Pensieve header: Full testing of the sl_2 portfolio. Continues pensieve://Projects/SL2Portfolio2/PortfolioTesting.nb. Time: 262.768.

Startup

```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 1;
<< Engine-210104.m
<< Objects.m
<< KT.m
```

(Alt) Out[∘]= {2021, 1, 4, 8, 32, 36.3874662}

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: April 2020. Original version: July 1994.
```

(Alt) In[∘]:=

```
$k = 2; (*ℏ=γ=1;*)
```

Utilities

(Alt) In[∘]:=

```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Testing

(Alt) In[∘]:=
```
Block[{$k = 1}, {
    am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j},
    aS → aS_i, aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k},
    dΔ → dΔ_{i→j,k}, C → C_i, C̄ → C̄_i, Kink → Kink_i, Kink̄ → Kink̄_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

(Alt) Out[∘]=

$$\mathrm{am} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,(\alpha_i+\alpha_j)+x_k\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right),\ 0\right]$$

$$\mathrm{bm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\,(\beta_i+\beta_j)+y_k\,(\eta_i+\eta_j),\ -y_k\,\beta_i\,\eta_j\right]$$

$$\mathrm{dm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\Big[a_k\,(\alpha_i+\alpha_j)+b_k\,\beta_i+b_k\,\beta_j+y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\frac{(1-B_k)\,\eta_j\,\xi_i}{\hbar}+x_k\,\xi_j,$$
$$-\frac{y_k\,\beta_i\,\eta_j}{\mathcal{A}_i}-\frac{x_k\,\beta_j\,\xi_i}{\mathcal{A}_j}+a_k\,B_k\,\eta_j\,\xi_i+\frac{\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j}+\frac{(1-3\,B_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i}+\frac{(1-3\,B_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j}+\frac{(1-4\,B_k+3\,B_k^2)\,\eta_j^2\,\xi_i^2}{4\,\hbar}\Big]$$

$$\mathrm{R} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i+\hbar\,x_j\,y_i,\ -\frac{1}{4}\,\hbar^3\,x_j^2\,y_i^2\right]$$

$$\overline{\mathrm{R}} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i-\frac{\hbar\,x_j\,y_i}{B_i},\ -\frac{\hbar^2\,a_j\,x_j\,y_i}{B_i}-\frac{3\,\hbar^3\,x_j^2\,y_i^2}{4\,B_i^2}\right]$$

$$\mathrm{P} \to \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar}+\frac{\eta_i\,\xi_j}{\hbar},\ \frac{\eta_i^2\,\xi_j^2}{4\,\hbar}\right]$$

$$\mathrm{aS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ -\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$$

$$\overline{\mathrm{aS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ \hbar\,x_i\,\mathcal{A}_i\,\xi_i-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$$

$$\mathrm{bS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ -\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$$

$$\overline{\mathrm{bS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ \frac{\hbar\,y_i\,\eta_i}{B_i}-\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$$

$$\mathrm{dS} \to \mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\,\alpha_i-b_i\,\beta_i-\frac{y_i\,\mathcal{A}_i\,\eta_i}{B_i}-x_i\,\mathcal{A}_i\,\xi_i+\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,B_i},$$
$$\frac{\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{B_i}-\frac{y_i\,\mathcal{A}_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,B_i^2}-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-x_i\,\mathcal{A}_i\,\beta_i\,\xi_i+\frac{a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i}-\frac{\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{B_i}+\frac{(-\mathcal{A}_i+B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{B_i}+$$
$$\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\beta_i\,\eta_i\,\xi_i}{\hbar\,B_i}+\frac{y_i\,(3\,\mathcal{A}_i^2-B_i\,\mathcal{A}_i^2)\,\eta_i^2\,\xi_i}{2\,B_i^2}-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2+\frac{x_i\,(3\,\mathcal{A}_i^2-B_i\,\mathcal{A}_i^2)\,\eta_i\,\xi_i^2}{2\,B_i}+\frac{(-3\,\mathcal{A}_i^2+4\,B_i\,\mathcal{A}_i^2-B_i^2\,\mathcal{A}_i^2)\,\eta_i^2\,\xi_i^2}{4\,\hbar\,B_i^2}\Big]$$

$$\mathrm{a}\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+x_j\,\xi_i+x_k\,\xi_i,\ -\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right]$$

$$\mathrm{b}\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[(b_j+b_k)\,\beta_i+B_k\,y_j\,\eta_i+y_k\,\eta_i,\ \frac{1}{2}\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\right]$$

$$\mathrm{d}\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\Big[a_j\,\alpha_i+a_k\,\alpha_i+(b_j+b_k)\,\beta_i+y_j\,\eta_i+B_j\,y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,$$
$$\frac{1}{2}\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2-\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\Big]$$

$$\mathrm{C} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2},\ -\frac{\hbar\,a_i}{2}\right]$$

$$\overline{\mathrm{C}} \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2},\ \frac{\hbar\,a_i}{2}\right]$$

$$\mathrm{Kink} \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2}+\hbar\,a_i\,b_i+\hbar\,x_i\,y_i,\ \frac{\hbar\,a_i}{2}-\frac{1}{4}\,\hbar^3\,x_i^2\,y_i^2\right]$$

$$\overline{\mathrm{Kink}} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2}-\hbar\,a_i\,b_i-\frac{\hbar\,x_i\,y_i}{B_i},\ -\frac{\hbar\,a_i}{2}-\frac{\hbar^2\,a_i\,x_i\,y_i}{B_i}-\frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^2}\right]$$

$$\mathrm{b2t} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i-t_i\,\beta_i+y_i\,\eta_i+x_i\,\xi_i,\ a_i\,\beta_i\right]$$

$$\mathrm{t2b} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i+y_i\,\eta_i+x_i\,\xi_i-b_i\,\tau_i,\ a_i\,\tau_i\right]$$

Check that on the generators this agrees with our conventions in the handout:

(Alt) In[∘]:=
```
𝔼2Λ[ℰ_] := Module[{k}, Sum[ℰ[k] ϵ^(k-1), {k, 0, ℰ[$]}]];
Timing@Block[{$k = 2}, {
   {
    "[a,x]" → 𝔼2Λ[𝔼_{}→{1,2}[0, a2 x1] // am1,2→1] - 𝔼2Λ[𝔼_{}→{1,2}[0, a1 x2] // am1,2→1],
    "[b,y]" → 𝔼2Λ[𝔼_{}→{1,2}[0, y2 b1, 0] // bm1,2→1] - 𝔼2Λ[𝔼_{}→{1,2}[0, y1 b2, 0] // bm1,2→1]
   } /. z_1 → z,
   {
    "Δ[y]" → Last[𝔼_{}→{1}[0, y1] // bΔ1→1,2],
    "Δ[b]" → Last[𝔼_{}→{1}[0, b1] // bΔ1→1,2],
    "Δ[a]" → Last[𝔼_{}→{1}[0, a1] // aΔ1→1,2],
    "Δ[x]" → Last[𝔼_{}→{1}[0, x1] // aΔ1→1,2]},
   {
    "S(a)" → ((𝔼_{}→{1}[0, a1] // aS1)[1]),
    "S(x)" → ((𝔼_{}→{1}[0, x1] // aS1)[1]),
    "S(b)" → ((𝔼_{}→{1}[0, b1] // bS1)[1]),
    "S(y)" → ((𝔼_{}→{1}[0, y1] // bS1)[1])
   } /. z_1 → z
  }]
```

(Alt) Out[∘]= $\{4.48438,$

$\{\{[a,x] \to -x,\ [b,y] \to -y\,\epsilon\},\ \{\triangle[y] \to B_2\, y_1 + y_2,\ \triangle[b] \to b_1 + b_2,\ \triangle[a] \to a_1 + a_2,\ \triangle[x] \to x_1 + x_2\},$

$\{S(a) \to -a,\ S(x) \to -x,\ S(b) \to -b,\ S(y) \to -\frac{y}{B}\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

(Alt) In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {(am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}
 ]
```

