EVERYTHING AROUND sl^ϵ_{2+} IS DPG. HOORAY!

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Abstract. We construct sl^ϵ_{2+}, a certain "lossless approximation" of sl_2, and show that "everything that matters" around its universal enveloping algebra and its quantization, namely the products, the co-products, the R-matrix, and other essential ingredients can be described in terms of a certain category **DPG** of "**Do**cile **P**erturbed **G**aussian differential operators".

Those essential ingredients are what one needs in order to construct powerful knot invariants with good algebraic properties. Also, as we show, **DPG** is "easy" in the sense of computational complexity. Hence we get (and implement and compute) powerful poly-time-computatble knot invariants with favourable algebraic properties. Hooray!

Similar constructions ought to exist for all semi-simple Lie algebras, but we do not pursue this here.

Contents

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1. Plan of the Paper

There is little we want to say by means of an introduction beyond what we said already in the abstract (please read it again). Instead, here's the plan:

In Section 2 MORE.

In Section 3 MORE.

Sections 2 and 3 completely commute and can be read in either order.

MORE.

1.1. **Acknowledgement.** We wish to thank M. Pugh for Footnote 9.

2. The Category **DPG**

sec:DoPeGDO

2.1. **Motivation, conventions, generating functions.** This section may seem like an awful way to start a topology paper — it's all about formula-based technicalities. Here are its redeeming features (beyond its usefulness for the later parts of the paper):

- Did you know that quadratic forms (aka "Gaussians") form a category in a natural way? (Theorem 2.9 thm:GDO).
- Did you know that Feynman diagrams arise in pure algebra in a completely natural way?

mot:PBW **Motivation 2.1.** The "PBW Principle" says that many algebras U are isomorphic, as vector spaces, to polynomial rings (hence as algebras they are "polynomial rings with funny multiplications"). Many times one needs to understand maps between algebras. Primarily, the algebra's own structure: the multiplication map $m\colon U\otimes U\to U$, perhaps a co-multiplication $\Delta\colon U\to U\otimes U$, and more. Sometimes one may care about specific special elements in U or some tensor power thereof; say, $R\in U\otimes U=\operatorname{Hom}(U^{\otimes\varnothing}\to U^{\otimes 2})$. So we need to understand the category of maps between algebras and their tensor powers, and hence, by PBW, the category of maps between polynomial rings. This category is way too big — one can encode an infinite amount of information into a map between polynomial rings (no matter the base fields) — and so no finite computer can fully store a general such map. Hence we develop a theory of "maps between polynomial rings that can be described using finite formulas (of a certain kind)" and we are lucky that the maps we care about later in this paper can indeed be described by formulas of that kind. Those maps/formulas are "**Do**cile **Pe**rturbed **G**aussian differential operators", and they make a category, **DPG**, which is the main object of study for this section.

Convention 2.2. Throughout this paper we will use lower case Latin letters such as z, y, b, a, x, and t to denote the generators of polynomial rings. Each such generator comes with a dual (whose purpose will be explained shortly), and the dual will always be denoted by the corresponding Greek letter: $z^*=\zeta$, $y^*=\eta$, $b^*=\beta$, $a^*=\alpha$, $x^*=\xi$, and $t^*=\tau$. If C is a finite set, we will denote by $z_C=\{z_c\}_{c\in C}$ the set of variables denoted by the letter z with an index $c\in C$; likewise there's y_C, x_C, etc. We will regard z_C sometimes as a set and sometimes as a column vector, as appropriate. We extend duality to indexed variables: $z_C^*=\zeta_C=\{z_c^*=\zeta_c\}_{c\in C}$. We will sometimes treat ζ_C (or η_C, etc) as a row vector.

Next, we establish a bijection

$$\mathcal{G}\colon\ \operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])\to\mathbb{Q}[z_B][\![\zeta_a]\!] \tag{1}$$ eq:calG

between linear maps from polynomials in variables z_A to polynomials in variables z_B (A and B are finite sets) and a certain class of power series in the output variables and the duals of the input variables (more precisely, power series in the Greek variables corresponding to the inputs, with coefficients that are polynomials in the Latin variables corresponding to the outputs).

Definition 2.3. Let A and B be finite sets and let $L\colon\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]$ be linear. Let

$$\mathcal{L}=\mathcal{G}(L):=\sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}L(z_A^n)\in\mathbb{Q}[z_B][\![\zeta_a]\!] \tag{2}$$ eq:calG1

be the exponential generating function of the values of L. Here $\mathbb{N}$ denotes the non-negative integers, $n=(n_a)_{a\in A}$ is a multi-index, $\zeta_A^n:=\prod_{a\in A}\zeta_a^{n_a}$ and likewise $z_A^n:=\prod_{a\in A}z_a^{n_a}$, and $n!:=$

$\prod_{a\in A} n_a!$. Extending L without changing its name to an operator $L\colon \mathbb{Q}[z_A][\![\zeta_a]\!] \to \mathbb{Q}[z_B][\![\zeta_a]\!]$ by treating the ζ_A's as scalars, and recalling the definition of the exponential function, we find that (2) can also be written as

eq:calG1

$$\mathcal{L} = \mathcal{G}(L) = L\left(\mathbb{e}^{\zeta_A\cdot z_A}\right),$$

where $\zeta_A \cdot z_A := \sum_{a\in A} \zeta_a z_a$.

Proposition 2.4. *$\mathcal{G}\colon$ Hom$(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B]) \to \mathbb{Q}[z_B][\![\zeta_a]\!]$ is a bijection. If $\mathcal{L} \in \mathbb{Q}[z_B][\![\zeta_a]\!]$ and $p \in \mathbb{Q}[z_A]$ then*

$$\mathcal{G}^{-1}(\mathcal{L})(p) = p(\partial_{\zeta_a})\mathcal{L}(\zeta_a, z_b)|_{\zeta_a=0} = \mathcal{L}(\partial_{z_a}, z_b)p(z_a)|_{z_a=0}$$

□

Example 2.5. Consider $L_i\colon \mathbb{Q}[z] \to \mathbb{Q}[z]$ for $i = 1,2,3,4$, where $L_1(p) = p$ is the identity, $L_2(p) = p(z+1)$ is the shift, $L_3(p) = p'$ is differentiation, and $L_4(p) = \int_0^z p$ is definite integration. Then

$$\mathcal{G}(L_1) = \mathbb{e}^{\zeta z}, \qquad \mathcal{G}(L_2) = \mathbb{e}^{\zeta(z+1)}, \qquad \mathcal{G}(L_3) = \zeta\mathbb{e}^{\zeta z}, \qquad \text{and} \qquad \mathcal{G}(L_4) = (\mathbb{e}^{\zeta z}-1)/\zeta.$$

□

Linear maps between polynomial rings can be composed, and it is useful to know how their corresponding generating functions compose[1]:

:LMcomposition **Proposition 2.6.** *Let A, B, and C be finite sets, and let $L \in$ Hom$(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$ and $M \in$ Hom$(\mathbb{Q}[z_B] \to \mathbb{Q}[z_C])$. Then, with b standing for all elements of B,*

$$\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0} = \left(\mathcal{G}(M)|_{\zeta_b\to\partial_{z_b}}\mathcal{G}(L)\right)_{z_b=0}. \tag{3}$$

eq:LMcompositi

□

Said differently, $\mathcal{G}$ is an isomorphism of categories from the category of polynomial rings in finitely many generators to the category $\mathfrak{G}$ whose objects are finite sets with morphisms $\mathrm{mor}_{\mathfrak{G}}(A \to B) = \mathbb{Q}[z_B][\![\zeta_A]\!]$ and compositions

$$\mathcal{L}/\!\!/\mathcal{M} = \left(\mathcal{L}|_{z_b\to\partial_{\zeta_b}}\mathcal{M}\right)_{\zeta_b=0} = \left(\mathcal{M}|_{\zeta_b\to\partial_{z_b}}\mathcal{L}\right)_{z_b=0}, \tag{4}$$

eq:frakGCompos

where $\mathcal{L} \in \mathrm{mor}_{\mathfrak{G}}(A \to B)$ and $\mathcal{M} \in \mathrm{mor}_{\mathfrak{G}}(B \to C)$.

