

Pensieve header: The Engine, with Zip3 encapsulation.

Canonical Forms:

```
CCF[ $\mathcal{E}_-$ ] := PPCCF@ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ]; (*Coefficient Canonical Form *)
CF[ $\mathcal{E}_-$ ] := PPCF@Module[
  { $\mathbf{vs} = \text{Cases}[\mathcal{E}, (\mathbf{y} \mid \mathbf{a} \mid \mathbf{x} \mid \eta \mid \beta \mid \tau \mid \xi)_-, \infty] \cup \{\mathbf{y}, \mathbf{a}, \mathbf{x}, \eta, \beta, \tau, \xi\}$ ,
  Total[(CCF[#][2]) (Times@@ $\mathbf{vs}^{\#}[1])$ ] & /@ CoefficientRules[ $\mathcal{E}$ ,  $\mathbf{vs}$ ]}
];
CF[ $\mathcal{E}_{-E}$ ] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}_{-List}$ ] := CF /@  $\mathcal{E}$ ;
CF[ $\mathbb{E}_{sp\_}[\mathcal{ES}\_\_\_\_]$ ] := CF /@  $\mathbb{E}_{sp}[\mathcal{ES}]$ ;
```

Variables and their duals:

```
In[ ]:= { $\mathbf{t}^*, \mathbf{b}^*, \mathbf{y}^*, \mathbf{a}^*, \mathbf{x}^*, \mathbf{z}^*, \tau^*, \beta^*, \eta^*, \alpha^*, \xi^*, \zeta^*$ } = { $\tau, \beta, \eta, \alpha, \xi, \zeta, \mathbf{t}, \mathbf{b}, \mathbf{y}, \mathbf{a}, \mathbf{x}, \mathbf{z}$ };
( $\mathbf{vs}_{-List}$ )* := ( $\mathbf{v} \mapsto \mathbf{v}^*$ ) /@  $\mathbf{vs}$ ;
( $\mathbf{u}_{-i}$ )* := ( $\mathbf{u}^*$ ) $i$ ;
```

Weights:

```
Clear[Wt];
Evaluate[Wt] /@ { $\mathbf{y}, \mathbf{b}, \mathbf{t}, \mathbf{a}, \mathbf{x}, \eta, \beta, \tau, \alpha, \xi$ } = {1, 0, 0, 2, 1, 1, 2, 2, 0, 1};
Wt[ $\mathbf{u}_{-i}$ ] := Wt[ $\mathbf{u}$ ];
```

The maximal weight \$n\$, i.e. the n of $gl(n)$. Initially and for a long while this will not be tested beyond $n == 2$.

```
In[ ]:= $n = 2;
```

Upper to lower and lower to Upper:

```
U2l[ $\mathcal{E}_-$ ] :=  $\mathcal{E}$  /. { $\mathbf{B}_{i-}^{p-} \mapsto \mathbf{e}^{-p \hbar \mathbf{b}_i}$ ,  $\mathbf{B}_{i-}^{p-} \mapsto \mathbf{e}^{-p \hbar \mathbf{b}}$ ,  $\mathbf{T}_{i-}^{p-} \mapsto \mathbf{e}^{p \hbar \mathbf{t}_i}$ ,  $\mathbf{T}_{i-}^{p-} \mapsto \mathbf{e}^{p \hbar \mathbf{t}}$ ,  $\mathcal{A}_{i-}^{p-} \mapsto \mathbf{e}^{p \alpha_i}$ ,  $\mathcal{A}_{i-}^{p-} \mapsto \mathbf{e}^{p \alpha}$ };
l2U[ $\mathcal{E}_-$ ] :=  $\mathcal{E}$  //. { $\mathbf{e}^{c- \cdot \mathbf{b}_i + d-} \mapsto \mathbf{B}_{i-}^{-c/\hbar} \mathbf{e}^d$ ,  $\mathbf{e}^{c- \cdot \mathbf{b} + d-} \mapsto \mathbf{B}_{i-}^{-c/\hbar} \mathbf{e}^d$ ,  $\mathbf{e}^{c- \cdot \mathbf{t}_i + d-} \mapsto \mathbf{T}_{i-}^{c/\hbar} \mathbf{e}^d$ ,  $\mathbf{e}^{c- \cdot \mathbf{t} + d-} \mapsto \mathbf{T}_{i-}^{c/\hbar} \mathbf{e}^d$ ,
 $\mathbf{e}^{c- \cdot \alpha_i + d-} \mapsto \mathcal{A}_{i-}^c \mathbf{e}^d$ ,  $\mathbf{e}^{c- \cdot \alpha + d-} \mapsto \mathcal{A}_{i-}^c \mathbf{e}^d$ ,  $\mathbf{e}^{\mathcal{K}-} \mapsto \mathbf{e}^{\text{Expand}[\mathcal{K}]}$ };
l2U[r_Rule] := Module[{ $\mathbf{U} = \mathbf{r}[1]$  /. { $\mathbf{b} \rightarrow \mathbf{B}$ ,  $\mathbf{t} \rightarrow \mathbf{T}$ ,  $\alpha \rightarrow \mathcal{A}$ }},  $\mathbf{U} \rightarrow \mathbf{l2U}[\mathbf{U2l}[\mathbf{U}] /. \mathbf{r}]$ ];
AlsoUpper[rs_List] :=  $\mathbf{rs} \cup (\mathbf{l2U} /@ \mathbf{rs})$ ;
```