(Alt) Out[∘]= {0.4375, {True, True}}

R and *P* are inverses:

(Alt) In[∘]:=
```
Timing@Block[{$k = 3}, {R_i,j, P_i,k, HL[(R_i,j // P_i,k) ≡ aσ_k→j]}]
```

(Alt) Out[∘]= $\{0.578125,\ \{\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\, a_j\, b_i + \hbar\, x_j\, y_i,\ -\frac{1}{4}\hbar^3 x_j^2 y_i^2,\ \frac{1}{9}\hbar^5 x_j^3 y_i^3,\ \frac{1}{48}\left(\hbar^5 x_j^2 y_i^2 - 3\hbar^7 x_j^4 y_i^4\right)\right],$

$\mathbb{E}_{\{i,k\}\to\{\}}\left[\frac{\alpha_k\,\beta_i}{\hbar} + \frac{\eta_i\,\xi_k}{\hbar},\ \frac{\eta_i^2\,\xi_k^2}{4\,\hbar},\ \frac{1}{8}\eta_i^2\,\xi_k^2 + \frac{5\,\eta_i^3\,\xi_k^3}{36\,\hbar},\ \frac{1}{24}\hbar\,\eta_i^2\,\xi_k^2 + \frac{1}{6}\eta_i^3\,\xi_k^3 + \frac{5\,\eta_i^4\,\xi_k^4}{48\,\hbar}\right],\ \text{True}\}\}$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

(Alt) In[∘]:=
```
Timing[HL /@ {(aS̄1 // aS1) ≡ aσ1→1, (bS̄1 // bS1) ≡ bσ1→1}]
```

(Alt) Out[∘]= {1.23438, {True, True}}

(co)-associativity on both sides

(Alt) In[∘]:=
```
Timing[
 HL /@ {(aΔ_{1→1,2} // aΔ_{2→2,3}) ≡ (aΔ_{1→1,3} // aΔ_{1→1,2}), (bΔ_{1→1,2} // bΔ_{2→2,3}) ≡ (bΔ_{1→1,3} // bΔ_{1→1,2}),
   (am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}]
```
(Alt) Out[∘]=
```
{1.09375, {True, True, True, True}}
```

Δ is an algebra morphism

(Alt) In[∘]:=
```
Timing[HL /@ {(am_{1,2→1} // aΔ_{1→1,2}) ≡ ((aΔ_{1→1,3} aΔ_{2→2,4}) // (am_{3,4→2} am_{1,2→1})),
   (bm_{1,2→1} // bΔ_{1→1,2}) ≡ ((bΔ_{1→1,3} bΔ_{2→2,4}) // (bm_{3,4→2} bm_{1,2→1}))}]
```
(Alt) Out[∘]=
```
{1.92188, {True, True}}
```

An explicit formula for aS_i

(Alt) In[ ]:=
$$\text{Timing@Block}\Big[\{\$k = 4\},\ \text{HL}\Big[aS_i \equiv \Big(\mathbb{E}_{\{i\}\to\{i,j\}}\Big[-\alpha_i\, a_j,\ -\xi_i\, x_i,\ \text{Sum}\Big[\text{Expand}\Big[\frac{e^{\xi_i x_i}(-\hbar\,\gamma\,\epsilon)^k}{2^k\, k!}\,\text{Nest}\big[\text{Expand}\big[x_i^2\,\partial_{\{x_i,2\}}\#\big]\,\&,\ e^{-\xi_i\, e^{\hbar\epsilon a_i}\, x_i},\ k\big]\Big],\ \{k, 0, \$k\}\Big]\Big]_{\$k}\ //\ am_{i,j\to i}\Big)\Big]\Big]$$