Later in this paper we will also want to consider power series in the mold of $\mathbb{e}^z \in \mathbb{Q}[\![z]\!]$ or $(1-z)^{-1}$. The generating function formalism does not extend to power series in the most naive way: the space Hom$\left(\mathbb{Q}[\![z_A]\!] \to \mathbb{Q}[\![z_B]\!]\right)$ is *not* isomorphic to some space of "generating functions" such as $\mathbb{Q}[\![\zeta_A, z_B]\!]$. Indeed, $\mathbb{Q}[\![z_A]\!]$ is of uncountable dimension over $\mathbb{Q}$, and Hom$\left(\mathbb{Q}[\![z_A]\!] \to \mathbb{Q}[\![z_B]\!]\right)$ is quite wild. One standard way to get around this is to introduce a "small" parameter $\hbar$ and insist that it be present in power series, as in $\mathbb{e}^{\hbar z}$ and $(1-\hbar z)^{-1}$. But first, a discussion and a convention.

In analysis the identity $(1-\hbar z)^{-1} = \sum \hbar^n z^n$ holds true even if $|z|$ isn't small, provided $\hbar$ is small enough[2]. In algebra, if we want to enrich $\mathbb{Q}[z]$ so as to allow such identities[3] we need to do two things:

[1]Below and throughout we use "$/\!\!/$" for left-to-right composition: $L/\!\!/M = M \circ L$.

[2]How small? $|h|$ must be smaller than $|z|^{-1}$, so $\hbar$ must be determined *after* z.

[3]And yet without making z small, that is, without switching to $Q[\![z]\!]$, which our formalism can't handle.

- Tensor multiply $\mathbb{Q}[z]$ with $\mathbb{Q}[\hbar]$ to get $\mathbb{Q}[z,\hbar]$, so as to allow coefficient depending on $\hbar$.
- Complete relative to the $\hbar$-adic topology so as to get $\mathbb{Q}[z][\![\hbar]\!]$, where series like $\sum \hbar^n z^n$ make sense.

disc:Qh **Convention 2.7** (and subtle point)**.** We slightly abuse notation and use $\mathbb{Q}_\hbar$ as a symbol for both steps:

$$\mathbb{Q}_\hbar[x,y,z] \coloneqq \mathbb{Q}[x,y,z][\![\hbar]\!].$$

Note that $\mathbb{Q}_\hbar$ is not a ring but a name for an operator: tensor with $\mathbb{Q}[\hbar]$ and complete relative to the $\hbar$-adic topology. In particular, $\mathbb{Q}_\hbar$ isn't $\mathbb{Q}[\![\hbar]\!]$ and $\mathbb{Q}_\hbar[z]$ isn't $\mathbb{Q}[\![\hbar]\!][z]$. Indeed, $\mathbb{e}^{\hbar z}$ and $(1-\hbar z)^{-1}$ are both members of $\mathbb{Q}_\hbar[z]$ but not of $\mathbb{Q}[\![\hbar]\!][z]$.

Yet we further abuse notation, and when $\mathbb{Q}_\hbar$ is on its own, we will regard it as the ring $\mathbb{Q}[\![\hbar]\!]$. So "$\omega \in \mathbb{Q}_\hbar$" means that ω is a power series in $\hbar$ with rational coefficients.

With all this said, in much of this paper one can read $\mathbb{Q}_\hbar$ to simply mean "$\mathbb{Q}$, also with a small parameter $\hbar$", with only a minor disloyalty to precission. disc:Qh $\boxed{2.7}$

Everything said so far work over $\mathbb{Q}_\hbar$ as well as over $\mathbb{Q}$. The same bijection as in (1) eq:calG,

$$\mathcal{G}\colon\ \mathrm{Hom}(\mathbb{Q}_\hbar[z_A] \to \mathbb{Q}_\hbar[z_B]) \to \mathbb{Q}_\hbar[z_B][\![\zeta_a]\!],$$

with the same definition (2) eq:calG1 and the same composition law (3) eq:LMcomposition.

MORE.

ssec:GDO

2.2. Gaussian Differential Operators.

In the examples we care about (see Motivation 2.1 mot:PBW) the generating functions turn out to be perturbed Gaussians, whose perturbations are in some sense "docile"[4]. Hence we seek to define a category **DPG** of docile perturbed Gaussian generating functions, with "differential operator" compositions as in Proposition ?? prop:LMcomposition. We start with the unperturbed version, **GDO**:

def:GDO **Definition 2.8. GDO** is the category with objects finite sets and, if A and B are finite, with $\mathrm{mor}(A \to B)$ the set of "Gaussians in $\zeta_A \cup z_B$":

$$\mathrm{mor}(A \to B) = \left\{\omega \mathbb{e}^{Q}\right\},$$

where $\omega \in \mathbb{Q}_\hbar$ is a scalar and where Q is a "small" quadratic expression in $\zeta_A \cup z_B$ with coefficients in $\mathbb{Q}_\hbar$. To define "small" and the composition law, we decompose quadratics in $\zeta_A \cup z_B$ into a Greek-Latin part E, and Greek-Greek part F, and a Latin-Latin part G:

$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j.$$

With this, "small" means that G must be a multiple of $\hbar$. Also, we define the composition of $\omega_1\mathbb{e}^{Q_1} \in \mathrm{mor}(A\to B)$ and $\omega_2\mathbb{e}^{Q_2}$ to be $\omega\mathbb{e}^{Q}$, with

$$\begin{aligned} E &= E_1(I-F_2G_1)^{-1}E_2, & F &= F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T, \\ G &= G_2 + E_2^TG_1(I-F_2G_1)^{-1}E_2, & \omega &= \omega_1\omega_2\det(I-F_2G_1)^{-1/2}, \end{aligned} \tag{5}$$

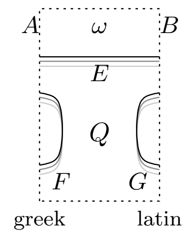
eq:gdocomposit

where (E,F,G) and (E_i,F_i,G_i) are the Greco-Roman decompositions of Q and of Q_i as above. Finally, the identity morphism in $\mathrm{mor}(A\to A)$ is declared to be $\mathbb{e}^{\zeta_A\cdot z_A}$. def:GDO $\boxed{2.8}$

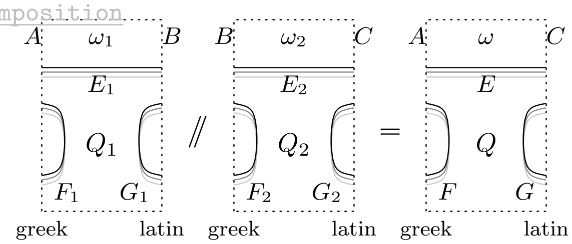
[4] Or perhaps, we care about those examples precisely because their generating functions are docile perturbed Gaussians.

Theorem 2.9. *(i)* **GDO** *is indeed a category (the composition law is associative, the identity morphisms are identity morphisms).*
(ii) The explicit composition law of (5) agrees with the "differential operator" one of (3).

