July 24, 2020

$$0 \to \mathbb{Z} \xrightarrow[g]{\cdot 2} \mathbb{Z} \xrightarrow[f]{} \mathbb{Z}/2 \to 0$$

$$\operatorname{im} f = \mathbb{Z}/2 \quad \ker f = 2\mathbb{Z}$$

$$\operatorname{im} f \oplus \ker f = \mathbb{Z}/_2 \oplus 2\mathbb{Z} \neq \mathbb{Z}$$

$\mathbb{Z}/_2 \ni (1,0) = \tau$

$2\tau = 0$

$G = \pi_1 = F_n$

$\tilde{J} = G \cdot d_0$

$$\frac{\mathbb{Z}\langle \pi_1 \rangle}{\mathbb{Z} \cdot e} \overset{?}{\simeq} \mathbb{Z}\pi_1 \langle \alpha_1 \ldots \alpha_n \rangle \qquad \tilde{H} = H_1(\tilde{\Sigma}, \tilde{J}) \quad \boxed{\otimes \times}$$

$\alpha_j - 1 \longleftarrow \alpha_j \qquad [g\alpha_j - g] \leftarrow g\alpha_j$

$[g\alpha_j] \longleftarrow g \cdot \alpha_j \quad g \in F_n \quad (g \in \pi_1)$

?

$$0 \to H_1(\tilde{\Sigma}) \longrightarrow \tilde{H} \longrightarrow \frac{\mathbb{Z}[t^{\pm 1}]}{\mathbb{Z}} \to 0$$

1 $\quad\Longleftrightarrow\quad$ 2

$2 \Rightarrow 1$ ✓ $\qquad 1 \Rightarrow 2$?

$$\mathbb{Z}F_n\langle \alpha_1 \ldots \alpha_n \rangle \longrightarrow \frac{\mathbb{Z}F_n}{\cancel{\mathbb{Z}\langle e \rangle}} \quad \ker ?$$

$$g\alpha_i \longmapsto [g\alpha_i - g]$$ isomorphism?

$$A \longrightarrow B \xrightarrow{\varphi} \mathbb{Z} \longrightarrow 0$$

$$\Updownarrow ?$$

$$A \longrightarrow B/\mathbb{Z} \longrightarrow 0$$

$$A \longrightarrow \ker\varphi \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow A \longrightarrow B$$

$$\Updownarrow$$

$$0 \longrightarrow A/\mathbb{Z} \longrightarrow B$$

$$a : \mathbb{Z}F_n \longrightarrow \mathbb{Z}$$

$$g_i \longmapsto 1$$

$$\sum \alpha_i g_i \longmapsto \sum \alpha_i$$

$$K := \ker(a) = \underbrace{\left\{\sum \alpha_i g_i : \sum \alpha_i = 0\right\}}_{\text{"augmentation ideal"}}$$

$$\mathbb{Z}F_n\langle x_1, \ldots, x_n\rangle \longrightarrow K$$

$$g x_i \longmapsto g x_i - g$$

isomorphism?

$n \cdot \infty = \infty - 1$!!

Aug 5, 2020

$\tilde{d} = \pi^{-1}(\{d\})$

$\langle \tilde{d} \rangle$ → ~~The free group generated by $\tilde{d}$~~ ?

→ The $\mathbb{Z}$-module generated by $\tilde{d}$?

!

~~$g \in \langle \tilde{d} \rangle$~~

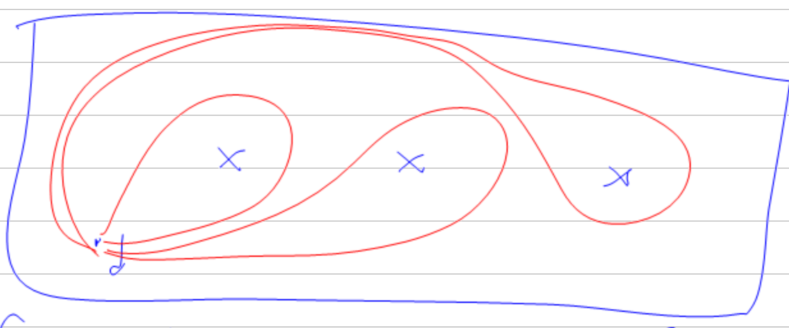
Fix $d_0 \in \tilde{d}$

$g \in \tilde{d}$ $\quad$ ~~$g \mapsto$~~ $\quad$ $g - d_0 \in \langle \tilde{d} \rangle$

$\ker \varphi = \langle g - d_0 : g \in \tilde{d} \rangle$

$\tilde{d} \longleftrightarrow \mathbb{Z}$

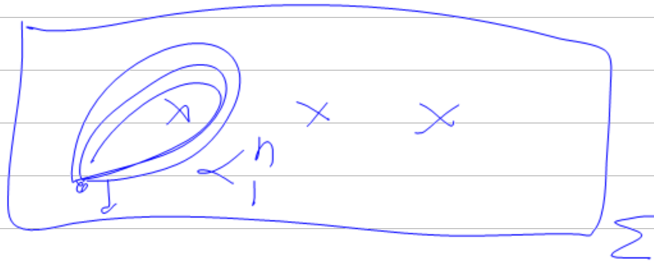
$d_0 \longleftrightarrow 0$



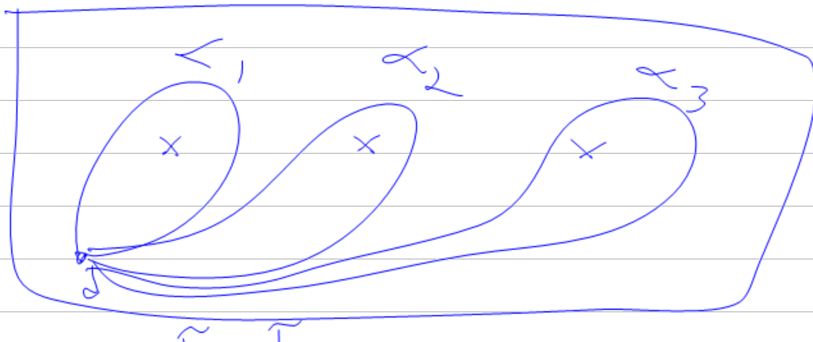
$\{\dots, d_{-1}, d_0, d_1, d_2, \dots\} = \tilde{d}$

$d_1 - d_0 = \partial \gamma \qquad \gamma : [0,1] \to \tilde{\Sigma}$

$\boxed{\tilde{\alpha}^n_i}$ ✓ $\qquad \tilde{\alpha}^n_i$ ✗ $\qquad$ ~~$\underset{d_0 \to d_1}{\tilde{\alpha}_i}$ $\underset{d_0 \to d_1}{\tilde{\alpha}_i}$~~