(Alt) Out[ ]=
$$\Big\{6.21875,\ \mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\,\alpha_i - x_i\,\mathcal{A}_i\,\xi_i,\ -\hbar\, a_i\, x_i\,\mathcal{A}_i\,\xi_i - \frac{1}{2}\hbar\, x_i^2\mathcal{A}_i^2\xi_i^2,$$
$$-\frac{1}{2}\hbar^2 a_i^2 x_i\mathcal{A}_i\xi_i + \frac{1}{4}\hbar^2 x_i^2\mathcal{A}_i^2\xi_i^2 - \hbar^2 a_i x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{1}{2}\hbar^2 x_i^3\mathcal{A}_i^3\xi_i^3,\ -\frac{1}{6}\hbar^3 a_i^3 x_i\mathcal{A}_i\xi_i - \frac{1}{12}\hbar^3 x_i^2\mathcal{A}_i^2\xi_i^2 +$$
$$\frac{1}{2}\hbar^3 a_i x_i^2\mathcal{A}_i^2\xi_i^2 - \hbar^3 a_i^2 x_i^2\mathcal{A}_i^2\xi_i^2 + \frac{2}{3}\hbar^3 x_i^3\mathcal{A}_i^3\xi_i^3 - \frac{3}{2}\hbar^3 a_i x_i^3\mathcal{A}_i^3\xi_i^3 - \frac{2}{3}\hbar^3 x_i^4\mathcal{A}_i^4\xi_i^4,\ -\frac{1}{24}\hbar^4 a_i^4 x_i\mathcal{A}_i\xi_i +$$
$$\frac{1}{48}\hbar^4 x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{1}{6}\hbar^4 a_i x_i^2\mathcal{A}_i^2\xi_i^2 + \frac{1}{2}\hbar^4 a_i^2 x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{2}{3}\hbar^4 a_i^3 x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{13}{24}\hbar^4 x_i^3\mathcal{A}_i^3\xi_i^3 +$$
$$2\hbar^4 a_i x_i^3\mathcal{A}_i^3\xi_i^3 - \frac{9}{4}\hbar^4 a_i^2 x_i^3\mathcal{A}_i^3\xi_i^3 + \frac{13}{8}\hbar^4 x_i^4\mathcal{A}_i^4\xi_i^4 - \frac{8}{3}\hbar^4 a_i x_i^4\mathcal{A}_i^4\xi_i^4 - \frac{25}{24}\hbar^4 x_i^5\mathcal{A}_i^5\xi_i^5\Big] \equiv$$
$$\mathbb{E}_{\{i,j\}\to\{i\}}\Big[a_i\,(\alpha_i+\alpha_j) + x_i\Big(\frac{\xi_i}{\mathcal{A}_j} + \xi_j\Big),\ 0,\ 0,\ 0,\ 0\Big]\Big[$$
$$\mathbb{E}_{\{i\}\to\{i,j\}}\Big[-a_j\,\alpha_i,\ -x_i\,\xi_i,\ e^{x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i} - \frac{1}{2}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma\,\epsilon\,\hbar\, x_i^2\xi_i^2 + \frac{1}{4}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}$$
$$\gamma^2\epsilon^2\hbar^2 x_i^2\xi_i^2 - \frac{1}{12}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^2\xi_i^2 + \frac{1}{48}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^2\xi_i^2 -$$
$$\frac{1}{2}e^{3\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^2\epsilon^2\hbar^2 x_i^3\xi_i^3 + \frac{2}{3}e^{3\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^3\xi_i^3 -$$
$$\frac{13}{24}e^{3\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^3\xi_i^3 + \frac{1}{8}e^{4\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^2\epsilon^2\hbar^2 x_i^4\xi_i^4 -$$
$$\frac{19}{24}e^{4\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^4\xi_i^4 + \frac{163}{96}e^{4\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^4\xi_i^4 +$$
$$\frac{1}{4}e^{5\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^5\xi_i^5 - \frac{3}{2}e^{5\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^5\xi_i^5 -$$
$$\frac{1}{48}e^{6\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^6\xi_i^6 + \frac{47}{96}e^{6\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^6\xi_i^6 -$$
$$\frac{1}{16}e^{7\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^7\xi_i^7 + \frac{1}{384}e^{8\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^8\xi_i^8\Big]_4\Big]\Big\}$$

S is convolution inverse of id

(Alt) In[ ]:=
$$\text{Timing}\big[\text{HL}[\# \equiv s\epsilon_1\, s\eta_1]\ \&\ /@\ \{(a\Delta_{1\to1,2}\ //\ aS_1)\ //\ am_{1,2\to1},\ (a\Delta_{1\to1,2}\ //\ aS_2)\ //\ am_{1,2\to1},$$
$$(b\Delta_{1\to1,2}\ //\ bS_1)\ //\ bm_{1,2\to1},\ (b\Delta_{1\to1,2}\ //\ bS_2)\ //\ bm_{1,2\to1}\}\big]$$

(Alt) Out[ ]= {1.8125, {True, True, True, True}}

But not with the opposite product:

(Alt) In[∘]:=
```
Timing[Short[# ≡ sϵ₁ sη₁] & /@ {
  (aΔ_{1→1,2} ~ B₁ ~ aS₁) ~ B_{1,2} ~ am_{2,1→1}, (aΔ_{1→1,2} ~ B₂ ~ aS₂) ~ B_{1,2} ~ am_{2,1→1},
  (bΔ_{1→1,2} ~ B₁ ~ bS₁) ~ B_{1,2} ~ bm_{2,1→1}, (bΔ_{1→1,2} ~ B₂ ~ bS₂) ~ B_{1,2} ~ bm_{2,1→1}}]
```