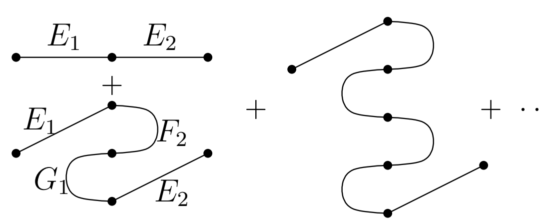
Proof. Part (i) can be verified by explicit matrix computations. It can also be implemented and tested, and seeing that we are committed to computability, we do that in Appendix 6.1. Finally, part (i) follows from part (ii) and the fact that the composition law of (3) is obviously associative. Hence we concentrate on proving (ii). We do it in two ways: pictorial, right below, for those who are familiar with diagrammatic algebra, and pure algebraic, on page **??**. □



Pictorial proof of Theorem 2.9, (ii). This proof assumes familiarity with the kind of diagrammatics that occurrs with Feynman diagrams in quantum field theory and/or with exponentials of connected diagrams as they occurr in, say, [BGRT]. Pictorially, we view morphisms in $\mathrm{mor}_{\mathbf{GDO}}(A \to B)$ as in the picture on the right: we put the Greek input variables corresonding to A on the left, the Latin output variable corresponding to B on the right, we indicate the scalar coefficient ω at the top, and we use the bulk of the picture to indicate Q and its Greco-Roman decomposition, with an obvious "Greek facing" placement of F, "Latin facing" placement of G, and "across the divide" placement of E. Note that Q is exponentiated and that exponentials are "reservoirs of multiple copies" $\mathbb{e}^x = 1 + x + xx/2 + xxx/6 + \ldots$. We emphasize this by drawing E, F, and G as having multiple shadows.



With this language, a composition as in (3) of a pair of morphisms as on the right is interpreted as "sum over all possible contractions of latin-side ends in $\mathbb{e}^{Q_1}$ with greek-side ends in $\mathbb{e}^{Q_2}$ (provided their labels, which are elements of B, agree)". Thus to figure out, say, the E part of the output, we need to figure out all the ways to travel from A to C across the composition of $\mathbb{e}^{Q_1}$ and $\mathbb{e}^{Q_2}$ by carrying out such contractions.

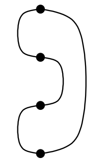


The most obvious way to travel across is the direct route: contract E_1 with E_2. This contributes a term proportional to E_1E_2 to the output E. Another possibility is to travel along E_1, then F_2, then G_1, then E_2, producing a term proportional to $E_1F_2G_1E_2$. Another possibility is to take the F_2G_1 detour twice, producing a term proportional to $E_1(F_2G_1)^2E_2$. In general, and with proper accounting of the combinatorial factors (it turns out that all proportionality factors are 1), we get

$$E = \sum_{r=0}^{\infty} E_1(F_2G_1)^r E_2 = E_1(I - F_2G_1)^{-1}E_2,$$

where the last equality was obtained by summing a geometric series.

Similar reasonings justify the formulas for F and for G.



Yet there is one further contribution to $\mathbb{e}^{Q_1} /\!\!/ \mathbb{e}^{Q_2}$, coming from closed F_2G_1 cycles as on the right (but of an arbitrary length r). This contribution is a scalar that modifies $\omega_1\omega_2$, and it is $\exp\left(\sum_{r=1}^{\infty}\frac{1}{2r}\operatorname{tr}(F_2G_1)^r\right) = \exp(-\frac{1}{2}\operatorname{tr}\log(1-F_2G_1)) = \det(1-F_2G_1)^{-1/2}$, justifying the last part of Equation (eq:gdocompositions 5). Note that in the last formula we used the familar quantum field theory dictum to "divide each diagram by the order of its symmetry group" to get the $1/2r$ factor, and that throughout the proof we regarded only connected diagrams and exponentiated the result, as per the dictum "the logarithm of the partition function is generated by connected diagrams". $\boxed{\text{pictorial}}$

2.3. **A Baby DPG and the Statement of the main DPG Theorem.** In this section we introduce a "baby" version of **DPG**, in which the most interesting features of the "mature" versions are present, yet some inconveniencies regarding weights are censored.

Definition 2.10. Let Ω be some ring of "scalars" and let ϵ be a formal parameter. Like **GDO**, let $\mathbf{DPG}_b$ be the category with objects finite sets and, if A and B are finite, with $\operatorname{mor}(A \to B)$ the set of "docile perturbed Gaussians in $\zeta_A \cup z_B$":

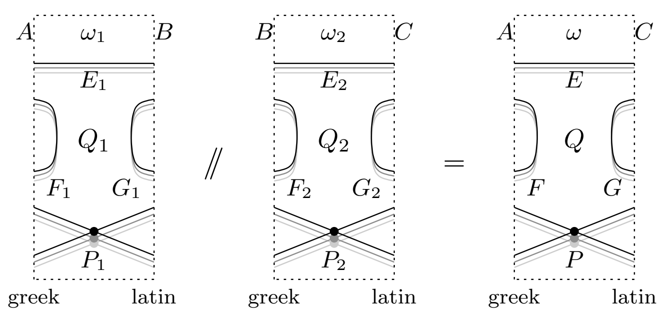
$$\operatorname{mor}(A \to B) = \left\{ \omega \mathbb{e}^{Q+P} \right\},$$

where ω and Q are ϵ-independent and otherwise as in Definition 2.8, and where P is a power series in ϵ of the form $P = \sum_{k\geqslant 1} P^{(k)}\epsilon^k$ and where each $P^{(k)}$ is a polynomial in $\zeta_A \cup z_B$ satisfying the "docility condition":

$$\deg P^{(k)} \leqslant 2k+2.$$

The composition law of $\mathbf{DPG}_b$ is "whatever is compatible with (3)" (so this definition becomes complete only following the discussion of Feynman diagrams below, or in Section 2.4). $\boxed{\text{Def. 2.10}}$

We now seek to understand compositions. With the same diagrammatic language as before, we seek to determine ω, $Q = (E, F, G)$ and P, so that the following would hold, where composition is "all possible contractions":

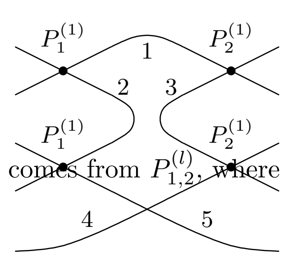


(6)

Looking only at the ϵ-independent part, it is clear that the composition law for ω and for Q is the same as for **GDO** (5) (so **DPG** is an "extension" of **GDO**). We just have to find $P = \sum_{k\geqslant 1} P^{(k)}\epsilon^k$ as a function of $Q_{1,2}$ and $P_{1,2}$.

Well, $P^{(k)}$ must get k factors of ϵ and it can only get them from P_1 and P_2. So $P^{(k)}$ is a sum of diagrams that have at most k vertices[5]. These vertices can be connected to each other (including self-connections), or to the outside, either directly, or by travelling along $E_{1,2}$ lines, or by travelling along F_2G_1 or G_1F_2 cycles as before. The latter cycles produce geometric series that sum to either $(I - F_2G_1)^{-1}$ or $(I - G_1F_2)^{-1}$. We arrive at the following theorem, which we state in a slightly informal manner as a more rigorous treatment follows in Section 2.4:

Theorem 2.11. *In a composition as in* (6) *the term $P^{(k)}$ in P is the sum of all connected Feynman diagrams as on the right, each divided by the order of its automorphism group, and in which the vertices are*



[5]Less than k if a single vertex brings along more than one factor of ϵ. Namely, if it comes from $P^{(l)}_{1,2}$, where $l \geqslant 2$.

determined by P_1 and P_2 and in which there are five types of propagators (all sampled on the right):

(1) A P_1-to-P_2 propagator which equals $(I - F_2G_1)^{-1}$.
(2) A P_1-to-P_1 propagator which equals $(I - F_2G_1)^{-1}F_2$.
(3) A P_2-to-P_2 propagator which equals $G_1(I - G_1F_2)^{-1}$.
(4) A greek-to-P_2 propagator which equals $E_1(I - F_2G_1)^{-1}$.
(5) A P_1-to-latin propagator which equals $(I - F_2G_1)^{-1}E_2$.