Derivatives in the presence of exponentiated variables:

```
In[ ]:= Db[ $f_-$ ] :=  $\partial_{\mathbf{b}} f - \hbar \mathbf{B} \partial_{\mathbf{B}} f$ ; Dbi[ $f_-$ ] :=  $\partial_{\mathbf{b}_i} f - \hbar \mathbf{B}_i \partial_{\mathbf{B}_i} f$ ;
Dt[ $f_-$ ] :=  $\partial_{\mathbf{t}} f + \hbar \mathbf{T} \partial_{\mathbf{T}} f$ ; Dti[ $f_-$ ] :=  $\partial_{\mathbf{t}_i} f + \hbar \mathbf{T}_i \partial_{\mathbf{T}_i} f$ ;
D $\alpha$ [ $f_-$ ] :=  $\partial_{\alpha} f + \mathcal{A} \partial_{\mathcal{A}} f$ ; D $\alpha_i$ [ $f_-$ ] :=  $\partial_{\alpha_i} f + \mathcal{A}_i \partial_{\mathcal{A}_i} f$ ;
Dv[ $f_-$ ] :=  $\partial_v f$ ;
```

E operations:

```
 $\mathcal{E}_{-E}[\$]$  := Length[ $\mathcal{E}$ ] - 1;  $\mathbb{E}_{-}[\mathcal{ES}\_\_\_\_][\$]$  :=  $\mathbb{E}[\mathcal{ES}][\$]$ ;
 $\mathcal{E}_{-E}[k\_Integer]$  :=  $\mathcal{E}[k+1]$ ;  $\mathbb{E}_{-}[\mathcal{ES}\_\_\_\_][k\_Integer]$  := { $\mathcal{ES}$ }[ $k+1$ ];
 $\mathbb{E} /: \mathcal{E1}_{-E} \equiv \mathcal{E2}_{-E} := \text{Inner}[\text{CF}@\#1 == \text{CF}@\#2 \&, \mathcal{E1}, \mathcal{E2}, \text{And}]$ ;
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E1s}\_\_\_\_] \equiv \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E2s}\_\_\_\_] \wedge := (d1 == d2) \wedge (r1 == r2) \wedge (\mathbb{E}[\mathcal{E1s}] \equiv \mathbb{E}[\mathcal{E2s}])$ ;
 $\mathbb{E} /: \mathcal{E1}_{-E} * \mathcal{E2}_{-E} := \mathbb{E}@\text{Table}[\text{CF}[\mathcal{E1}[\mathbf{kk}] + \mathcal{E2}[\mathbf{kk}]], \{\mathbf{kk}, 0, \text{Min}[\mathcal{E1}[\$], \mathcal{E2}[\$]]\}]$ ;
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E1s}\_\_\_\_] \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E2s}\_\_\_\_] \wedge := \mathbb{E}_{(d1 \cup d2) \rightarrow (r1 \cup r2)}@\text{Table}[\mathcal{E1s} \mathbb{E}[\mathcal{E2s}]]$ ;
```