(Alt) Out[∘]=

$$\{0.015625,$$
$$\{B_{1,2}[B_1[\mathbb{E}_{\{1\}\to\{1,2\}}[a_1\,\alpha_1 + a_2\,\alpha_1 + x_1\,\xi_1 + x_2\,\xi_1,\ \ll 1\gg + \ll 1\gg,\ \ll 1\gg],\ \ll 1\gg],\ \ll 1\gg] \equiv \ll 1\gg,$$
$$B_{1,2}[B_2[\mathbb{E}_{\{1\}\to\{1,2\}}[a_1\,\alpha_1 + a_2\,\alpha_1 + x_1\,\xi_1 + x_2\,\xi_1,\ \ll 1\gg + \ll 1\gg,\ \ll 1\gg],\ \ll 1\gg],\ \ll 1\gg] \equiv \ll 1\gg,$$
$$B_{1,2}\left[B_1\left[\mathbb{E}_{\{1\}\to\{1,2\}}[\ll 1\gg],\ \mathbb{E}_{\{1\}\to\{1\}}\left[-b_1\,\beta_1 - \frac{\ll 1\gg}{\ll 1\gg},\ -\ll 1\gg - \ll 1\gg,\ \ll 1\gg\right]\right],\ \ll 1\gg\right] \equiv \ll 1\gg,$$
$$B_{1,2}\left[B_2\left[\mathbb{E}_{\{1\}\to\{1,2\}}[\ll 1\gg],\ \mathbb{E}_{\{2\}\to\{2\}}\left[-b_2\,\beta_2 - \frac{\ll 1\gg}{\ll 1\gg},\ -\ll 1\gg - \ll 1\gg,\ \ll 1\gg\right]\right],\ \ll 1\gg\right] \equiv \ll 1\gg\}\}$$

S is an algebra anti-(co)morphism

(Alt) In[∘]:=
```
Timing[HL /@ {(am_{1,2→1} // aS₁) ≡ ((aS₁ aS₂) // am_{2,1→1}), (bm_{1,2→1} // bS₁) ≡ ((bS₁ bS₂) // bm_{2,1→1}),
  (aS₁ // aΔ_{1→1,2}) ≡ (aΔ_{1→2,1} // (aS₁ aS₂)), (bS₁ // bΔ_{1→1,2}) ≡ (bΔ_{1→2,1} // (bS₁ bS₂))}]
```

(Alt) Out[∘]= {2.23438, {True, True, True, True}}

Pairing axioms

(Alt) In[∘]:=
```
Timing[HL /@ {((bm_{1,2→1} sϒ_{3→0,0,3,3} // sϵ₀) // P_{1,3}) ≡
   (((sϒ_{1→1,1,0,0} // sϵ₀) (sϒ_{2→2,2,0,0} // sϵ₀) aΔ_{3→4,5}) // P_{1,4} // P_{2,5}),
  ((bΔ_{1→1,2} (sϒ_{3→0,0,3,3} // sϵ₀) (sϒ_{4→0,0,4,4} // sϵ₀)) // P_{1,3} // P_{2,4}) ≡
   (((sϒ_{1→1,1,0,0} // sϵ₀) am_{3,4→3}) // P_{1,3})}]
```

(Alt) Out[∘]= {2.09375, {True, True}}

(Alt) In[∘]:=
```
Timing[HL /@ {((bS₁ aσ_{2→2}) // P_{1,2}) ≡ ((bσ_{1→1} aS₂) // P_{1,2}),
  ((overline(bS)₁ aσ_{2→2}) // P_{1,2}) ≡ ((bσ_{1→1} overline(aS)₂) // P_{1,2})}]
```

(Alt) Out[∘]= {1.125, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

(Alt) In[∘]:=
```
(*Timing@{{
      "[a,y]"→ ((𝔼_{}→{1,2}[0,0,y₂a₁]~B_{1,2}~dm_{1,2→1})[3]-(𝔼_{}→{1,2}[0,0,y₁a₂]~B_{1,2}~dm_{1,2→1})[3]),
      "[b,x]"   →
       ((𝔼_{}→{1,2}[0,0,x₂b₁]~B_{1,2}~dm_{1,2→1})[3]-(𝔼_{}→{1,2}[0,0,x₁b₂]~B_{1,2}~dm_{1,2→1})[3]),"xy-qyx"→
       ((𝔼_{}→{1,2}[0,0,x₁y₂]~B_{1,2}~dm_{1,2→1})[3]-(1+ϵ)(𝔼_{}→{1,2}[0,0,y₁x₂]~B_{1,2}~dm_{1,2→1})[3])
     }/.{z_1→z}//Expand//Factor,
  {
    "Δ(a)"→((𝔼_{}→{1}[0,0,a₁]~B₁~dΔ_{1→1,2})[3]),
    "Δ(x)"→((𝔼_{}→{1}[0,0,x₁]~B₁~dΔ_{1→1,2})[3]),
    "Δ(b)"→((𝔼_{}→{1}[0,0,b₁]~B₁~dΔ_{1→1,2})[3]),
    "Δ(y)"→((𝔼_{}→{1}[0,0,y₁]~B₁~dΔ_{1→1,2})[3])
   }//Simplify,
  {
     "S(a)"→((𝔼_{}→{1}[0,0,a₁]~B₁~dS₁)[3]),
     "S(x)"→((𝔼_{}→{1}[0,0,x₁]~B₁~dS₁)[3]),
     "S(b)"→((𝔼_{}→{1}[0,0,b₁]~B₁~dS₁)[3]),
     "S(y)"→((𝔼_{}→{1}[0,0,y₁]~B₁~dS₁)[3])
    }/.{z_1→z}//Simplify
 }*)
```

(Alt) In[∘]:=
```
{HL[((sϒ_{1→0,0,1,1} // sϵ₀) (sϒ_{2→0,0,2,2} // sϵ₀) // dm_{1,2→1}) ≡ am_{1,2→1}],
 HL[((sϒ_{1→1,1,0,0} // sϵ₀) (sϒ_{2→2,2,0,0} // sϵ₀) // dm_{1,2→1}) ≡ bm_{1,2→1}]}
```