The figure here depicts a contribution to $P^{(4)}$. In general the valencies of vertices may be higher and self-contractions of two edges coming out of the same vertex are allowed. □

def:DoPeGDO eq:DoPeGDOCompositions

Proposition 2.12. $\mathbf{DPG}_b$*, as defined in Definition 2.10 and with composition as in the above theorem, is indeed a category. Namely, with notation as in Equation (6) and with P as in the theorem, if P_1 and P_2 are docile then so is P.*

Proof. Consider a diagram contributing to P that has m vertices $v_1, \ldots, v_m$. Each v_i comes from either P_1 or P_2 and brings along some power k_i of ϵ, so the diagram overall contributes a term T in which the power of ϵ is $k = \sum_{i=1}^m k_i$. We need to show that the degree of T in the Greek and Latin variables satisfies $\deg T \leqslant 2k + 2$. Indeed, by the docility of P_1 and P_2 each v_i contributes at most $2k_i + 2$ to that degree. Also, the diagram is connected[6] so it has at least $m - 1$ edges, and each one contracts to variables, so each one reduces the overall degree by 2. So $\deg T \leqslant (\sum_{i=1}^m 2k_i + 2) - 2(m - 1) = 2k + 2$. □

The full **DPG** category needed in this paper is merely a "garnished" version of $\mathbf{DPG}_b$, in which every variable has a "weight", and some weight restriction apply. We now turn to its formal definitioni, which we give in a slightly informal manner.

Context 2.13. Let $n > 0$ be a positive integer, and let us work in some universe of Latin and Greek variables in which every variable z (or ζ) has a weight $\mathrm{wt}(z)$ (or $\mathrm{wt}(\zeta)$) with $0 \leqslant \mathrm{wt}(z), \mathrm{wt}(\zeta) \leqslant n$, so that if z and ζ are dual then $\mathrm{wt}(z) + \mathrm{wt}(\zeta) = n$. Every monomial in our universe now has a weight, the sum of the weights of all the variables appearing in it, counted with multiplicity. The variables $\hbar$ and ϵ are special and do not carry a weight.

sec:Everything

Example 2.14. In the main context of this paper, that of Section 4, we will have variables y_i, b_i, a_i, and x_i (where i can run in some sets of labels), and their duals η_i, β_i, α_i, and ξ_i, with weights $\mathrm{wt}(y_i, b_i, a_i, x_i) = (1, 0, 2, 1)$ and $\mathrm{wt}(\eta_i, \beta_i, \alpha_i, \xi_i) = (1, 2, 0, 1)$. In this context, $\mathrm{wt}(\alpha_3^{62} a_1^8 y_{41}^3 \hbar^1 \epsilon^8) = 62 \cdot 0 + 8 \cdot 2 + 3 \cdot 1 + 0 + 0 = 19$.

Definition 2.15. A power series $P = \sum P^{(k)} \epsilon^k$ is called "docile" if for every k every monomial appearing in $P^{(k)}$ has weight less than $n(k + 1)$ (with slight imprecision, this is $\mathrm{wt}(P^{(k)}) \leqslant n(k + 1)$). The same P is called "G-docile" if it is docile and in addition the following "Condition G_{n0}" holds:

> **Condition G_{n0}.** For any weight-n variable z, $\partial_z P^{(0)}$ is affine-linear in the weight-0 variables.

Comment 2.16. Note that if P is docile then $\mathrm{wt}(P^{(0)}) \leqslant n$ so if also $wt(z) = n$, then $\mathrm{wt}(\partial_z P^{(0)}) = 0$.

MORE: State up front a full EDDO/**DPG** theorem.

[6]Da liegt der Hund begraben. Had we used $\omega e^Q P$ instead of ωe^{Q+P} for the morphisms of **DPG** we'd have had no connectedness here and the docility bound would have been $\deg P^{(k)} \leqslant 4k$, leading to slower computations.

The diagrammatic discussion of this section can be continued and extended to the full $\mathbf{DPG}_n$ category of Section [ssec:FullDPG] 2.5 but we prefer the more solid grounds of pure algebra as in the next section, Section [ssec:PDE] 2.4.

2.4. **Algebra by means of Partial Differential Equations.** Much as we love intuitive graphical reasonings such as in the previous sections, we also like the more solid grounds of algebra. Hence we repeat the content of Sections 2.2 and 2.3 in a purely algebraic language (as it turns out, it is the language of partial differential equations, though they are only used with power series, and hence we remain in pure algebra).

We recall the recipe (4) for the composition of generating functions $A \xrightarrow{\mathcal{L}} B \xrightarrow{\mathcal{M}} C$ and add a third version, the rightmost formula below, which treats $\mathcal{L}$ and $\mathcal{M}$ and Greek and Latin letters more symmetrically:

$$\mathcal{L}/\!\!/\mathcal{M} = \left(\mathcal{L}|_{z_b\to\partial_{\zeta_b}}\mathcal{M}\right)_{\zeta_b=0} = \left(\mathcal{M}|_{\zeta_b\to\partial_{z_b}}\mathcal{L}\right)_{z_b=0} = \mathbb{e}^{\sum\partial_{z_b}\partial_{\zeta_b}}(\mathcal{L}\cdot\mathcal{M})\big|_{z_b=\zeta_b=0}, \tag{7}$$

where the indices b run through the set B. Here $\mathcal{L}\cdot\mathcal{M}$ stands for the ordinary product of power series $\mathbb{Q}[z_B][\![\zeta_A]\!]\otimes\mathbb{Q}[z_C][\![\zeta_B]\!]\to\mathbb{Q}[z_{A\cup B}][\![\zeta_{B\cup C}]\!]$.[7] Thus we come to the following

Definition 2.17. Let B be a finite set, let F be a $B\times B$ matrix, and let $\mathcal{E}$ be a power series in variables that include the variable z_B. Set the "partial contraction" and the "full contraction" of $\mathcal{E}$ using F to be

$$[F\colon\mathcal{E}]_B := \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad\text{and}\quad \langle F\colon\mathcal{E}\rangle_B := [F\colon\mathcal{E}]_B|_{z_B\to 0}.$$

Note 2.18. To ensure convergence one must assume some "smallness" condition on either F or $\mathcal{E}$. We defer this to a later point.

Note 2.19. In the above definition, $\mathcal{E}$ replaces the product $\mathcal{L}\cdot\mathcal{M}$ of (7), we restrict to a single "type" of variables z_B instead of the $z_B\cup\zeta_B$ of (7) (so B here is "twice" the B of (7)), and instead of a pairing matrix of the form $\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$ as in (7), we allow a general matrix F. This will become beneficial soon.

Note 2.20. The computations of $[F\colon\cdot]_B$ and of $\langle F\colon\cdot\rangle_B$ are equivalent by "soft" means: $[F\colon\cdot]_B$ clearly determines $\langle F\colon\cdot\rangle_B$, and we also have $[F\colon\mathcal{E}]_B = \left\langle F\colon\ \mathcal{E}|_{z_b\to z_b+z'_b}\right\rangle\Big|_{z'_b\to z_b}$, where z'_B is a new set of variables indexed by B. The full contraction $\langle F\colon\cdot\rangle_B$ is used in (7), yet the partial contraction $[F\colon\cdot]_B$ is easier to manipulate as below.