Aug 11, 2020.



$$h'([g - d_0]) := [\underbrace{\alpha_1^m}_{\gamma}]$$

$$g = d_0 * m$$

$$h'\left(\sum c_i [g_i - d_0]\right) = \sum c_i h'([g_i - c_0])$$

Is this a $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$-module homomorphism?

$$\boxed{h'(t^k[g_i - d_0]) \overset{?}{=} t^k h'([g_i - d_0])}$$

Ex. First do this for $\mathbb{R} \xrightarrow{p} S^1$

$$h'([g + g' - 2d_0]) = h'([g - d_0] + [g' - d_0])$$

$$h'([g - g']) = ?$$

Aug 16, 2020 Comment:

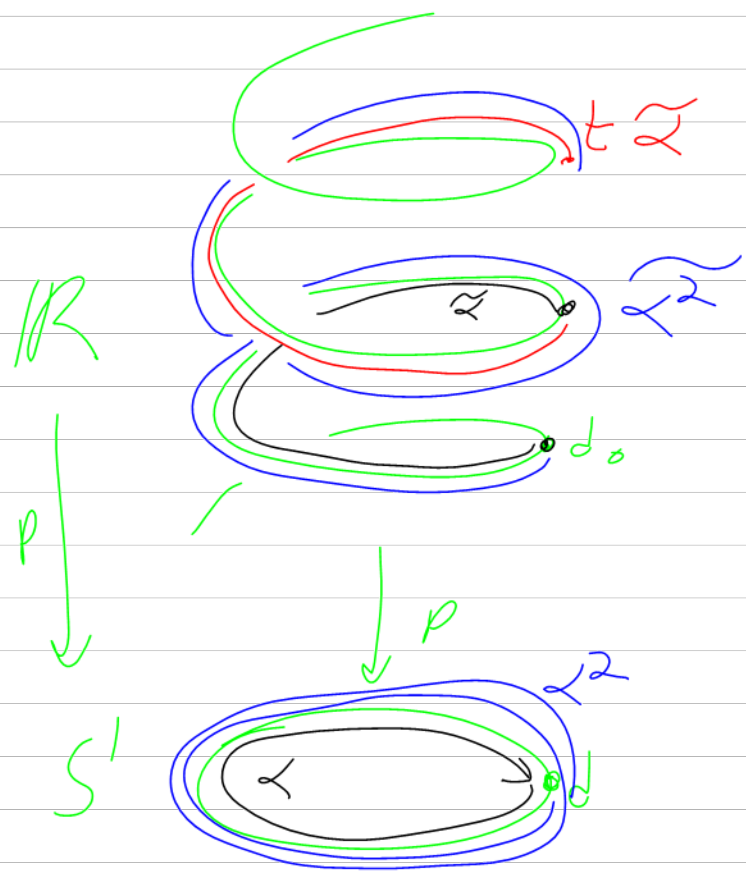
$$\{\underbrace{\text{Non-empty words}}_{\infty - 1}\} = \{\underbrace{\text{letters}}_{26}\} \times \{\underbrace{\text{all words}}_{\infty}\}$$

$$\infty - 1 = 26 \times \infty$$

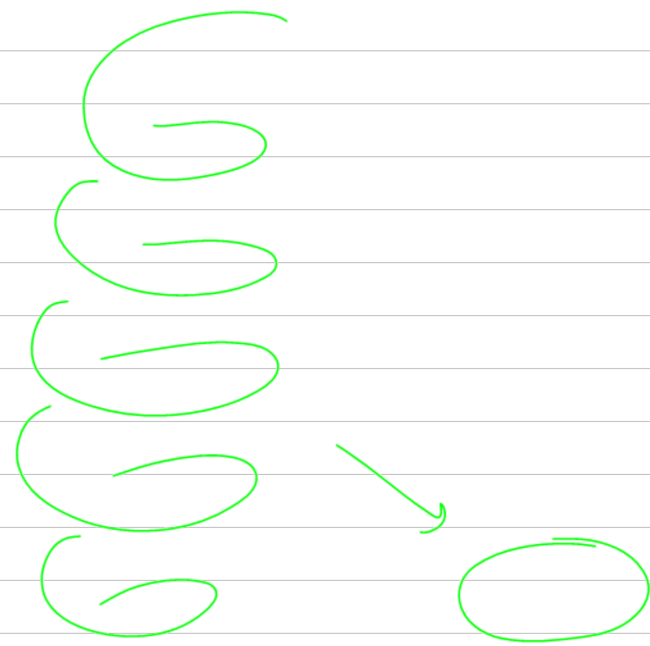
Aug 19, 2020

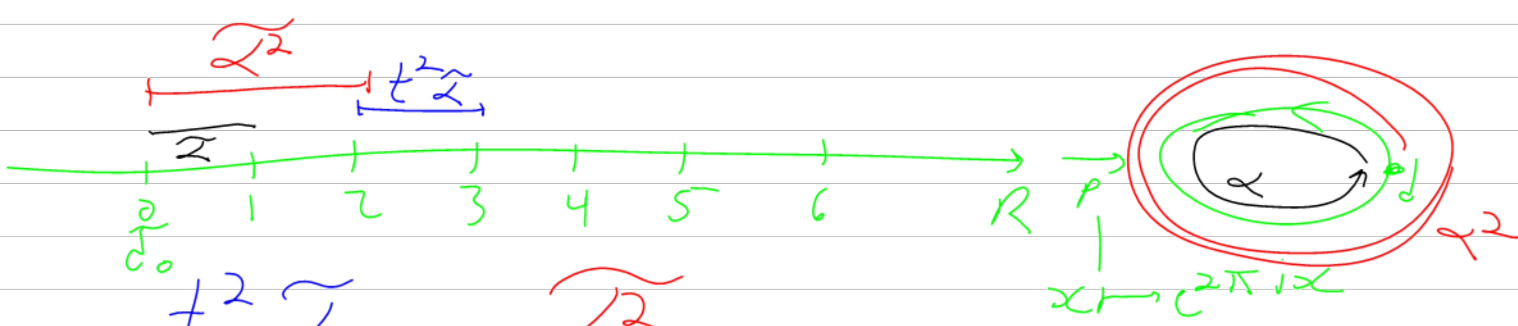
$\widetilde{\alpha_k^i}$ makes sense.