(Alt) Out[∘]=
```
{True, True}
```

(co)-associativity

(Alt) In[∘]:=
```
Timing[Block[{$k = 1},
  HL /@ {(dΔ_{1→1,2} // dΔ_{2→2,3}) ≡ (dΔ_{1→1,3} // dΔ_{1→1,2}), (dm_{1,2→1} // dm_{1,3→1}) ≡ (dm_{2,3→2} // dm_{1,2→1})}]
]
```

(Alt) Out[∘]=
```
{0.84375, {True, True}}
```

Δ is an algebra morphism

(Alt) In[∘]:=
```
Timing@HL[(dm_{1,2→1} // dΔ_{1→1,2}) ≡ ((dΔ_{1→1,3} dΔ_{2→2,4}) // (dm_{3,4→2} dm_{1,2→1}))]
```

(Alt) Out[∘]=
```
{3.9375, True}
```

dS and $\overline{\mathrm{dS}}$ are inverses:

(Alt) In[∘]:=
```
Timing@HL[(d̄S₁ // dS₁) ≡ dσ_{1→1}]
```

(Alt) Out[∘]=
```
{3.84375, True}
```

S_2 inverts R, but not S_1:

(Alt) In[∘]:= `Timing@{(R_{1,2} // dS_1) ≡ R̄_{1,2}, HL[(R_{1,2} // dS_2) ≡ R̄_{1,2}]}`

(Alt) Out[∘]=
$$\left\{0.65625, \left\{\frac{\hbar^2 x_2 y_1}{B_1} - \frac{\hbar^2 a_2 x_2 y_1}{B_1} - \frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2} == -\frac{\hbar^2 a_2 x_2 y_1}{B_1} - \frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2} \;\&\& \right.\right.$$
$$-\frac{\hbar^3 x_2 y_1}{2 B_1} + \frac{\hbar^3 a_2 x_2 y_1}{B_1} - \frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1} + \frac{2\hbar^4 x_2^2 y_1^2}{B_1^2} - \frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2} - \frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3} ==$$
$$\left.\left. -\frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1} + \frac{\hbar^4 x_2^2 y_1^2}{2 B_1^2} - \frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2} - \frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3}, \text{True}\right\}\right\}$$

dS is convolution inverse of id

(Alt) In[∘]:= `Timing[HL[# ≡ dϵ_1 dη_1] & /@ {(dΔ_{1→1,2} // dS_1) // dm_{1,2→1}, (dΔ_{1→1,2} // dS_2) // dm_{1,2→1}}]`

(Alt) Out[∘]= `{4.95313, {True, True}}`

dS is a (co)-algebra anti-morphism

(Alt) In[∘]:=
```
Timing[HL /@
  Expand /@ {(dm_{1,2→1} // dS_1) ≡ ((dS_1 dS_2) // dm_{2,1→1}), (dS_1 // dΔ_{1→1,2}) ≡ (dΔ_{1→2,1} // (dS_1 dS_2))}]
```

(Alt) Out[∘]= `{11.0469, {True, True}}`

Quasi-triangular axiom 1:

(Alt) In[∘]:=
```
Timing[
 HL /@ {(R_{1,3} // dΔ_{1→1,2}) ≡ ((R_{1,4} R_{2,3}) // dm_{3,4→3}), (R_{1,2} // dΔ_{2→2,3}) ≡ ((R_{1,2} R_{4,3}) // dm_{1,4→1})}]
```

(Alt) Out[∘]= `{1.0625, {True, True}}`

Quasi-triangular axiom 2:

(Alt) In[∘]:= `Timing@HL[((dΔ_{1->1,2} R_{3,4}) // (dm_{1,3→1} dm_{2,4→2})) ≡ ((R_{1,2} dΔ_{1→3,4}) // (dm_{1,4→1} dm_{2,3→2}))]`

(Alt) Out[∘]= `{3.28125, True}`

The Drinfel'd element inverse property, $(u_1 \bar{u}_2) // \mathrm{dm}_{1,2\to 1} \equiv \mathrm{d}\epsilon_i$:

(Alt) In[∘]:= `Timing@HL[(((R_{1,2} // dS_1 // dm_{2,1→i}) (R_{1,2} // dS_2 // dS_2 // dm_{2,1→j})) // dm_{i,j→i}) ≡ dη_i]`

(Alt) Out[∘]=
$$\left\{5.67188, \frac{1}{2}\left(-\mathrm{Log}\left[\frac{1}{B_i^2}\right] - \mathrm{Log}\left[B_i^2\right]\right) == 0\right\}$$

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

(Alt) In[∘]:=
```
Timing@
 Block[{$k = 2}, (((R_{1,2} // dS_1 // dm_{2,1→i}) // dS_i) (R_{1,2} // dS_2 // dS_2 // dm_{2,1→j})) // dm_{i,j→i}]
```

(Alt) Out[∘]=
$$\left\{10.2188, \mathbb{E}_{\{\}\to\{i\}}\left[\frac{1}{2}\left(-\mathrm{Log}\left[\frac{1}{B_i^2}\right] - 2\,\mathrm{Log}\left[B_i^2\right]\right), \hbar\, a_i, 0\right]\right\}$$

(Alt) In[∘]:= $\text{Timing@Block}\left[\{\$k = 2\}, \text{HL} \;/@\; \left\{\left(\left(C_i\,\overline{C}_j\right) \;//\; dm_{i,j\to i}\right) \equiv d\eta_i,\; \left(\left(\overline{C}_i\,\overline{C}_j\right) \;//\; dm_{i,j\to i}\right) \equiv \left(\left(\left(R_{1,2} \;//\; dS_1 \;//\; dm_{2,1\to i}\right) \;//\; dS_i\right)\left(R_{1,2} \;//\; dS_2 \;//\; dS_2 \;//\; dm_{2,1\to j}\right) \;//\; dm_{i,j\to i}\right)\right\}\right]$

(Alt) Out[∘]= $\left\{10.1719, \left\{\text{True}, \hbar\, b_i == \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - 2\,\text{Log}\left[B_i^2\right]\right)\right\}\right\}$