Let λ be a formal variable and let $\mathcal{Z}_\lambda := [\lambda F\colon\mathcal{E}]_B$. Then $\mathcal{Z}_\lambda$ (and hence all we care about in this section) is determined by the following initial value problem, a heat equation:

$$\mathcal{Z}_0 = \mathcal{E} \qquad\text{and}\qquad \partial_\lambda\mathcal{Z}_\lambda = \frac{1}{2}\sum_{i,j\in B}F_{ij}\partial_{z_iz_j}\mathcal{Z}_\lambda. \tag{8}$$

Yet we like to write generating functions as exponentials[8], and hence the following proposition:

Proposition 2.21. *With $E=\log\mathcal{E}$ and $Z_\lambda=\log\mathcal{Z}_\lambda$ Equation (8) becomes*

$$Z_0 = E \qquad\textit{and}\qquad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B}F_{ij}\left(\partial_{z_iz_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right). \tag{9}$$

[7] Strictly speaking this is valid only if there are no name clashes, namely if $A\cap B = B\cap C=\varnothing$. That's a non-issue — if needed the labels in B can be temporarily renamed before the formula is applied.

[8] The equations become non-linear, but as we will see later, their solutions lie in smaller spaces, allowing for

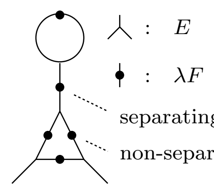
Proof. Simply substitute $\mathcal{Z}_\lambda = e^{Z_\lambda}$ into (8) and carry out the differentiations. $\square$

A sometimes-useful alternative to (9) is to allow F to be implicitly dependent on λ in an arbitrary (differentiable) manner with the condition $F|_{\lambda=0} = 0$ and to suppress the λ subscript in Z_λ. The resulting equation is

$$Z|_{\lambda=0} = E \qquad \text{and} \qquad \partial_\lambda Z = \frac{1}{2}\sum_{i,j\in B} (\partial_\lambda F_{ij})\left(\partial_{z_i z_j} Z + (\partial_{z_i} Z)(\partial_{z_j} Z)\right). \tag{9'}$$

We call Equation (9) (and its variant Equation (9')) "the synthesis equation", as it governs how the "vertices" in E merge and contract to sythesize larger and larger connected diagrams, as in the interpretation below.[9]

Interpretation 2.22. For the initiated, we cannot resist including a Feynman-diagran interpretation of Equation (9). With E "the vertices" and F "the contraction tensor" (roughly, "the propagator"), $\mathcal{Z}_\lambda = \langle \lambda F \colon e^E \rangle$ is the sum of all Feynman diagrams that can be made with vertices in E and contractions as dictated by F, with each contraction multiplied by an additional factor of λ. Then $Z_\lambda = \log \mathcal{Z}_\lambda$ is the same, except restricting to connected Feynman diagrams. And then $\partial_\lambda Z_\lambda$ picks out one contraction in Z_λ. If it is "separating", it contributes an F-weighted product of two connected diagrams — the term $(\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)$. If it not separating, it can be seen to contribute the $\partial_{z_i z_j} Z_\lambda$ term. See the picture on the right. 2.22



Lemma 1. $\left\langle F\colon \mathcal{E}\, e^{\sum_{i\in B} y_i z_i}\right\rangle_B = e^{\frac12 \sum_{i,j\in B} F_{ij} y_i y_j} \left\langle F\colon\ \mathcal{E}|_{z_B \to z_B + F y_B}\right\rangle_B$ and

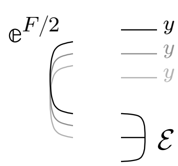
$$\left[F\colon \mathcal{E}\, e^{\sum_{i\in B} y_i z_i}\right]_B = e^{\frac12 \sum_{i,j\in B} F_{ij} y_i y_j + \sum_{i\in B} y_i z_i} \left[F\colon\ \mathcal{E}|_{z_B\to z_B+F y_B}\right]_B$$
$$= e^{\frac12 \sum_{i,j\in B} F_{ij} y_i y_j + \sum_{i\in B} y_i z_i} \left(\left[F\colon \mathcal{E}\right]_B|_{z_B \to z_B + F y_B}\right).$$

Lemma 2. With convergences left to the reader,

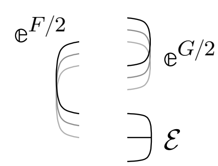
$$\left\langle F\colon \mathcal{E}\, e^{\frac12 \sum_{i,j\in B} G_{ij} z_i z_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}\colon \mathcal{E}\right\rangle_B,$$

and

$$\left[F\colon \mathcal{E}\, e^{\frac12 \sum_{i,j\in B} G_{ij} z_i z_j}\right]_B = \det(1-GF)^{-1/2} e^{\frac12 \sum_{i,j\in B} (G(I-FG)^{-1})_{ij} z_i z_j}$$
$$\cdot \left(\left[F(1-GF)^{-1}\colon \mathcal{E}\right]_B\right)_{z_B \to (I-FG)^{-1} z_B}.$$



Lemma 1



Lemma 2

MORE.

more efficient manipulations.

[9]M. Pugh told us that Equation (9) is a variant of "Burger's equation", and that it's relationship with the heat equation (8) is a variant of the "Cole-Hopf transformation".

ssec:FullDPG

2.5. **Full DPG.** MORE.

3. sl_{2+}^{ϵ}, CU, and QU

sec:U

For a minimalistic reading of this paper it is enough to know the definitions and some basic propeties of the Lie algebra sl_{2+}^{ϵ} and its associated associative algebras CU, and QU. Hence we start this section by declaring these algebras by fiat and listing some of their properties, postponing some of their proofs to Section 3.3. In Section 3.2 we explain the motivation behind sl_{2+}^{ϵ} and find that it extends to arbitrary semi-simple Lie algebras.

In anticipation of Section **??**, in which we show that everything that matters around sl_{2+}^{ϵ} is **DPG**, we emphasize the first occurrence of every object in this section that is later shown to be **DPG** with a lollipop symbol. Within the context of the current section the lollipops are purely motivational.

3.1. Definitions and Basic Properties.

Our ground ring throughout this section is $\mathbb{Q}[\epsilon]$, the ring of polynomials with rational coefficients over a formal parameter ϵ. Quantum algebra people should note that ϵ is distinct from $\hbar$.

Definition 3.1. Let sl_{2+}^{ϵ} be the Lie algebra $L\langle y,b,a,x\rangle$ with generators $\{y,b,a,x\}$ and with commutation relations

$$[a,x]=x,\quad [b,y]=-\epsilon y,\quad [a,b]=0,\quad [a,y]=-y,\quad [b,x]=\epsilon x,\quad [x,y]=b+\epsilon a. \tag{10}$$

eq:slepsrelati

Remark 3.2. It is easy to verify that $t:=b-\epsilon a$ is central in sl_{2+}^{ϵ}, and that if ϵ is invertible[10] then sl_{2+}^{ϵ} splits as a direct sum: $sl_{2+}^{\epsilon}\cong sl_2\oplus\langle t\rangle$, explaining its name. (Though we will mostly care about the vicinity of $\epsilon=0$, and at $\epsilon=0$[11] our algebra is not a direct sum).