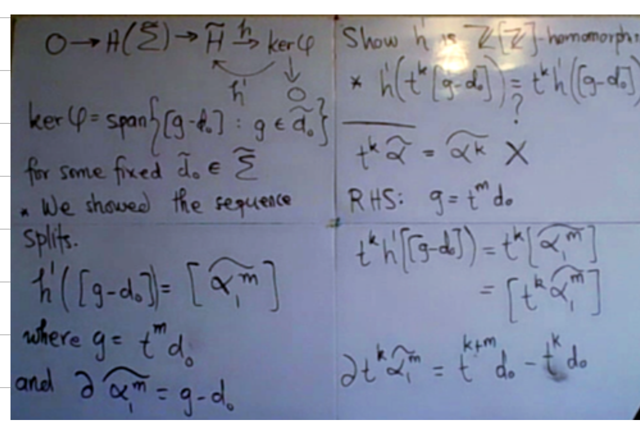
$\widetilde{\alpha_k}^i$ makes no sense b/c $\widetilde{\alpha}_k$ isn't closed.



$$t^2\widetilde{\alpha} \neq \widetilde{\alpha^2}$$







$$0 \to H(\widetilde{\Sigma}) \to \widetilde{H} \xrightarrow{h} \ker\varphi \to 0$$

h'

$\ker\varphi = \mathrm{span}\{[g-d_0] : g \in \widetilde{d}_0\}$

for some fixed $\tilde{d}_0 \in \widetilde{\Sigma}$

* We showed the sequence splits.

$h'([g-d_0]) = [\widetilde{\alpha_1^m}]$

where $g = t^m d_0$

and $\partial \widetilde{\alpha_1^m} = g - d_0$

Show h' is $\mathbb{Z}[\mathbb{Z}]$-homomorphism

* $h'(t^k[g-d_0]) \overset{?}{=} t^k h'([g-d_0])$

$\overline{t^k\widetilde{\alpha} = \widetilde{\alpha^k}}$ X

RHS: $g = t^m d_0$

$t^k h'([g-d_0]) = t^k[\widetilde{\alpha_1^m}]$

$= [t^k\widetilde{\alpha_1^m}]$

$\partial t^k \widetilde{\alpha_1^m} = t^{k+m} d_0 - t^k d_0$

$$x_{i_1}^{\pm 1} x_{i_2}^{\pm 1} \cdots x_{i_k}^{\pm 1} - 1 \quad \leftarrow \text{For simplicity: all powers are } +1$$

What if not?

$$= x_{i_1} x_{i_2} \cdots x_{i_k} - 1 = \sum \alpha_a (g_a x_{i_a} - g_a)$$

$$= \sum \alpha_a g_a (x_{i_a} - 1) \ ?$$

$$xy - 1 = (\ldots)(x-1) + (\ldots)(y-1)$$

$$\underline{xy - x} + \underline{x - 1} = x(y-1) + (x-1)$$

Sep 11, 2020 $\quad w \in FG(x_1 \ldots x_n)$

$$w - 1 = \sum \frac{\partial w}{\partial x_i}(x_i - 1) \quad ? \ ?$$

$w = 1 \quad x_1$

$$x_1 - 1 = 1 \cdot (x_1 - 1) + 0 \ldots \checkmark$$

$$x_i^{-1} - 1 = -x_i^{-1}(x_i - 1) + \ldots$$

$$0 = \frac{\partial}{\partial x_i}(x_i x_i^{-1}) = 1 + x_i \frac{\partial x_i^{-1}}{\partial x_i}$$

$$\frac{\partial}{\partial x_i}(x_i^{-1}) = -x_i^{-1}$$

$$W_k = \underline{W_{k-1}} x_{j_k}^{\epsilon_k}$$

Assume

$$W_{k-1} - 1 = \sum_{i=1}^{n} \frac{\partial W_{k-1}}{\partial x_i}(x_i - 1)$$

$$W_k - 1 = W_{k-1} x_{i_k}^{\epsilon_k}$$

$$= (W_{k-1} - 1) x_{i_k}^{\epsilon_k} + (x_{i_k}^{\epsilon_k} - 1)$$

$$= \left(\sum_{i=1}^{n} \frac{\partial W_{k-1}}{\partial x_i} (x_i - 1) \right) x_{i_k}^{\epsilon_k} + (x_{i_k}^{\epsilon_k} - 1)$$

$W_{k-1} x_{i_k}^{\epsilon_k}$

$$\overset{?}{=} \sum_{i=1}^{k} \frac{\partial W_k}{\partial x_i} (x_i - 1)$$

$$F \longrightarrow \tilde{\Sigma} \longrightarrow \Sigma$$

$$F \longrightarrow E \longrightarrow B$$

$$H_n(F) \longrightarrow H_n(\tilde{\Sigma}) \longrightarrow H_n(\Sigma) \longrightarrow H_{n-1}(F) \longrightarrow \cdots$$

"Long exact seq for a cover"

"Long exact seq of a fibration"

$$H_1(\tilde{\Sigma}, F) \qquad H_1(\tilde{\Sigma}) \quad H_0(F)$$

$$\forall w \qquad w-1=\sum_{i=1}^{n}\frac{\partial w}{\partial x_i}(x_i-1)$$

$$x_k^{\epsilon}w'=w \qquad w'-1=\sum_{i=1}^{n}\frac{\partial w'}{\partial x_i}(x_i-1)$$

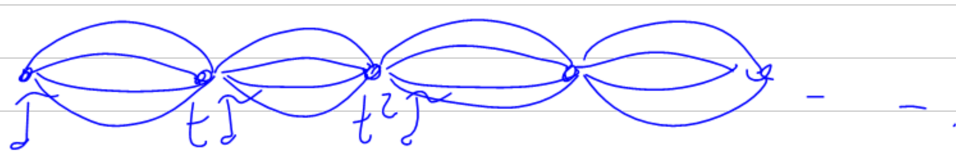
$$w-1=x_k^{\epsilon}w'-1=x_k^{\epsilon}(w'-1)+(x_k^{\epsilon}-1)$$

$$=x_k^{\epsilon}\sum_{i=1}^{n}\frac{\partial w'}{\partial x_i}(x_i-1)+(x_k^{\epsilon}-1)=\#_1$$

$$w=x_k^{\epsilon}w' \qquad \frac{\partial w}{\partial x_i}=\begin{cases} x_k^{\epsilon}\dfrac{\partial w'}{\partial x_i} & i\neq k\\ \dfrac{\partial x_k^{\epsilon}}{\partial x_k}+x_k^{\epsilon}\dfrac{\partial w'}{\partial x_k} & i=k\end{cases}$$

$$\sum_{i=1}^{n}\frac{\partial w}{\partial x_i}(x_i-1)=x_k^{\epsilon}\sum\frac{\partial w'}{\partial x_i}(x_i-1)+\frac{\partial x_k^{\epsilon}}{\partial x_k}(x_k-1)=\#_2$$

$n=4$



Back to Kassel-Turaev!

