ_{1→1,3} // cΔ_{1→1,2}),
    (cm_{1,2→1} // cΔ_{1→1,2}) ≡ ((cΔ_{1→1,3} cΔ_{2→2,4}) // (cm_{3,4→2} cm_{1,2→1}))}]
```
Out[ ]=
```
{2.67188, {True, True, True}}
```

S is convolution inverse of id:

In[ ]:=
```
Timing@Block[{$k = 3}, HL[# ≡ cϵ_1 cη_1] & /@ {
    (cΔ_{1→1,2} // cS_1) // cm_{1,2→1}, (cΔ_{1→1,2} // cS_2) // cm_{1,2→1}}]
```
Out[ ]=
```
{2.5625, {True, True}}
```

S is an algebra anti-(co)morphism

In[ ]:=
```
Timing@Block[{$k = 3},
  HL /@ {(cm_{1,2→1} // cS_1) ≡ ((cS_1 cS_2) // cm_{2,1→1}), (cS_1 // cΔ_{1→1,2}) ≡ (cΔ_{1→2,1} // (cS_1 cS_2))}]
```
Out[ ]=
```
{2.79688, {True, True}}
```

Classical is the $\hbar \to 0$ limit of quantum:

In[ ]:=
```
ClassicalLimit[f_] := Normal@Series[Normal[f] // U2l, {ℏ, 0, 0}] // l2U;
Timing[HL /@ Simplify /@
   {cm_{1,2→3} ≡ ClassicalLimit /@ dm_{1,2→3},
    (cΔ_{1→2,3} /. τ_1 → 0) ≡ ClassicalLimit /@ dΔ_{1→2,3}, cS_1 ≡ ClassicalLimit /@ dS_1}]
```
Out[ ]=
```
{1.3125, {True, True, True}}
```

In[ ]:=
```
PrintProfile[]
```
Out[ ]=
```
ProfileRoot is root. Profiled time: 136.744
  (     1)   0.203/  32.688 above Z
  (    59)   0.748/  26.735 above Boot
  (  1329)   2.448/   6.761 above CF
  (   198)   1.201/  30.127 above EZip3
  (     1)   0.015/   0.015 above RVK
  (   198)   2.486/   3.758 above Zip1
  (   198)   2.628/  11.073 above Zip2
  (   198)   6.767/  25.587 above Zip3
CF: called 118488 times, time in 52.769/97.221
  (   214)   1.060/   1.980 under Z
  (   407)   0.499/   1.138 under Boot
  (  1269)   7.015/  18.475 under EZip3
  (  1329)   2.448/   6.761 under ProfileRoot
  (   682)   0.720/   2.367 under Zip1
  ( 27027)   8.826/  14.585 under Zip2
  ( 87560)  32.201/  51.915 under Zip3
  ( 88475)  44.452/  44.452 above CCF
CCF: called 88475 times, time in 44.452/44.452
  ( 88475)  44.452/  44.452 under CF
Zip3: called 682 times, time in 26.626/78.541
  (    57)   1.786/  10.298 under Z
  (    86)   2.769/   7.754 under Boot
  (   341)  15.304/  34.902 under EZip3
```

```
  (   198)   6.767/ 25.587 under ProfileRoot
  ( 87560)  32.201/ 51.915 above CF
Zip1: called 341 times, time in 4.853/7.22
  (    57)   0.581/  1.113 under Z
  (    86)   1.786/  2.349 under Boot
  (   198)   2.486/  3.758 under ProfileRoot
  (   682)   0.720/  2.367 above CF
Zip2: called 341 times, time in 4.702/19.287
  (    57)   0.795/  3.591 under Z
  (    86)   1.279/  4.623 under Boot
  (   198)   2.628/ 11.073 under ProfileRoot
  ( 27027)   8.826/ 14.585 above CF
EZip3: called 341 times, time in 2.235/55.612
  (    57)   0.676/ 15.284 under Z
  (    86)   0.358/ 10.201 under Boot
  (   198)   1.201/ 30.127 under ProfileRoot
  (  1269)   7.015/ 18.475 above CF
  (   341)  15.304/ 34.902 above Zip3
Boot: called 86 times, time in 0.889/42.311
  (     3)      0/  0.219 under Z
  (    24)   0.141/ 15.357 under Boot
  (    59)   0.748/ 26.735 under ProfileRoot
  (    24)   0.141/ 15.357 above Boot
  (   407)   0.499/  1.138 above CF
  (    86)   0.358/ 10.201 above EZip3
  (    86)   1.786/  2.349 above Zip1
  (    86)   1.279/  4.623 above Zip2
  (    86)   2.769/  7.754 above Zip3
Z: called 1 times, time in 0.203/32.688
  (     1)   0.203/ 32.688 under ProfileRoot
  (     3)      0/  0.219 above Boot
  (   214)   1.060/  1.980 above CF
  (    57)   0.676/ 15.284 above EZip3
  (    57)   0.581/  1.113 above Zip1
  (    57)   0.795/  3.591 above Zip2
  (    57)   1.786/ 10.298 above Zip3
RVK: called 1 times, time in 0.015/0.015
  (     1)   0.015/  0.015 under ProfileRoot
```