Definition 3.3. Let $CU:=\mathcal{U}(sl_{2+}^{\epsilon})$ be the universal enveloping algebra of sl_{2+}^{ϵ}. Namely, CU is the associative algebra $A\langle y,b,a,x\rangle$ generated by the same $\{y,b,a,x\}$, subject to the same relations as in (10). We denote the multiplication map of CU with ${}^cm\colon CU\otimes CU\to CU$. CU is a Hopf algebra in the standard way; namely, with its given associative algebra structure and with unit ${}^c\eta\colon\mathbb{Q}\to CU$, counit ${}^c\varepsilon\colon CU\to\mathbb{Q}$[12], antipode ${}^cS\colon CU\to CU$, and coproduct ${}^c\Delta\colon CU\to CU\otimes CU$ given as follows:

$$\begin{aligned}
{}^c\eta(\lambda) &= \lambda\cdot 1,\\
{}^c\varepsilon(1,y,b,a,x) &= (1,0,0,0,0),\\
{}^cS(y,b,a,x) &= (-y,-b,-a,-x),\\
{}^c\Delta(y,b,a,x) &= (y\otimes 1+1\otimes y,\, b\otimes 1+1\otimes b,\, a\otimes 1+1\otimes a,\, x\otimes 1+1\otimes x).
\end{aligned} \tag{11}$$

eq:CUDef

Convention 3.4. Throughout this paper we often put labels on tensor factors in a tensor product instead of ordering them; hence we often write $U^{\otimes A}$, where U is a vector space and A is a finite set, instead of $U^{\otimes n}$, where n is a natural number[13]. If U has a prescribed unit $1\in U$ and if $z\in U$ and $i\in A$, we write z_i for "z placed in tensor factor i (with 1 in all other tensor factors)". If $\psi\colon U^{\otimes A}\to U^{\otimes B}$ is a map, we often emphasize its domain and range by

[10] E.g., if the ring of scalars is extended to $\mathbb{Q}(\epsilon)$ via $sl_{2+}^{\epsilon}\mapsto\mathbb{Q}(\epsilon)\otimes_{\mathbb{Q}[\epsilon]}sl_{2+}^{\epsilon}$.

[11] Evaluation at $\epsilon=\epsilon_0\in\mathbb{Q}$ makes sense via $sl_{2+}^{\epsilon}\mapsto(\mathbb{Q}[\epsilon]/(\epsilon-\epsilon_0))\otimes_{\mathbb{Q}[\epsilon]}sl_{2+}^{\epsilon}$, a Lie algebra over $\mathbb{Q}$.

[12] We use `\epsilon` (ϵ) for a perturbation parameter and `\varepsilon` (ε) for counits. There's rarely a reason for confusion.

writing "ψ^A_B". Thus for example, using these conventions (eq:CUDef 11) becomes:

$$\begin{aligned}
&{}^c\eta_i\colon \mathbb{Q}\to CU^{\otimes\{i\}}, && {}^c\eta_i(\lambda)=\lambda\cdot 1_i,\\
&{}^c\varepsilon^i\colon CU^{\otimes\{i\}}\to\mathbb{Q}, && {}^c\varepsilon^i(1_i,y_i,b_i,a_i,x_i)=(1,0,0,0,0),\\
&{}^cS_i:={}^cS^i_i\colon CU^{\otimes\{i\}}\to CU^{\otimes\{i\}}, && {}^cS_i(y_i,b_i,a_i,x_i)=(-y_i,-b_i,-a_i,-x_i),\\
&{}^c\Delta^i_{jk}\colon CU^{\otimes\{i\}}\to CU^{\otimes\{j,k\}}, && {}^c\Delta^i_{jk}(y_i,b_i,a_i,x_i)=(y_j+y_k,b_j+b_k,a_j+a_k,x_j+x_k).
\end{aligned}\tag{12}$$

eq:CUDefId

def:QU **Definition 3.5.** Let QU, a "quantization" of CU, be the associative algebra $A\langle y,b,a,x\rangle[\![\hbar]\!]$ over the ring $\mathbb{Q}[\![\hbar]\!]$ modulo to the relations

$$[a,x]=x,\quad [b,y]=-\epsilon y,\quad [a,b]=0,\quad [a,y]=-y,\quad [b,x]=\epsilon x,\quad xy-qyx=\frac{1-AB}{\hbar},$$

where $q:=\mathbb{e}^{\hbar\epsilon}$, $A:=\mathbb{e}^{-\hbar\epsilon a}$, and $B:=\mathbb{e}^{-\hbar b}$. We denote the multiplication map of QU with ${}^qm\colon QU\otimes QU\to QU$. We also set

$$\begin{aligned}
{}^q\eta_i(\lambda)&=\lambda\cdot 1_i, \\
{}^q\varepsilon^i(1_i,y_i,b_i,a_i,x_i)&=(1,0,0,0,0), \\
{}^qS_i(y_i,b_i,a_i,x_i)&=(-B_i^{-1}y_i,-b_i,-a_i,-A_i^{-1}x_i), \\
{}^q\Delta^i_{jk}(y_i,b_i,a_i,x_i)&=(y_j+B_jy_k,b_j+b_k,a_j+a_k,x_j+A_jx_k).
\end{aligned}\tag{13}$$

eq:QUDefId

The following claim can be verified easily by explicit computations:

Claim 3.6. *With the above operations and relative to the $\hbar$-adic topology, QU is a complete topological*[14] *Hopf algebra over the ring $\mathbb{Q}[\epsilon][\![\hbar]\!]$.* □

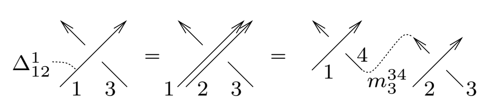
def:R **Definition 3.7.** Let R be the element of $QU\otimes QU$[15] given by the following formula:

$$R=\sum_{m,n\geqslant 0}\frac{y^nb^m\otimes(\hbar a)^m(\hbar x)^n}{m![n]_q!},\quad\text{alternatively}\quad R_{ij}=\sum_{m,n\geqslant 0}\frac{y_i^nb_i^m(\hbar a_j)^m(\hbar x_j)^n}{m![n]_q!}\in\mathbb{B}_i\otimes\mathbb{A}_j,$$

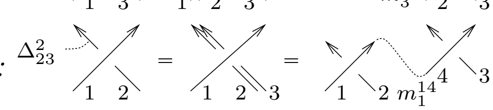
where $[n]_q!:=[1]_q[2]_q\cdots[n]_q$ and $[k]_q:=\frac{q^k-1}{q-1}=1+q+q^2+\ldots+q^{k-1}$ (recall that $q=\mathbb{e}^{\hbar\epsilon}$).

prop:R **Proposition 3.8** (proof in Section ssec:UProofs 3.3)**.** *R is an R-matrix. Namely, it has the following properties: (This algebra section can be self contained, yet when we can, we can't resist including knot-theoretic interpretations, prefixed with "KT". Pure algebraists can ignore.)*

$$R_{13}/\!\!/{}^q\Delta^1_{12}=(R_{14}R_{23})/\!\!/{}^qm^{34}_3$$

KT: 

$$R_{12}/\!\!/{}^q\Delta^2_{23}=(R_{12}R_{43})/\!\!/{}^qm^{14}_1$$

KT: 

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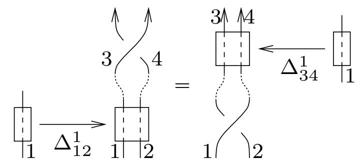
[13]These conventions only make sense in strict monoidal categories. They are consistent with the "identity" world view as oppossed to the "geography" view; see Talk:Toronto-1912 [BN].

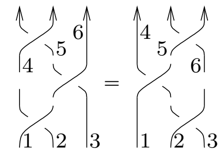
[14]Most people can safely ignore the "topological" language: it just means that everything can be a power series in $\hbar$, and only reasonable things are done to such series.