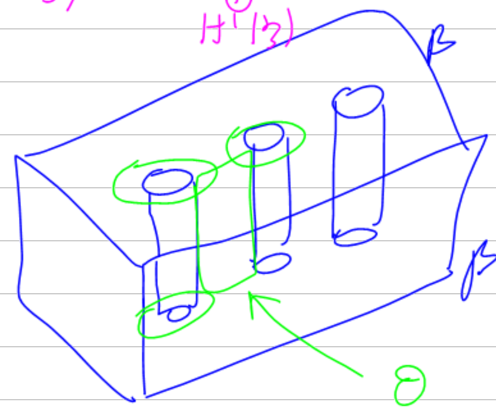
$$(d\alpha)(\gamma) \qquad \beta(\gamma)$$

$$\shortparallel \qquad\qquad \shortparallel$$

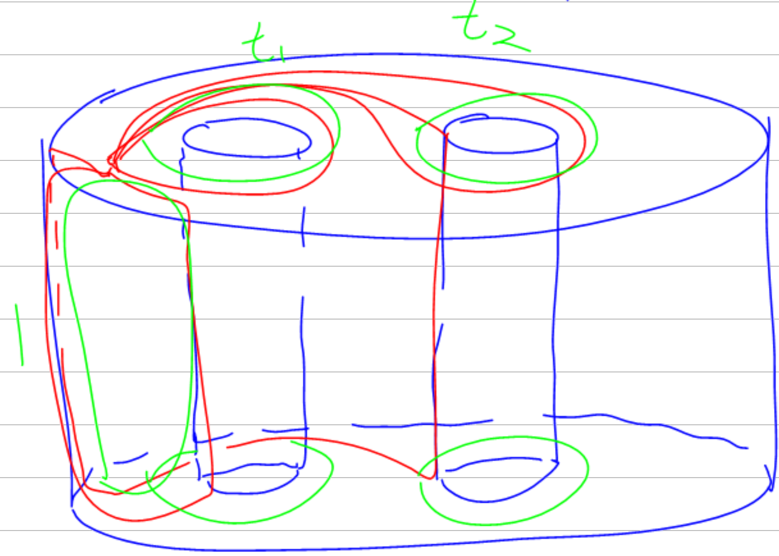
$$1-t \qquad\qquad 1$$

$$d\alpha=(1-t)\beta$$

$$H^1(A\cup B) \to H^1(A)\oplus H^1(B) \to H^1(A\cap B)$$



$$H^1(\partial X) = \left\langle \begin{matrix} \beta_1 \dots \beta_n \\ \beta_1' \dots \beta_n' \end{matrix} \right\rangle \Big/ \begin{matrix} \sum (t_i-1)\beta_i = 0 \\ \sum (t_i-1)\beta_i' = 0 \end{matrix}$$



$$\pi_1(\partial X) \to \pi_1(X) \xrightarrow{\phi} \mathbb{Q}(t_1, t_2)$$

$$U \cup V = \partial X,\quad U,\quad V,\quad U\cap V = \cup S^1$$

$$F : \pi_1(Y) \to G \xrightarrow{\varphi} K$$

$$C_n(Y, F) = C_n(\tilde{Y}) \otimes_{\pi_1(Y)} K$$
$$\cong$$

$$C_n(\hat{Y})\otimes_G \mathbb{R}$$

$$G \to \hat{Y} = (\tilde{Y}\times G)/\pi_1 \to Y$$

If $n=2$

$$\underline{2(n-1)} = \underline{n} = 2$$

$n=3$

$$\begin{array}{cc} \downarrow & \downarrow \\ 4 & 3 \end{array}$$

$A_1,\ A_2,\ A_3,$

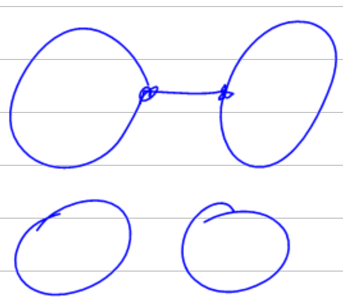
$+(t_2-1)B_1 - (t_1-1)B_2, \quad (t_3-1)$

$-(t_3-1)B_1 - (t_1-1)B_3, \quad (t_2-1)$

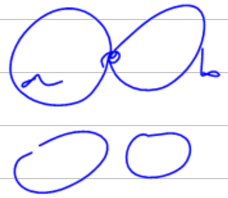
$+(t_3-1)B_2 - (t_2-1)B_3 \quad (t_1-1)$

$$0 \to \underline{H^1(\partial X)} \xrightarrow{i} \begin{array}{c} \underline{H^1(X_0)} \\ \oplus \\ \underline{H^1(X_1)} \end{array} \longrightarrow H^1(\cancel{S^1})$$