(Alt) In[∘]:= $\text{Timing@Block}\left[\{\$k = 2\}, \text{HL} \;/@\; \left\{\left(\left(C_i\,\overline{C}_j\right) \;//\; dm_{i,j\to i}\right) \equiv d\eta_i,\; \left(\left(\overline{C}_i\,\overline{C}_j\right) \;//\; dm_{i,j\to i}\right) \equiv \left(\left(\left(\left(R_{1,2} \;//\; dS_1 \;//\; dm_{2,1\to i}\right) \;//\; dS_i\right)\left(R_{1,2} \;//\; dS_2 \;//\; dS_2 \;//\; dm_{2,1\to j}\right)\right) \;//\; dm_{i,j\to i}\right)\right\}\right]$

(Alt) Out[∘]= $\left\{10.7813, \left\{\text{True}, \hbar\, b_i == \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - 2\,\text{Log}\left[B_i^2\right]\right)\right\}\right\}$

Reidemeister 2:

(Alt) In[∘]:= $\text{Timing}\left[\text{HL}[\# \equiv d\eta_1\, d\eta_2]\;\&\;/@\;\left\{\left(\overline{R}_{1,2}\,R_{3,4}\right) \;//\; (dm_{1,3\to 1}\,dm_{2,4\to 2}),\; \left(R_{1,2}\,\overline{R}_{3,4}\right) \;//\; (dm_{1,3\to 1}\,dm_{2,4\to 2})\right\}\right]$

(Alt) Out[∘]= $\{2.53125, \{\text{True}, \text{True}\}\}$

Cyclic Reidemeister 2:

(Alt) In[∘]:= $\text{Timing@HL}\left[\left(\left(R_{1,4}\,\overline{R}_{5,2}\,\overline{C}_3\right) \;//\; dm_{2,4\to 2} \;//\; dm_{1,3\to 1} \;//\; dm_{1,5\to 1}\right) \equiv \overline{C}_1\, d\eta_2\right]$

(Alt) Out[∘]= $\{1.64063, \text{True}\}$

Reidemeister 3:

(Alt) In[∘]:= $\text{Timing@HL}[(R_{1,2}\,R_{6,3}\,R_{4,5} \;//\; dm_{1,6\to 1}\,dm_{2,4\to 2}\,dm_{3,5\to 3}) \equiv (R_{2,3}\,R_{1,4}\,R_{5,6} \;//\; dm_{1,5\to 1}\,dm_{2,6\to 2}\,dm_{3,4\to 3})]$

(Alt) Out[∘]= $\{4.95313, \text{True}\}$

Relations between the four kinks:

(Alt) In[∘]:= $\text{Timing}\left[\text{HL} \;/@\; \left\{\text{Kink}_i \equiv ((R_{3,1}\,C_2) \;//\; dm_{1,2\to 1} \;//\; dm_{1,3\to i}),\; \overline{\text{Kink}}_j \equiv \left(\left(\overline{R}_{3,1}\,\overline{C}_2\right) \;//\; dm_{1,2\to 1} \;//\; dm_{1,3\to j}\right),\; \left(\left(\text{Kink}_i\,\overline{\text{Kink}}_j\right) \;//\; dm_{i,j\to 1}\right) \equiv d\eta_1\right\}\right]$

(Alt) Out[∘]= $\left\{6.71875, \left\{\frac{\hbar\, b_i}{2} + \hbar\, a_i\, b_i + \hbar\, x_i\, y_i == \hbar\, a_i\, b_i + \frac{1}{2}\left(-\text{Log}\left[B_i^2\right] - \hbar\, b_i\right) + \hbar\, x_i\, y_i,\; -\frac{\hbar\, b_j}{2} - \hbar\, a_j\, b_j - \frac{\hbar\, x_j\, y_j}{B_j} == -\hbar\, a_j\, b_j + \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_j^2}\right] + \hbar\, b_j\right) - \frac{\hbar\, x_j\, y_j}{B_j}, \text{True}\right\}\right\}$