[15]Tensor products are completed relative to the $\hbar$-adic topology with no further mention.

$$({}^q\Delta^1_{12}R_{34})/\!\!/({}^qm_1^{13}\,{}^qm_2^{24}) = (R_{12}\ {}^q\Delta^1_{34})/\!\!/({}^qm_1^{14}\,{}^qm_2^{23}) \qquad KT:$$



$$(R_{12}R_{63}R_{45}/\!\!/({}^qm_1^{16}\,{}^qm_2^{24}\,{}^qm_3^{35}) = (R_{23}R_{14}R_{56})/\!\!/({}^qm_1^{15}\,{}^qm_2^{26}\,{}^qm_3^{34}) \qquad KT:$$



We have finished listing the atomic pieces we need for the purpose of knot theory. Yet these pieces in themselves are assembled from even lower level pieces — perhaps "quarks" — and we need to introduce those as they are necessary for both the proof of Proposition 3.8 and for the proofs in Section 4 that all the lollipopped items above are indeed in **DPG**. Here we go:

Definition 3.9. Let $\mathfrak{a}$ be the 2-dimensional Lie algebra $L\langle a,x\rangle/[a,x]=x)$ and let $\mathbb{A} := \mathcal{U}(\mathfrak{a})[\![\hbar]\!]$ be the $\hbar$-adic completed universal enveloping algebra of the two dimensional Lie algebra with generators a and x and with the same bracket as in Definition 3.5. We turn $\mathbb{A}$ into a complete topological Hopf algebra with the obvious definitions for ${}^am$, ${}^a\varepsilon$, and ${}^a\eta$ (all), and with the definitions for ${}^aS$ and ${}^a\Delta$ (both) induced from (13). Namely,

$$\begin{aligned} {}^aS_i(a_i,x_i) &= (-a_i, -A_i^{-1}x_i), \\ {}^a\Delta^i_{jk}(a_i,x_i) &= (a_j+a_k, x_j+A_jx_k). \end{aligned} \tag{14}$$

Let $\mathbb{A}'$ be the subalgebra of $\mathbb{A}$ generated by $\hbar a$ and by $\hbar x$[16]. It is easy to check that $\mathbb{A}'$ is a sub-Hopf-algebra of $\mathbb{A}$.

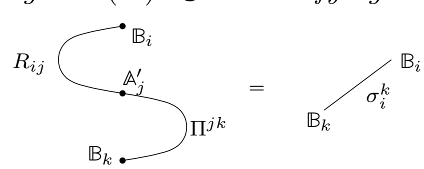
Definition 3.10. Similarly let $\mathbb{B} := \mathcal{U}(L\langle y,b\rangle/[b,y]=-\epsilon y)[\![\hbar]\!]$ be the $\hbar$-adic completed universal enveloping algebra of the two dimensional Lie algebra with generators y and b and with the same bracket as in Definition 3.5. We turn $\mathbb{B}$ into a complete topological Hopf algebra with the obvious definitions for ${}^bm$, ${}^b\varepsilon$, and ${}^b\eta$ (all), with ${}^bS$ taken to be the inverse of ${}^qS$ (but only on y and b) and with ${}^b\Delta$ taken to be the opposite of ${}^q\Delta$ (but only on y and b). Namely,

$$\begin{aligned} {}^bS_i(y_i,b_i) &= (-y_iB_i^{-1}, -b_i), \\ {}^b\Delta^i_{jk}(y_i,b_i) &= (B_ky_j+y_k, b_j+b_k). \end{aligned} \tag{15}$$

Clearly, $R \in \mathbb{B}\otimes\mathbb{A}'$. We claim that it has an inverse, a pairing $\Pi \in (\mathbb{A}')^*\otimes\mathbb{B}^*$:

Proposition 3.11. *There is a unique pairing* $\Pi \in (\mathbb{A}')^*\otimes\mathbb{B}^*$ *satisfying*

$$R_{ij}/\!\!/\Pi^{jk} = \sigma_i^k, \qquad FD:$$



[16]Elements of $\mathbb{A}$ are infinite series $\sum w_n\hbar^n$ where $w_n \in \mathcal{U}(\mathfrak{a})$. Elements of $\mathbb{A}'$ are such series in which each w_n is a (non-commutative) polynomial in a and x of degree at most n. So using language similar to the language of Section 2, $\mathbb{A}'$ is the "docile" subspace of $\mathbb{A}$.

where $\sigma_i^k\colon \mathbb{B}_k \to \mathbb{B}_i$ *is the identity map (more preciely, the factor renaming map) and where "FD" stands for "Flow Diagram(s)", a rather standard graphical language for representing compositions of tensors (e.g.* [ES, Lecture 12]*) which nevertheless seems not to have a standard name.*

EtingofSchiffman:QuantumGroups

defined on the generators by

$$\Pi\langle\hbar a, b\rangle = \Pi\langle\hbar x, y\rangle = 1, \qquad \Pi\langle\hbar a, y\rangle = \Pi\langle\hbar x, b\rangle = 0,$$

MORE.
MORE.

ec:UMotivation

3.2. Motivation for sl^ϵ_{2+}, CU, and QU. MORE.

ssec:UProofs

3.3. Proofs. MORE.

DRAFT! See http://drorbn.net/DPG/

4. Everything Around sl_{2+}^{ϵ} is DPG

sec:Everything

MORE.

5. Tangles and Knots and Algebraic Knot Theory

MORE.

6. Computational Appendices

We believe in implementing as much as possible. Actually, we hardly believe ourselves unless we implement.

All code in these appendices is written in *Mathematica* [Wo].

6.1. Computational Verification of Theorem 2.9, (i). We test that the composition law of **GDO** is indeed associative, by defining it general and verifying associativity on random (and hence likely generic) morphisms. First, we define the composition law of two morphisms. The program first determines E_i, F_i, and G_i from Q_i ($i = 1, 2$) by taking partial derivatives, and then outputs the scalar ω and quadratic Q, with equations (5) converted nearly literally into code:

```
𝕄_{A_→B_}[ω1_, Q1_] // 𝕄_{B_→C_}[ω2_, Q2_] := Module[{ζA, zC, E1, F1, G1, E2, F2, G2, I},
  ζA = Table[ζ_i, {i, A}]; zC = Table[z_i, {i, C}]; I = IdentityMatrix@Length@B;
  E1 = Table[∂_{ζ_i,z_j} Q1, {i, A}, {j, B}]; E2 = Table[∂_{ζ_i,z_j} Q2, {i, B}, {j, C}];
  F1 = Table[∂_{ζ_i,ζ_j} Q1, {i, A}, {j, A}]; F2 = Table[∂_{ζ_i,ζ_j} Q2, {i, B}, {j, B}];
  G1 = Table[∂_{z_i,z_j} Q1, {i, B}, {j, B}]; G2 = Table[∂_{z_i,z_j} Q2, {i, C}, {j, C}];
  Expand /@ 𝕄_{A→C}[ω1 ω2 Det[I - F2.G1]^(-1/2), ζA.E1.Inverse[I - F2.G1].E2.zC
     + 1/2 ζA.(F1 + E1.F2.Inverse[I - G1.F2].E1^T).ζA +
     1/2 zC.(G2 + E2^T.G1.Inverse[I - F2.G1].E2).zC] ]
```