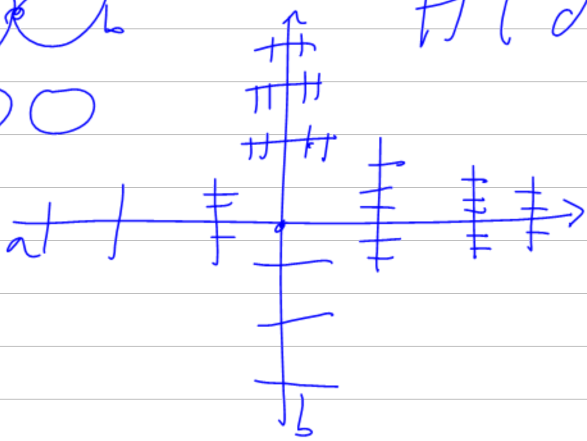
$n=2$



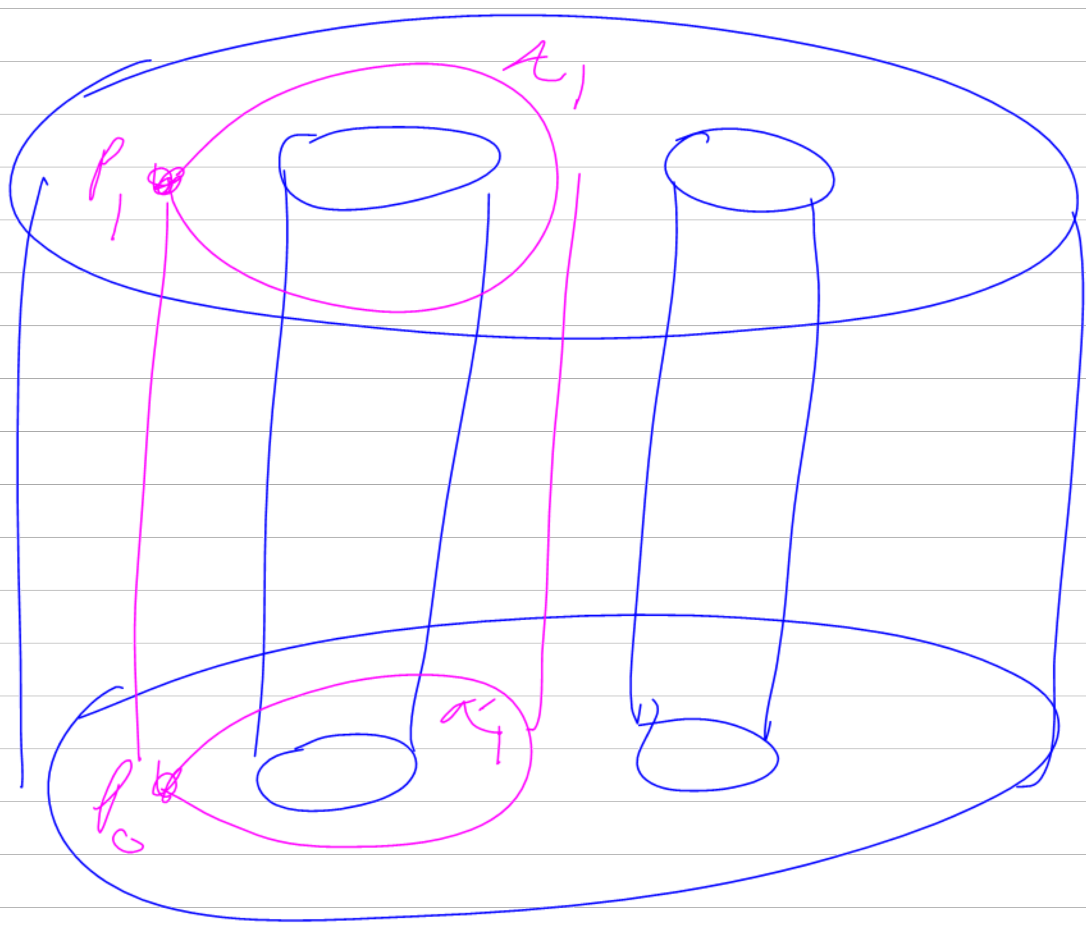
X_0

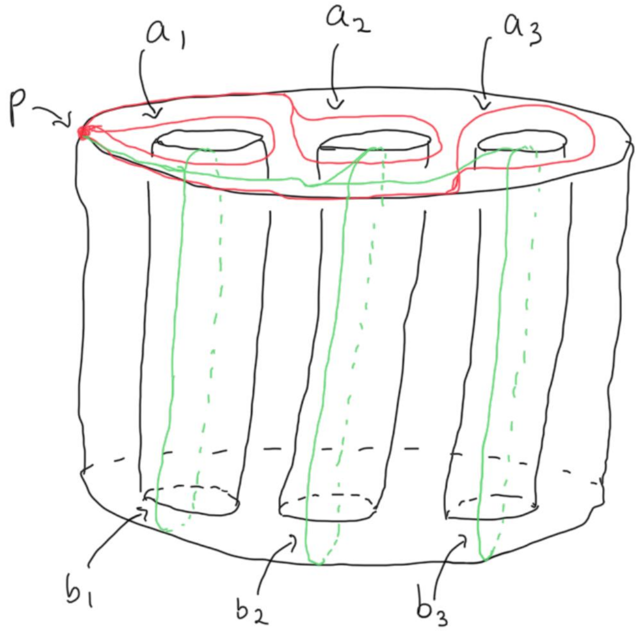


$H^1(\partial X)$



$A_i \neq A_i'$





Cell structure

0-cell : p

1-cells: a_i, b_i, $i = 1, 2, 3$

2-cell : e

$$\partial e = a_1 b_1 a_1^{-1} b_1^{-1}$$

$$\left[\begin{array}{l} C_0 = \langle \tilde{p} \rangle, \quad C_1 = \langle \tilde{a}_i, \tilde{b}_i \rangle_{i=1,2,3}, \quad C_2 = \langle \tilde{e} \rangle \\ 0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \end{array}\right]$$

$$\left[\begin{array}{l} C^0 = \langle P \rangle, \quad C^1 = \langle A_i, B_i \rangle_{i=1,2,3}, \quad C^2 = \langle E \rangle \\ 0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} 0 \end{array}\right]$$

We compute $\mathcal{H}^1(\partial X, F)$, $F = \mathbb{Q}(t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1})$

The twisting

$$\phi : \pi_1(X, p) \longrightarrow F$$

$a_i \longmapsto t_i$ (taking meridians to t_i) $i = 1, 2, 3$

$b_i \longmapsto 1$, for $i = 1, 2, 3$.

$$\mathcal{H}^1(\partial X, F) = \ker d^1 / \operatorname{im} d^0$$

① $\ker d^1$

let $S \in C^1$. Then $d^1(S) \in C^2$, so

$$d^1(S)(\tilde{e}) = S(\partial_2(\tilde{e})).$$

What is $\partial_2(\tilde{e})$ (Find $\partial_2(\tilde{e})$ using ∂e)

$$\partial e = \underbrace{a_1 b_1 a_1^{-1} b_1^{-1}}_{1} \underbrace{a_2 b_2 a_2^{-1} b_2^{-1}}_{2} \underbrace{a_3 b_3 a_3^{-1} b_3^{-1}}_{3}.$$

Boundary of $\tilde{e}$:

$$a_1 + b_1\phi(a_1) - a_1\phi(a_1b_1a_1^{-1}) - b_1\phi(a_1b_1a_1^{-1}b_1^{-1}) \quad (1)$$

$$+ a_2\phi(a_1b_1a_1^{-1}b_1^{-1}) + b_2\phi(a_1b_1a_1^{-1}b_1^{-1}a_2) - a_2\phi(a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1})$$

$$- b_2\phi(a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}) \quad (2)$$

$$+ a_3\phi(a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}) + b_3\phi(a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}a_3)$$

$$- a_3\phi(a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}a_3b_3a_3^{-1}) - b_3\phi(a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}a_3b_3a_3^{-1}b_3^{-1}) \quad (3)$$

$$= \cancel{a_1} + t_1b_1 - \cancel{a_1} - b_1$$
$$+ \cancel{a_2} + t_2b_2 - \cancel{a_2} - b_2$$
$$+ \cancel{a_3} + t_3b_3 - \cancel{a_3} - b_3$$

$$\phi : \begin{cases} a_i \longmapsto t_i \\ b_i \longmapsto 1 \end{cases}$$

$$= (t_1-1)b_1 + (t_2-1)b_2 + (t_3-1)b_3.$$ Thus,

$$\partial_2(\tilde{e}) = (t_1-1)\tilde{b}_1 + (t_2-1)\tilde{b}_2 + (t_3-1)\tilde{b}_3$$

$d^1(S)(\tilde{e}) = S(\partial_2(\tilde{e})) = 0$ for what $S \in C^1$?