The Trefoil

(Alt) In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = R1,5 R6,2 R3,7 C̄4 Kink̄8 Kink̄9 Kink̄10;
  Do[Z31 = Z31 // dm1,r→1, {r, 2, 10}];
  {Simplify /@ Z31, Simplify /@ (Z31 // b2t1 /. T1 → T)}]
```

(Alt) Out[∘]=

$$\left\{6.65625, \left\{\mathbb{E}_{\{\}\to\{1\}}\left[-\frac{1}{2}\,\mathrm{Log}\left[\left(1-B_1+B_1^2\right)^2\right]-\hbar\, b_1,\; -\frac{\hbar\left(B_1-2B_1^2-2B_1^4-a_1\left(-1+B_1-B_1^3+B_1^4\right)+2\hbar\, x_1 y_1+B_1^3\left(3+2\hbar\, x_1 y_1\right)\right)}{\left(1-B_1+B_1^2\right)^2}\right],\right.\right.$$
$$\left.\left.\mathbb{E}_{\{\}\to\{1\}}\left[-\frac{1}{2}\,\mathrm{Log}\left[\left(1-T_1+T_1^2\right)^2\right]+\hbar\, t_1,\; -\frac{\hbar\left(T_1-2T_1^2-2T_1^4-2a_1\left(-1+T_1-T_1^3+T_1^4\right)+2\hbar\, x_1 y_1+T_1^3\left(3+2\hbar\, x_1 y_1\right)\right)}{\left(1-T_1+T_1^2\right)^2}\right]\right\}\right\}$$

b2t, t2b, knot tensors.

(Alt) In[∘]:= `HL[(b2ti // t2bi) ≡ dσi→i]`

(Alt) Out[∘]= True

(Alt) In[∘]:= `t2bi // b2ti`

(Alt) Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}[a_i\,\alpha_i + y_i\,\eta_i + x_i\,\xi_i + t_i\,\tau_i,\ 0,\ 0]$

Reidemeister 2:

(Alt) In[∘]:=
```
Timing[HL[# ≡ dη1 dη2] & /@ {(kR̄1,2 kR3,4) // (km1,3→1 km2,4→2), (kR1,2 kR̄3,4) // (km1,3→1 km2,4→2)}]
```

(Alt) Out[∘]= {3.875, {True, True}}

Cyclic Reidemeister 2:

(Alt) In[∘]:=
```
Timing@HL[((kR1,4 kR̄5,2 kC̄3) // km2,4→2 // km1,3→1 // km1,5→1) ≡ kC̄1 dη2]
```

(Alt) Out[∘]= {1.32813, True}

Reidemeister 3:

(Alt) In[∘]:=
```
Timing@HL[(kR1,2 kR4,3 kR5,6 // km1,4→1 // km2,5→2 // km3,6→3 ) ≡
    (kR1,6 kR2,3 kR4,5 // km1,4→1 // km2,5→2 // km3,6→3)]
```

(Alt) Out[∘]= {2.60938, True}

Relations between the four kinks:

(Alt) In[∘]:=
```
Timing[HL /@ {kKink_i ≡ ((kR_{3,1} kC_2) // km_{1,2→1} // km_{1,3→i}),
   \overline{kKink}_j ≡ ((\overline{kR}_{3,1} \overline{kC}_2) // km_{1,2→1} // km_{1,3→j}), ((kKink_i \overline{kKink}_j) // km_{i,j→1}) ≡ dη_1}]
```

(Alt) Out[∘]=
$$\left\{3.53125, \left\{-\frac{t\hbar}{2} - t\hbar a_i + \hbar x_i y_i == \frac{1}{2}\left(t\hbar - \mathrm{Log}\left[T^2\right]\right) - t\hbar a_i + \hbar x_i y_i,\ \frac{t\hbar}{2} + t\hbar a_j - \frac{\hbar x_j y_j}{T} == \frac{1}{2}\left(-t\hbar - \mathrm{Log}\left[\frac{1}{T^2}\right]\right) + t\hbar a_j - \frac{\hbar x_j y_j}{T},\ \mathrm{True}\right\}\right\}$$

The Trefoil

(Alt) In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = kR_{1,5} kR_{6,2} kR_{3,7} \overline{kC}_4 \overline{kKink}_8 \overline{kKink}_9 \overline{kKink}_{10};
  Do[Z31 = Z31 // km_{1,r→1}, {r, 2, 10}];
  Simplify /@ Z31]
```

(Alt) Out[∘]=
$$\left\{5.375, \mathbb{E}_{\{\}\to\{1\}}\left[t\hbar - \frac{1}{2}\mathrm{Log}\left[\left(1 - T + T^2\right)^2\right], -\frac{\hbar\left(T - 2T^2 + 3T^3 - 2T^4 - 2\left(-1 + T - T^3 + T^4\right) a_1 + 2\left(1 + T^3\right)\hbar x_1 y_1\right)}{\left(1 - T + T^2\right)^2}\right]\right\}$$

(Alt) In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = kR_{1,5} kR_{6,2} kR_{3,7} \overline{kC}_4 \overline{kKink}_8 \overline{kKink}_9 \overline{kKink}_{10};
  Do[Z31 = Z31 // km_{1,r→1}, {r, 2, 10}];
  Simplify /@ Z31]
```

(Alt) Out[∘]=
$$\left\{4.32813, \mathbb{E}_{\{\}\to\{1\}}\left[t\hbar - \frac{1}{2}\mathrm{Log}\left[\left(1 - T + T^2\right)^2\right], -\frac{\hbar\left(T - 2T^2 + 3T^3 - 2T^4 - 2\left(-1 + T - T^3 + T^4\right) a_1 + 2\left(1 + T^3\right)\hbar x_1 y_1\right)}{\left(1 - T + T^2\right)^2}\right]\right\}$$

(Alt) In[∘]:=
```
Timing@Block[{$k = 1}, Z[Knot[8, 17]]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

(Alt) Out[∘]=
$$\left\{101.578,\ \mathbb{E}_{\{\}\to\{0\}}\left[\frac{1}{2}\left(-2\,t\,\hbar-\mathrm{Log}\left[\left(-1-\frac{1}{T^4}+\frac{4}{T^3}-\frac{6}{T^2}+\frac{5}{T}\right)^2\right]-\mathrm{Log}\left[\left(1+\frac{T}{1-4T+6T^2-5T^3+T^4}-\frac{2T^2}{1-4T+6T^2-5T^3+T^4}+\frac{T^3}{1-4T+6T^2-5T^3+T^4}\right)^2\right]-\mathrm{Log}\left[\left(1-\frac{T}{1-3T+4T^2-4T^3+T^4}+\frac{4T^2}{1-3T+4T^2-4T^3+T^4}-\frac{7T^3}{1-3T+4T^2-4T^3+T^4}+\frac{7T^4}{1-3T+4T^2-4T^3+T^4}-\frac{4T^5}{1-3T+4T^2-4T^3+T^4}+\frac{T^6}{1-3T+4T^2-4T^3+T^4}\right)^2\right]\right),\ \frac{-3\hbar+8T\hbar-8T^2\hbar+8T^4\hbar-8T^5\hbar+3T^6\hbar}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}+\frac{a\left(-6\hbar+16T\hbar-16T^2\hbar+16T^4\hbar-16T^5\hbar+6T^6\hbar\right)}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}+\frac{x\,y\left(-6\hbar^2+10T\hbar^2-6T^2\hbar^2-6T^3\hbar^2+10T^4\hbar^2-6T^5\hbar^2\right)}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}\right]\right\}$$

CU

Associativity of CU:

(Alt) In[∘]:=
$$\mathtt{Timing@Block[\{\$k = 3\},\ HL[(cm_{1,2\to1}\ //\ cm_{1,3\to1}) \equiv (cm_{2,3\to2}\ //\ cm_{1,2\to1})]]}$$