Next we implement “random morphisms” (RM) by picking their quadratic parts to have small random integer coefficients. We also set M_1, M_2, and M_3 to be random morphisms in $\mathrm{mor}(\{1,2\} \to \{1,2,3\})$, $\mathrm{mor}(\{1,2,3\} \to \{1,2,3\})$, and $\mathrm{mor}(\{1,2,3\} \to \{1,2\})$, respectively:

```
RM_{A_→B_} := Module[{vs = Table[ζ_i, {i, A}] ⋃ Table[z_i, {i, B}]},
    𝕄_{A→B}[1, Sum[RandomInteger[{-3, 3}] vi vj, {vi, vs}, {vj, vs}]]];
{M1 = RM_{{1,2}→{1,2,3}}, M2 = RM_{{1,2,3}→{1,2,3}}, M3 = RM_{{1,2,3}→{1,2}}} // Column
```

$$\mathbb{M}_{\{1,2\}\to\{1,2,3\}}\big[1,\ 3z_1^2 + 4z_1z_2 - z_2^2 - 2z_2z_3 - z_1\zeta_1 - 3z_2\zeta_1 - 4z_3\zeta_1 + \zeta_1^2 + 3z_1\zeta_2 - 6z_2\zeta_2 + 4z_3\zeta_2 + 4\zeta_1\zeta_2 + \zeta_2^2\big]$$
$$\mathbb{M}_{\{1,2,3\}\to\{1,2,3\}}\big[1,\ 3z_1^2 + z_1z_2 + 2z_2^2 - 2z_1z_3 + 4z_2z_3 - 3z_3^2 - z_1\zeta_1 - z_2\zeta_1 + 6z_3\zeta_1 - \zeta_1^2 + 2z_1\zeta_2 + z_2\zeta_2 - z_3\zeta_2 - 2\zeta_1\zeta_2 - 3\zeta_2^2 - z_1\zeta_3 - z_2\zeta_3 + z_3\zeta_3 - 3\zeta_1\zeta_3 + 2\zeta_2\zeta_3 + 2\zeta_3^2\big]$$
$$\mathbb{M}_{\{1,2,3\}\to\{1,2\}}\big[1,\ -2z_1^2 + 2z_2^2 - 5z_1\zeta_1 - z_2\zeta_1 + 2\zeta_1^2 - z_1\zeta_2 - 6z_2\zeta_2 + \zeta_1\zeta_2 - 2\zeta_2^2 - 2z_2\zeta_3 - 2\zeta_1\zeta_3 + 3\zeta_2\zeta_3 - \zeta_3^2\big]$$

Just to get an appreciation of what compositions look like, we compute $(M_1/\!\!/M_2)/\!\!/M_3$:

```
(M1 // M2) // M3
```

$$\mathbb{M}_{\{1,2\}\to\{1,2\}}\Big[\frac{1}{2\sqrt{47913}},\ \frac{655148\,z_1^2}{15971} + \frac{9600305\,z_1z_2}{31942} + \frac{47930587\,z_2^2}{95826} - \frac{1241140\,z_1\zeta_1}{15971} - \frac{12434423\,z_2\zeta_1}{47913} + \frac{1792724\,\zeta_1^2}{47913} + \frac{2520132\,z_1\zeta_2}{15971} + \frac{16827871\,z_2\zeta_2}{31942} - \frac{2104097\,\zeta_1\zeta_2}{15971} + \frac{2273807\,\zeta_2^2}{15971}\Big]$$

Finally, we verify that composition is associative:

```
((M1 // M2) // M3) == (M1 // (M2 // M3))
```

```
True
```

The last `True` above is an in-practice proof of Theorem thm:GDO 2.9, (i).

References

k:Toronto-1912 [BN] D. Bar-Natan, *Geography vs. Identity,* talk given at the Canadian Mathematical Society winter 2019 meeting in Toronto, handout and video at `http://drorbn.net/to19`. See pp. 20.

hurston:Aarhus [BGRT] D. Bar-Natan, S. Garoufalidis, L. Rozansky, and D. P. Thurston, *The Århus integral of rational homology 3-spheres I–III,* Selecta Math., New Series **8** (2002) 315–339, arXiv:q-alg/9706004, **8** (2002) 341–371, arXiv:math.QA/9801049, **10** (2004) 305–324, arXiv:math.QA/9808013. See pp. 8.

:QuantumGroups [ES] P. Etingof and O. Schiffman, *Lectures on Quantum Groups,* International Press, Boston, 1998. See pp. 22.

actionOfGroups [IW] E. Inonu and E. P. Wigner, *On the Contraction of Groups and their Representations,* Proc. Nat. Acad. Sci. **39** (1953) 510–524. See pp. .

alVirtualKnots [Ka] L. H. Kauffman, *Rotational Virtual Knots and Quantum Link Invariants,* arXiv:1509.00578. See pp. .

ntumInvariants [Oh] T. Ohtsuki, *Quantum Invariants,* Series on Knots and Everything **29**, World Scientific 2002. See pp. .

Overbay:Thesis [Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov. See pp. .

Rozansky:Flat1 [Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061. See pp. .

Rozansky:Burau [Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005. See pp. .

Rozansky:U1RCC [Ro3] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv: math/0201139. See pp. .

am:Mathematica [Wo] *Wolfram Language & System Documentation Center,* ωεβ/Wolf. See pp. 27.

An ∞-dimensional DD Theorem, take 1.

Suppose $\mathbb{A} = \mathcal{U}(\mathfrak{a})$ and $\mathbb{B} = \mathcal{U}(\mathfrak{b})$ are Hopf algebras with their native products and with aS, $a\Delta$, bS, $b\Delta$, etc. Suppose $\langle\cdot,\cdot\rangle\colon \mathfrak{b}\otimes\mathfrak{a} \to \mathbb{Q}$ is a pairing such that:

- Compatibility of $[]_b$ and $a\Delta$ etc.
- Non degeneracy.

Then

(1) $\langle\cdot,\cdot\rangle$ extends uniquely to a non-degenerate pairing $\mathbb{B}\otimes\mathbb{A} \to \mathbb{Q}$ such that m and Δ are compatible.
(2) $\mathbb{D} = \mathbb{B}\otimes\mathbb{A}$ is a Hopf algebra with the DD formulas and $\mathbb{A} \to \mathbb{D}$ and $\mathbb{B} \to \mathbb{D}$ are Hopf morphisms.
(3) If b_i and a_i are dual bases of $\mathbb{B}$ and $\mathbb{A}$ relative to our pairing, then $R = \sum b_{\otimes} a_i$ satisfies the quasi-triangularity axioms.

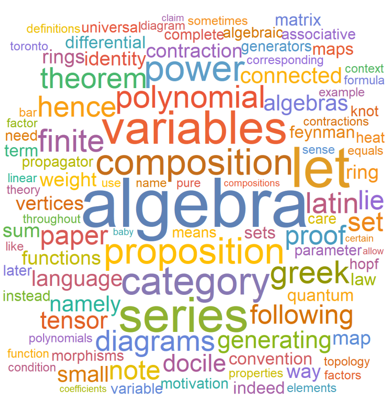
An ∞-dimensional DD Theorem, take 2.

Step 1. Everybody knows that if H is a finite dimensional Hopf algebra then $D = H^*\otimes$ is a quasi-triangular Hopf algebra, with R, m, Δ, S given by the following formulas…

Step 2. If A and B are Hopf algebras over a ring Ω with a Hopf pairing $P\colon A\otimes B \to \Omega$ and $R \in B\otimes A$ contracts P to the identity, the same conclusion holds for $D = B\otimes A$.

Step 3. Over $\Omega = \mathbb{Q}[\![\hbar]\!]$ let $\mathbb{A} = \mathcal{U}(\mathfrak{a})[\![\hbar]\!]$, $\mathbb{A}' = \langle\hbar\mathfrak{a}\rangle \subset \mathbb{A}$, and $\mathbb{B} = \mathcal{U}(\mathfrak{b})$, with $P\colon \mathbb{A}'\otimes\mathbb{B} \to \Omega$ be given by $\langle\hbar a, b\rangle = \langle\hbar x, y\rangle = 1$, and let $R = \sum_{m,n} y^n b^m \otimes (\hbar a)^m (\hbar x)^n / m![n]_q!$. Then we're in the situation of Step 2, with $A = \mathbb{A}'$ and $B = \mathbb{B}$, and hence $\mathbb{D}' = \mathbb{B}\otimes\mathbb{A}'$ is a quasi-triangular Hopf algebra.

Step 4. All the formulas extend Ω-linearly to $\mathbb{D} = \mathbb{B}\otimes\mathbb{A}$ and hence all identities hold there too.



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