Pensieve header: Testing the associativity of compositions in GDO. Based on GenericDoPeGDO.nb in pensieve://Talks/DaNang-1905/.

tex

We test that the composition law of \gdo\ is indeed associative, by defining it general and verifying associativity on random (and hence likely generic) morphisms. First, we define the composition law of two morphisms. The program first determines E_i, F_i, and G_i from Q_i ($i=1,2$) by taking partial derivatives, and then outputs the scalar ω and quadratic Q, with equations~\eqref{eq:gdocompositions} converted nearly literally into code:

pdfli

```
In[∘]:= 𝕄A_→B_[ω1_, Q1_] // 𝕄B_→C_[ω2_, Q2_] := Module[{ζA, zC, E1, F1, G1, E2, F2, G2, I},
   ζA = Table[ζi, {i, A}]; zC = Table[zi, {i, C}]; I = IdentityMatrix@Length@B;
   E1 = Table[∂ζi,zj Q1, {i, A}, {j, B}]; E2 = Table[∂ζi,zj Q2, {i, B}, {j, C}];
   F1 = Table[∂ζi,ζj Q1, {i, A}, {j, A}]; F2 = Table[∂ζi,ζj Q2, {i, B}, {j, B}];
   G1 = Table[∂zi,zj Q1, {i, B}, {j, B}]; G2 = Table[∂zi,zj Q2, {i, C}, {j, C}];
   Expand /@ 𝕄A→C[ω1 ω2 Det[I - F2.G1]^(-1/2), ζA.E1.Inverse[I - F2.G1].E2.zC
      + 1/2 ζA.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).ζA +
      1/2 zC.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).zC] ]
```

tex

Next we implement ``random morphisms'' (\verb"RM") by picking their quadratic parts to have small random integer coefficients. We also set M_1, M_2, and M_3 to be random morphisms in $\mor(\{1,2\}\to\{1,2,3\})$, $\mor(\{1,2,3\}\to\{1,2,3\})$, and $\mor(\{1,2,3\}\to\{1,2\})$, respectively:

pdfli

```
In[∘]:= RM A_→B_ := Module[{vs = Table[ζi, {i, A}] ⋃ Table[zi, {i, B}]},
     𝕄A→B[1, Sum[RandomInteger[{-3, 3}] vi vj, {vi, vs}, {vj, vs}]]];
   {M1 = RM{1,2}→{1,2,3}, M2 = RM{1,2,3}→{1,2,3}, M3 = RM{1,2,3}→{1,2}} // Column
```

pdfli

Out[∘]=

$\mathbb{M}_{\{1,2\}\to\{1,2,3\}}\big[1,\ 3 z_1^2 + 4 z_1 z_2 - z_2^2 - 2 z_2 z_3 - z_1 \zeta_1 - 3 z_2 \zeta_1 - 4 z_3 \zeta_1 + \zeta_1^2 + 3 z_1 \zeta_2 - 6 z_2 \zeta_2 + 4 z_3 \zeta_2 + 4 \zeta_1 \zeta_2 + \zeta_2^2\big]$

$\mathbb{M}_{\{1,2,3\}\to\{1,2,3\}}\big[1,\ 3 z_1^2 + z_1 z_2 + 2 z_2^2 - 2 z_1 z_3 + 4 z_2 z_3 - 3 z_3^2 - z_1 \zeta_1 - z_2 \zeta_1 + 6 z_3 \zeta_1 - \zeta_1^2 + 2 z_1 \zeta_2 + z_2 \zeta_2 - z_3 \zeta_2 - 2 \zeta_1 \zeta_2 - 3 \zeta_2^2 - z_1 \zeta_3 - z_2 \zeta_3 + z_3 \zeta_3 - 3 \zeta_1 \zeta_3 + 2 \zeta_2 \zeta_3 + 2 \zeta_3^2\big]$

$\mathbb{M}_{\{1,2,3\}\to\{1,2\}}\big[1,\ -2 z_1^2 + 2 z_2^2 - 5 z_1 \zeta_1 - z_2 \zeta_1 + 2 \zeta_1^2 - z_1 \zeta_2 - 6 z_2 \zeta_2 + \zeta_1 \zeta_2 - 2 \zeta_2^2 - 2 z_2 \zeta_3 - 2 \zeta_1 \zeta_3 + 3 \zeta_2 \zeta_3 - \zeta_3^2\big]$

tex

Just to get an appreciation of what compositions look like, we compute $(M_1\act M_2)\act M_3$:

pdf

```
In[∘]:= (M1 // M2) // M3
```

pdf

Out[∘]=

$$\mathbb{M}_{\{1,2\}\to\{1,2\}}\Big[\frac{1}{2\sqrt{47913}},\ \frac{655148 z_1^2}{15971} + \frac{9600305 z_1 z_2}{31942} + \frac{47930587 z_2^2}{95826} - \frac{1241140 z_1 \zeta_1}{15971} - \frac{12434423 z_2 \zeta_1}{47913} + \frac{1792724 \zeta_1^2}{47913} + \frac{2520132 z_1 \zeta_2}{15971} + \frac{16827871 z_2 \zeta_2}{31942} - \frac{2104097 \zeta_1 \zeta_2}{15971} + \frac{2273807 \zeta_2^2}{15971}\Big]$$

tex

Finally, we verify that composition is associative:

pdf

In[]:=
```
((M1 // M2) // M3) == (M1 // (M2 // M3))
```

pdf

Out[]=
```
True
```

tex

The last \verb”True” above is an in-practice proof of Theorem~\ref{thm:GDO},~(i).