$S(\partial_2(\tilde{e})) = 0$ if

$$S \in \operatorname{Span}\left\{ \begin{array}{l} A_1,\ A_2,\ A_3, \\ (t_2-1)B_1 - (t_1-1)B_2, \\ (t_3-1)B_1 - (t_1-1)B_3, \\ \cancel{(t_3-1)B_2 - (t_2-1)B_3} \end{array} \right\} = \ker d^1$$

Image of d^0

$d^0(\ell) \in C^1$.

$d^0(\ell)(a_i) = \ell(\partial_1(\tilde{a}_i)) = \ell((t_i - 1)\tilde{p}) = t_i - 1, \quad i = 1, 2, 3$

$d^0(\ell)(b_i) = \ell(\partial_1(\tilde{b}_i)) = \ell(0) = 0$

So $d^0(P) = \sum_{i=1}^{3} (t_i - 1) A_i$

$\operatorname{Im} d^0 = \left\langle \sum_{i=1}^{3} (t_i - 1) A_i \right\rangle$

$\ker d^1 = \{ A_1, A_2, \cancel{A_3}, (t_2 - 1) B_1 - (t_1 - 1) B_2, (t_3 - 1) B_1 - (t_1 - 1) B_3, \cancel{(t_3 - 1) B_2 - (t_2 - 1) B_3} \}$

$\operatorname{Im} d^0 = \left\langle \sum_{i=1}^{3} (t_i - 1) A_i \right\rangle$

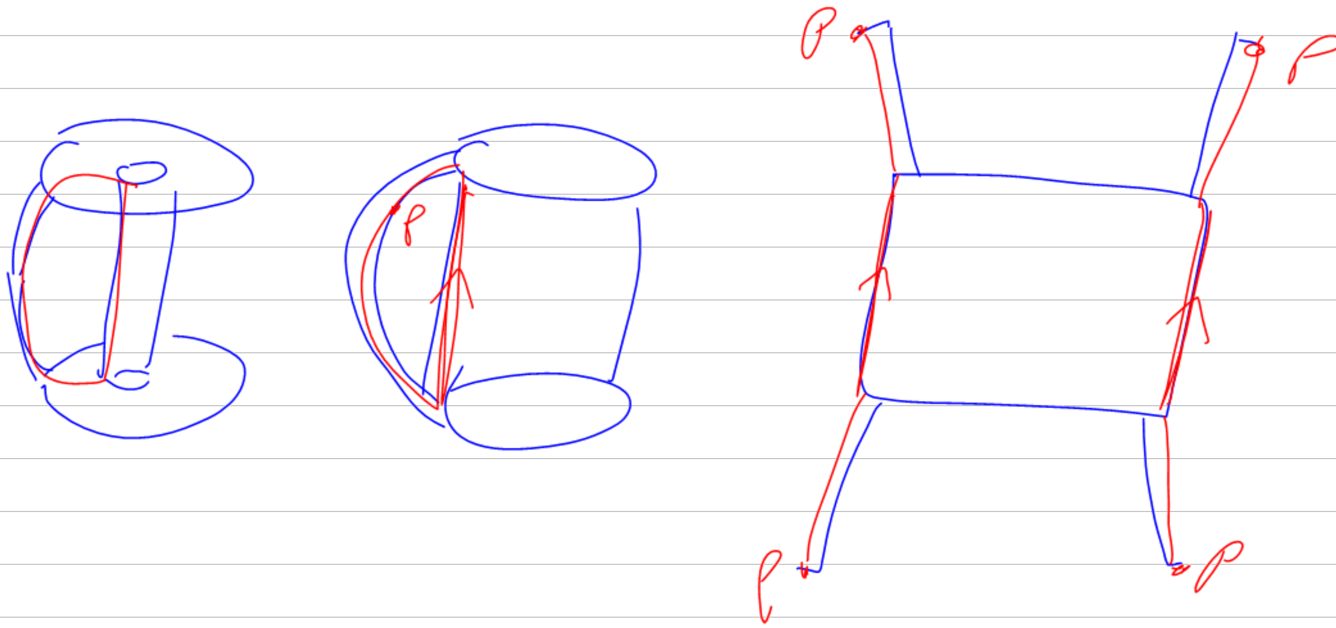
$H^1(X_{\partial, 1})$

This implies

$$\mathcal{H}^1(\partial X, F) \cong F^5$$

My computation still shows that the B_i's are involved for 3 holes and instead of dimension 4, I get 5.

I am not sure if I am doing something wrong.