(Alt) Out[∘]= {2.01563, True}

Associativity, co-associativity, and Δ is an algebra morphism:

(Alt) In[∘]:=
$$\mathtt{Timing@Block[\{\$k = 3\},\ HL\ /@\ \{(cm_{1,2\to1}\ //\ cm_{1,3\to1}) \equiv (cm_{2,3\to2}\ //\ cm_{1,2\to1}),\ (c\Delta_{1\to1,2}\ //\ c\Delta_{2\to2,3}) \equiv (c\Delta_{1\to1,3}\ //\ c\Delta_{1\to1,2}),\ (cm_{1,2\to1}\ //\ c\Delta_{1\to1,2}) \equiv ((c\Delta_{1\to1,3}\ c\Delta_{2\to2,4})\ //\ (cm_{3,4\to2}\ cm_{1,2\to1}))\}]}$$

(Alt) Out[∘]= {3.54688, {True, True, True}}

S is convolution inverse of id:

(Alt) In[∘]:=
$$\mathtt{Timing@Block[\{\$k = 3\},\ HL[\# \equiv c\epsilon_1\ c\eta_1]\ \&\ /@\ \{(c\Delta_{1\to1,2}\ //\ cS_1)\ //\ cm_{1,2\to1},\ (c\Delta_{1\to1,2}\ //\ cS_2)\ //\ cm_{1,2\to1}\}]}$$

(Alt) Out[∘]= {3.45313, {True, True}}

S is an algebra anti-(co)morphism

(Alt) In[∘]:=
$$\mathtt{Timing@Block[\{\$k = 3\},\ HL\ /@\ \{(cm_{1,2\to1}\ //\ cS_1) \equiv ((cS_1\ cS_2)\ //\ cm_{2,1\to1}),\ (cS_1\ //\ c\Delta_{1\to1,2}) \equiv (c\Delta_{1\to2,1}\ //\ (cS_1\ cS_2))\}]}$$

(Alt) Out[∘]= {5.53125, {True, True}}

Classical is the $\hbar\to 0$ limit of quantum:

(Alt) In[∘]:=
```
ClassicalLimit[f_] := Normal@Series[Normal[f] // U2l, {ħ, 0, 0}] // l2U;
Timing[HL /@ Simplify /@
  {cm_{1,2→3} ≡ ClassicalLimit /@ dm_{1,2→3},
   (cΔ_{1→2,3} /. τ_1 → 0) ≡ ClassicalLimit /@ dΔ_{1→2,3}, cS_1 ≡ ClassicalLimit /@ dS_1}]
```

(Alt) Out[∘]= {1.95313, {True, True, True}}

(Alt) In[∘]:=
```
PrintProfile[]
```

(Alt) Out[∘]=
```
ProfileRoot is root. Profiled time: 262.768
  (      1)   0.360/ 101.530 above Z
  (     59)   1.189/  33.596 above Boot
  (   1314)   3.086/   9.406 above CF
  (    197)   6.817/  31.713 above EZip3
  (      1)       0/       0 above RVK
  (    197)   4.110/   5.906 above Zip1
  (    394)   4.843/  41.981 above Zip2
  (    197)  10.741/  38.635 above Zip3
CCF: called 115780 times, time in 105.884/105.884
  ( 115780) 105.880/ 105.880 under CF
CF: called 88372 times, time in 91.391/197.275
  (    214)   1.390/   3.066 under Z
  (    407)   0.421/   1.481 under Boot
  (   1263)   8.786/  20.797 under EZip3
  (   1314)   3.086/   9.406 under ProfileRoot
  (    680)   1.402/   3.360 under Zip1
  (   2526)  35.363/  97.405 under Zip2
  (  81968)  40.943/  61.760 under Zip3
  ( 115780) 105.880/ 105.880 above CCF
Zip3: called 680 times, time in 28.242/90.002
  (     57)   3.268/  17.156 under Z
  (     86)   4.596/  11.610 under Boot
  (    340)   9.637/  22.601 under EZip3
  (    197)  10.741/  38.635 under ProfileRoot
  (  81968)  40.943/  61.760 above CF
EZip3: called 340 times, time in 18.696/62.094
  (     57)  10.861/  22.017 under Z
  (     86)   1.018/   8.364 under Boot
  (    197)   6.817/  31.713 under ProfileRoot
  (   1263)   8.786/  20.797 above CF
  (    340)   9.637/  22.601 above Zip3
Zip2: called 680 times, time in 8.489/105.894
  (    114)   1.602/  56.540 under Z
  (    172)   2.044/   7.373 under Boot
  (    394)   4.843/  41.981 under ProfileRoot
  (   2526)  35.363/  97.405 above CF
Zip1: called 340 times, time in 8.344/11.704
  (     57)   1.246/   2.033 under Z
  (     86)   2.988/   3.765 under Boot
  (    197)   4.110/   5.906 under ProfileRoot
  (    680)   1.402/   3.360 above CF
Boot: called 86 times, time in 1.362/48.422
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  (     3)      0/  0.359 under Z
  (    24)  0.173/ 14.467 under Boot
  (    59)  1.189/ 33.596 under ProfileRoot
  (    24)  0.173/ 14.467 above Boot
  (   407)  0.421/  1.481 above CF
  (    86)  1.018/  8.364 above EZip3
  (    86)  2.988/  3.765 above Zip1
  (   172)  2.044/  7.373 above Zip2
  (    86)  4.596/ 11.610 above Zip3
Z: called 1 times, time in 0.36/101.531
  (     1)  0.360/ 101.530 under ProfileRoot
  (     3)      0/  0.359 above Boot
  (   214)  1.390/  3.066 above CF
  (    57) 10.861/ 22.017 above EZip3
  (    57)  1.246/  2.033 above Zip1
  (   114)  1.602/ 56.540 above Zip2
  (    57)  3.268/ 17.156 above Zip3
RVK: called 1 times, time in 0./0.
  (     1)      0/     0 under ProfileRoot
```