Pensieve header: The Engine; passes all tests.

Canonical Forms:

```
CCF[ℰ_] := PP_CCF@ExpandDenominator@ExpandNumerator@Together[ℰ]; (*Coefficient Canonical Form *)
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (y | a | x | η | β | τ | ξ)_, ∞] ⋃ {y, a, x, η, β, τ, ξ}},
    Total[(CCF[#〚2〛] (Times @@ vs^#〚1〛)) & /@ CoefficientRules[ℰ, vs]]
   ];
CF[ℰ_𝔼] := CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

Variables and their duals:

```
In[∘]:= {t*, b*, y*, a*, x*, z*, τ*, β*, η*, α*, ξ*, ζ*} = {τ, β, η, α, ξ, ζ, t, b, y, a, x, z};
(vs_List)* := (v ↦ v*) /@ vs;
(u_i_)* := (u*)_i;
```

Weights:

```
Clear[Wt];
Evaluate[Wt /@ {y, b, t, a, x, η, β, τ, α, ξ}] = {1, 0, 0, 2, 1, 1, 2, 2, 0, 1};
Wt[u_i_] := Wt[u];
```

The maximal weight $n, i.e. the *n* of gl(*n*). Initially and for a long while this will not be tested beyond $n == 2.

```
In[∘]:= $n = 2;
```

Upper to lower and lower to Upper:

```
U2l[ℰ_] := ℰ /. {B_i_^p_. :→ e^(-p ℏ b_i), B^p_. :→ e^(-p ℏ b), T_i_^p_. :→ e^(p ℏ t_i), T^p_. :→ e^(p ℏ t), 𝒜_i_^p_. :→ e^(p α_i), 𝒜^p_. :→ e^(p α)};
l2U[ℰ_] := ℰ //. {e^(c_. b_i_+d_.) :→ B_i^(-c/ℏ) e^d, e^(c_. b+d_.) :→ B^(-c/ℏ) e^d, e^(c_. t_i_+d_.) :→ T_i^(c/ℏ) e^d, e^(c_. t+d_.) :→ T^(c/ℏ) e^d,
    e^(c_. α_i_+d_.) :→ 𝒜_i^c e^d, e^(c_. α+d_.) :→ 𝒜^c e^d, e^𝒳_ :→ e^(Expand@𝒳)};
l2U[r_Rule] := Module[{U = r〚1〛 /. {b → B, t → T, α → 𝒜}}, U → l2U[U2l[U] /. r]];
AlsoUpper[rs_List] := rs ⋃ (l2U /@ rs);
```

Derivatives in the presence of exponentiated variables:

```
In[∘]:= D_b[f_] := ∂_b f - ℏ B ∂_B f; D_b_i_[f_] := ∂_b_i f - ℏ B_i ∂_B_i f;
D_t[f_] := ∂_t f + ℏ T ∂_T f; D_t_i_[f_] := ∂_t_i f + ℏ T_i ∂_T_i f;
D_α[f_] := ∂_α f + 𝒜 ∂_𝒜 f; D_α_i_[f_] := ∂_α_i f + 𝒜_i ∂_𝒜_i f;
D_v_[f_] := ∂_v f;
```

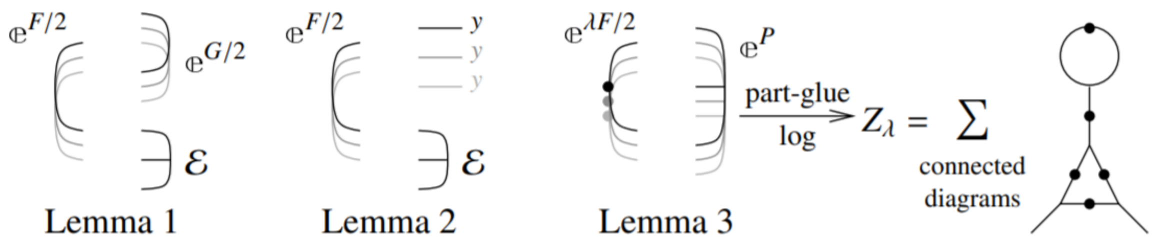
𝔼 operations:

```
ℰ_𝔼[$] := Length[ℰ] - 1; 𝔼_[ℰs___][$] := 𝔼[ℰs][$];
ℰ_𝔼[k_Integer] := ℰ〚k + 1〛; 𝔼_[ℰs___][k_Integer] := {ℰs}〚k + 1〛;
𝔼 /: ℰ1_𝔼 ≡ ℰ2_𝔼 := Inner[CF@#1 == CF@#2 &, ℰ1, ℰ2, And];
𝔼_d1_→r1_[ℰ1s___] ≡ 𝔼_d2_→r2_[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼 /: ℰ1_𝔼 * ℰ2_𝔼 := 𝔼 @@ Table[CF[ℰ1[kk] + ℰ2[kk]], {kk, 0, Min[ℰ1[$], ℰ2[$]]}];
𝔼_d1_→r1_[ℰ1s___] 𝔼_d2_→r2_[ℰ2s___] ^:= 𝔼_(d1⋃d2)→(r1⋃r2) @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
```

In[∘]:=
```
𝔼d1_→r1_[ℰ1s___] // 𝔼d2_→r2_[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs},
  lvs = Flatten@Table[{y$@i, b$@i, t$@i, a$@i, x$@i}, {i, is}];
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ (Zip lvs⋃lvs* [{lvs*.lvs, Times[
       𝔼[ℰ1s] /. Table[(v : b | B | t | T | a | x | y)i → v$@i, {i, is}],
       𝔼[ℰ2s] /. Table[(v : β | τ | α | 𝒜 | ξ | η)i → v$@i, {i, is}]
     ]}])
 ]
```

```
Λ2𝔼d_→r_[Λ_] := Module[{k}, 𝔼d→r @@ l2U@Table[SeriesCoefficient[Λ, {ϵ, 0, k}], {k, 0, $k}]];
```

Zipping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.




```
Zipvs_[{ℱ_, ℰ_}] := {ℱ, ℰ} // Zip1vs // Zip2Select[vs, (0<Wt[#]<$n)&] // Zip3Select[vs, (0<Wt[#]<$n)&] //
    Zip2Select[vs, (Wt[#]==0∨Wt[#]==$n)&] // Zip3Select[vs, (Wt[#]==0∨Wt[#]==$n)&] // Last;
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\,e^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_i z_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1} : \mathcal{E}\right\rangle_B$$

In[∘]:=
```
Zip1{} = Identity;
Zip1vs_@{ℱ_, 𝔼[Q_, P___]} := PPZip1@Module[{ℐ, F, G, u, v},
   ℐ = IdentityMatrix@Length@vs;
   F = Table[If[Wt[u] + Wt[v] == $n, ∂u*,v* ℱ, 0], {u, vs}, {v, vs}];
   G = Table[If[Wt[u] + Wt[v] == $n, ∂u,v Q, 0], {u, vs}, {v, vs}];
   {CF[vs*.(F.Inverse[ℐ - G.F]).vs* / 2], 𝔼[CF[Q - Log[Det[ℐ - G.F]] / 2 - vs.G.vs / 2], P]}
  ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E}\,e^{\sum_{i\in B} y_i z_i}\right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j}\left\langle F :\ \mathcal{E}|_{z_B \to z_B + F y_B}\right\rangle_B.$

```
Zip2{} = Identity;
Zip2vs_@{ℱ_, 𝔼[Q_, P___]} := PPZip2@Module[{F, Y, u, v},
   F = Table[If[Wt[u] + Wt[v] == $n, ∂u*,v* ℱ, 0], {u, vs}, {v, vs}];
   Y = Table[∂v Q, {v, vs}] /. AlsoUpper@Table[v → 0, {v, vs}];
   CF /@ ({ℱ, 𝔼[Q - Y.vs + Y.F.Y / 2, P]} /. AlsoUpper@Thread[vs → vs + F.Y])
  ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

Note that the power m of λ is at most $k-1+\frac{2k+2}{2}=2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

```
Zip3vs_@{𝓕_, ℰ_𝔼} := PPZip3@Module[{F, u, v, Z, $k, kk, jj, $m = 0, m, n},
  $k = Length[ℰ] - 1;
  Do[Z[0, kk] = ℰ⟦kk + 1⟧, {kk, 0, $k}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == $n, ∂u*,v* 𝓕, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ $k, ++kk,
    Z[m + 1, kk] = CF@Sum[
       If[F[u, v] === 0, 0, F[u, v] / (2 (m + 1))
         (Z[m, kk, u, v] + Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
       {u, vs}, {v, vs}];
    If[Z[m + 1, kk] =!= 0, $m = m + 1]
   ]];
  CF /@ ({
     𝓕 - Sum[F[u, v] u* v* / 2, {u, vs}, {v, vs}],
     𝔼 @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, $k}]
    } /. AlsoUpper@Table[v → 0, {v, vs}])
 ]
```