p
p
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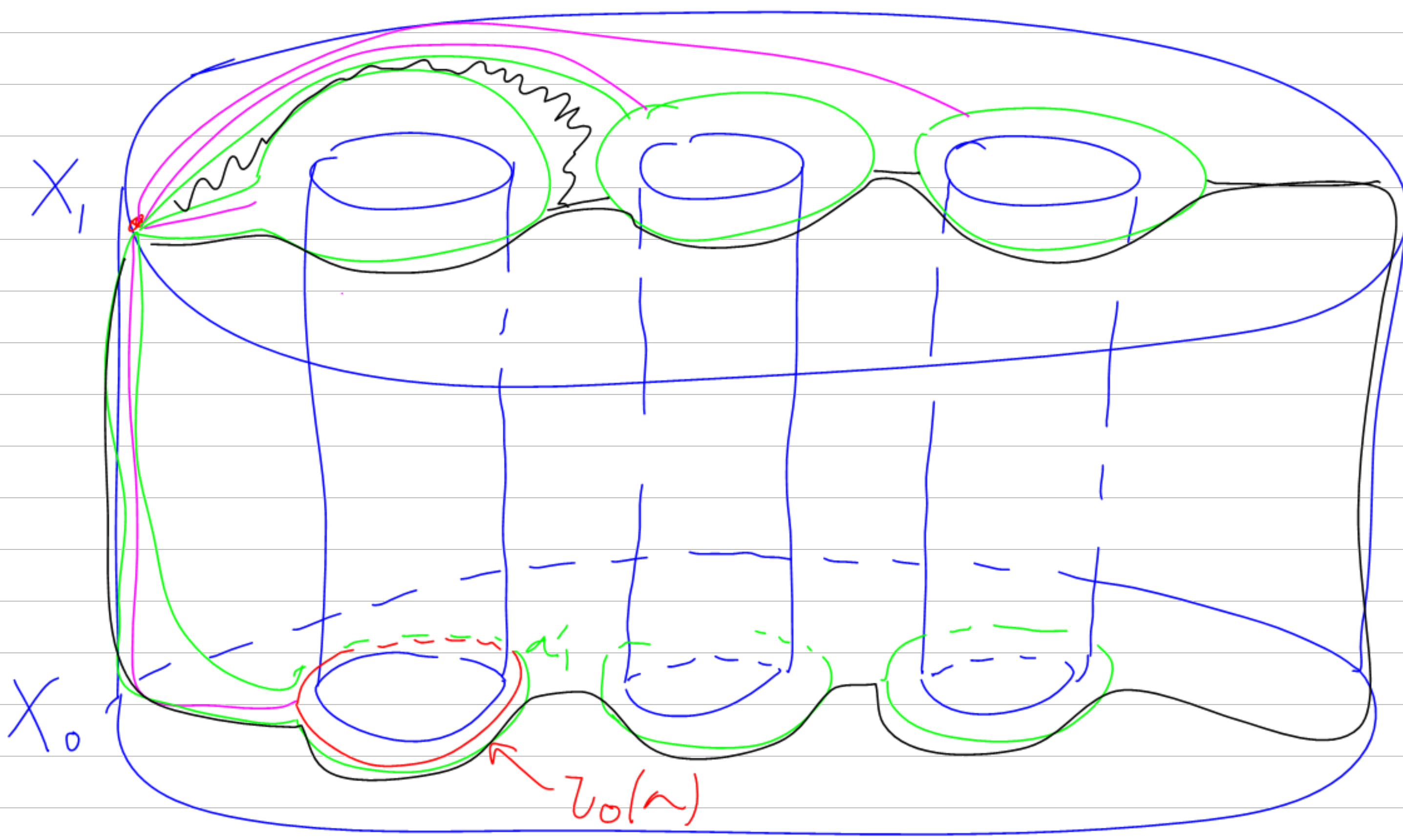
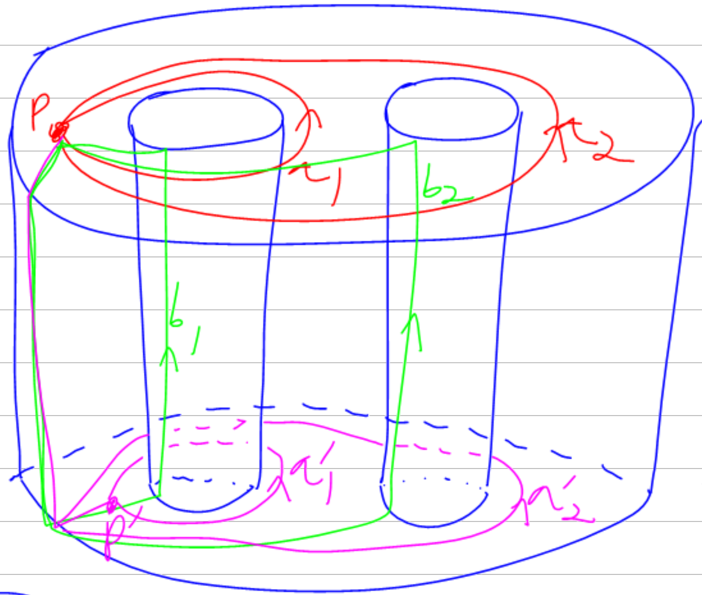

X_1
X_0
α_1
τ_0(α)

$$\partial \tilde{a}_i = (t_i - 1)\tilde{p}$$

X_1



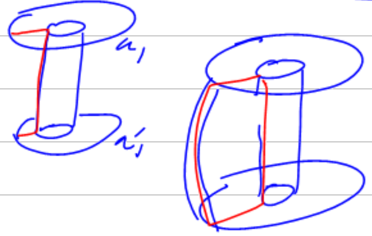
X_0

$$c = [(t_2-1)\tilde{a}_1 - (t_1-1)\tilde{a}_2]$$

$$c' = [(t_2-1)\tilde{a}'_1 - (t_1-1)\tilde{a}'_2]$$

$$\partial D \cong a_1^{-1} b_1^{-1} a'_1 b_1 \sim 1$$

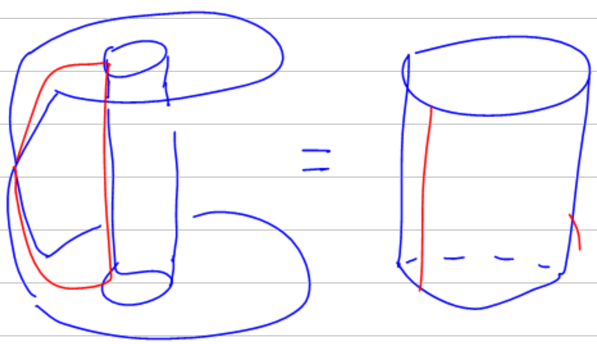
$$\partial \tilde{D} \cong -[\tilde{a}_1] + [\tilde{a}'_1] - t_1 \tilde{b}_1 + \tilde{b}_1$$



$$c - c' = (t_2-1)(t_1-1)(b_1 - b_2)$$

In $\tilde{C}_1$: $a'_1 = a_1 + (t_1-1)b_1$

$A_1(a'_1) = 1 + 0 = 1$



$$\begin{array}{l} H_1(X_0)\langle a_{0i}\rangle \longrightarrow a'_i = b_i^{-1} a_i b_i \\ \oplus \qquad\qquad \xrightarrow{\ \cong\ } H_1(\partial X) \\ H_1(X_1)\langle a_{1i}\rangle \longrightarrow a_i \end{array}$$

$$V = \langle x, y, z \rangle / x + y + z = 0$$

$$V^* = \langle X, Y, Z \rangle / X + Y + Z = 0$$

$$X(x) = 1 \quad X(y) = 0 \quad X(z) = 0$$

$$0 = X(x + y + z) = 1$$

$\langle a \ \ a' \ \ b \rangle / a - a' = (t-1)b$

$X \quad A \quad A' \quad B$

A

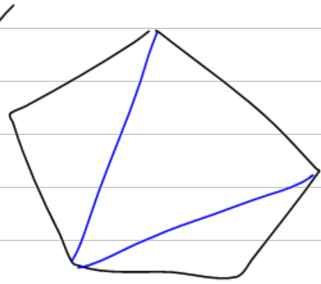
$V = \langle x, y, z \rangle / x + y + z = 0$