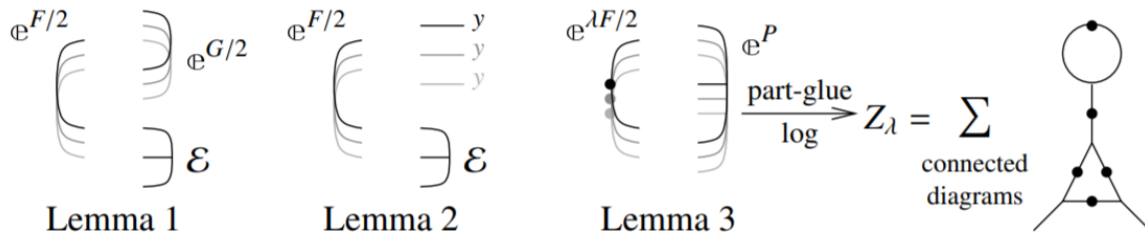
```

In[ ]:=
Module[{is = r1 ∩ d2, lvs},
  lvs = Flatten@Table[{y$ei, b$ei, t$ei, a$ei, x$ei}, {i, is}];
  E[d1 ∪ Complement[d2, is] → (r2 ∪ Complement[r1, is])] @@ (Ziplvs ∪ lvs*[{lvs*.lvs, Times[
    E[ $\mathcal{E}1s$ ] /. Table[(v : b | B | t | T | a | x | y)i → v$ei, {i, is}],
    E[ $\mathcal{E}2s$ ] /. Table[(v : β | τ | α | ℱ | ξ | η)i → v$ei, {i, is}]
  ]])
]

Λ2Ed→r[A_] := Module[{k}, Ed→r @@ l2U@Table[SeriesCoefficient[A, {ε, 0, k}], {k, 0, $k}]];

```

Ziping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.



```

In[ ]:=
Zipvs [{ $\mathcal{F}$ _,  $\mathcal{E}$ _}] := { $\mathcal{F}$ _,  $\mathcal{E}$ _} // Zip1vs // Zip2Select[vs, (0 < Wt[#] < $n) &] // EZip3Select[vs, (0 < Wt[#] < $n) &] //
Zip2Select[vs, (Wt[#] == 0 ∨ Wt[#] == $n) &] // Zip3Select[vs, (Wt[#] == 0 ∨ Wt[#] == $n) &] // Last;

```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \oplus^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

```

In[ ]:=
Zip1_{ } = Identity;
Zip1vs @ { $\mathcal{F}$ _, E[Q_, P_]} := PPZip1@Module[{I, F, G, u, v},
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == $n, ∂u*,v*  $\mathcal{F}$ , 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == $n, ∂u,v Q, 0], {u, vs}, {v, vs}];
  {CF[vs*. (F.Inverse[I - G.F]).vs* / 2], E[CF[Q - Log[Det[I - G.F]] / 2 - vs.G.vs / 2], P]}
]

```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E} \oplus^{\sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{E}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

```

In[ ]:=
Zip2_{ } = Identity;
Zip2vs @ { $\mathcal{F}$ _, E[Q_, P_]} := PPZip2@Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == $n, ∂u*,v*  $\mathcal{F}$ , 0], {u, vs}, {v, vs}];
  Y = Table[∂v Q, {v, vs}] /. AlsoUpper@Table[v → 0, {v, vs}];
  CF /@ ({ $\mathcal{F}$ _, E[Q - Y.vs + Y.F.Y / 2, P]} /. AlsoUpper@Thread[vs → vs + F.Y])
]

```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathbb{C}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

```

In[ ]:= Zip3vs@{F_, E_E} := PPZip3@Module[
  {F, u, v, Z, $k, kk, jj, $m = 0, m, n},
  $k = Length[E] - 1;
  Do[Z[0, kk] = E[kk + 1], {kk, 0, $k}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == $n, Du*,v*F, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ $k, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2(m+1)}$ 
        (Z[m, kk, u, v] + Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}]),
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1];
  ];
  CF/@({
    F - Sum[F[u, v] u* v* / 2, {u, vs}, {v, vs}],
    E@@Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, $k}]
  }) /. AlsoUpper@Table[v → 0, {v, vs}]
]

```

Encapsulation.

```

In[ ]:= EZip3vs@{F_, E_E} := PPEZip3@Module[
  {n8, nF, rc, ps, rr = {(*release rules*)}},
  rc = 0; n8 = Total[
    CoefficientRules[#, vs] /. (ps_ → c_) => (AppendTo[rr, c8[++rc] → c]; c8[rc] (Times@@ vsps))
  ] & /@ E;
  rc = 0; nF = Total[CoefficientRules[F, vs*] /.
    (ps_ → c_) => (AppendTo[rr, cF[++rc] → c]; cF[rc] (Times@@ (vs*)ps))];
  CF[Expand[{nF, n8} // Zip3vs] /. rr]
]

```