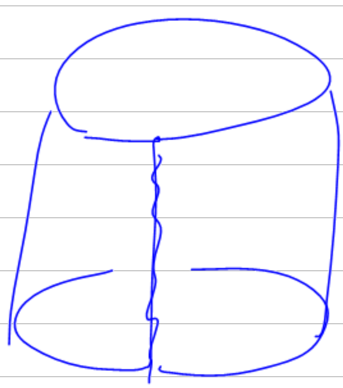
$\beta_1 = (x, y) \qquad \beta_1^* = (\varphi_1, \psi_1) \qquad \varphi_1(x) = 1, \ \varphi_1(y) = 0, \ \psi_1(x) = 0, \ \psi_1(y) = 1$

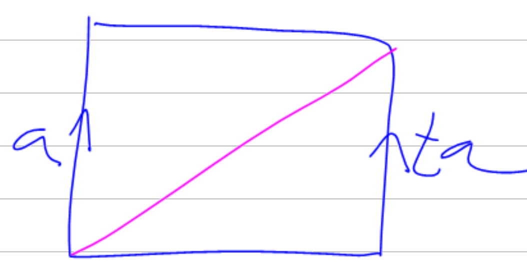
$=$ $\qquad\qquad \times$

$\beta_2 = (x, z) \qquad \beta_2^* = (\varphi_2, \psi_2) \qquad \varphi_2(x) = 1, \ \varphi_2(z) = 0, \ \psi_2(x) = 0, \ \psi_2(z) = 1$

$\varphi_1(y) = 0 \qquad \varphi_2(y) = \varphi_2(-x-z) = -1$

$(\alpha \cup \beta)\Big($  $\Big)$





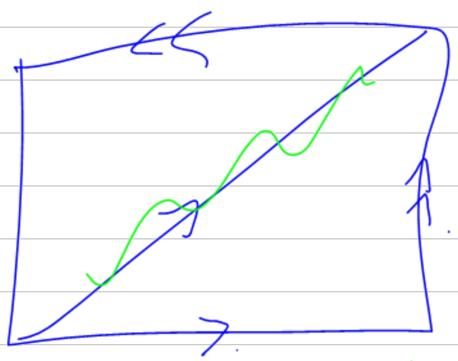
$$C^*(X;F) = \mathrm{Hom}_\pi(C_*(\tilde{X}) \to F)$$

$\varphi \in C^k(X;F) \quad \psi \in C^\ell(X;F)$

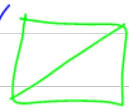
$\sigma \in C_k(\tilde{X}) \quad \varphi \cup \psi \in H^*(X)$

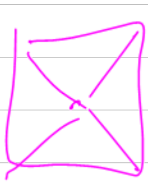
$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[0..k]})\,\psi(\sigma|_{[k,..k+\ell]})$$

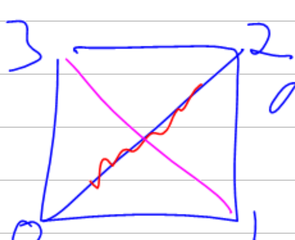
$g\sigma \quad g\cdot \quad \sigma \quad g\cdot \quad \sigma$



$\delta\phi = 0$

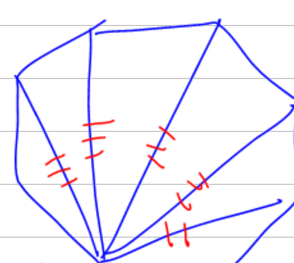
$\delta\psi = 0 \quad \phi \cup \psi($  $)$





$a_1 \vee b_1$

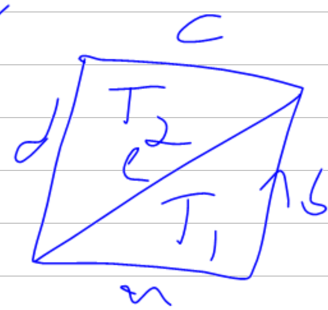
$H^1 = \langle a_1, b_1 \rangle$

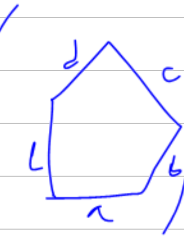


$H^1 = \langle a_1, b_1, a_2, b_2 \rangle$

$$(a_1 \smile b_1)(0123) = (a_1 \smile b_1)(012) + (a_1 \smile b_1)(023)$$

$$= -\ a_1(01)\ \cancel{a_1(02)}\ \ldots$$
$$b_1(23)\ \ldots$$

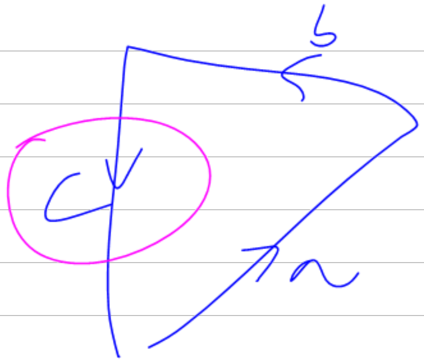
$(\alpha \smile \beta)$ () $= \ldots$

$(\alpha \smile \beta)$ () $= ?$

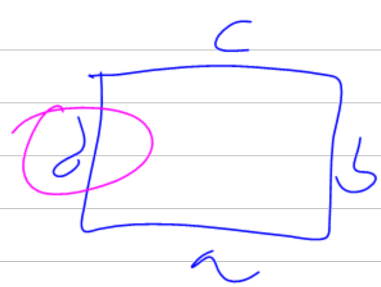
$$d\alpha = 0$$

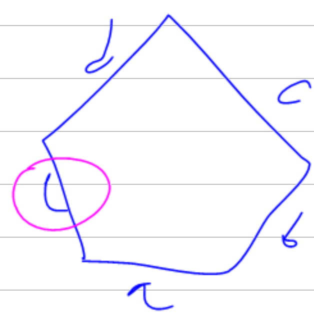
$$\alpha(\partial T_1) = 0$$

$$\alpha(l) - \alpha(a) - \alpha(b) = 0$$



$$\phi(c) = \phi(a) + \phi(b)$$
$$\phi(a)\alpha(b)$$





$$\sum_{1 \le i < j \le n-1} \alpha(a_i)\beta(a_j)$$

$$\underline{\underline{A_i}} \overset{\varphi}{\longmapsto} (A_i', \underline{\underline{A_i}})$$

$$\underset{\langle v_i \rangle}{V} \oplus \underset{\langle w_j \rangle}{W} = \left\langle \begin{matrix} (v_i, 0) \\ (0, w_j) \end{matrix} \right\rangle$$

$$(v_1, w_1) + (v_2, w_2) = (v_1, w_2) + (v_2, w_1)$$

$$(v_1 + v_2, w_1 + w_2) =$$