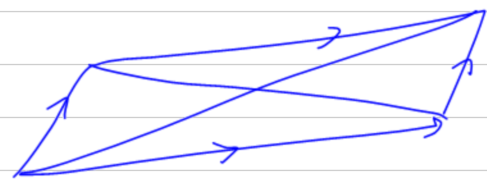
Def $\langle x, y\rangle = \frac{1}{4}\left(|x+y|^2 - |x-y|^2\right)$

$$\langle x+y, z\rangle \overset{?}{=} \langle x, z\rangle + \langle y, z\rangle$$

$$|x+y|^2 + |x-y|^2 = 2\left(|x|^2 + |y|^2\right)$$

not inside

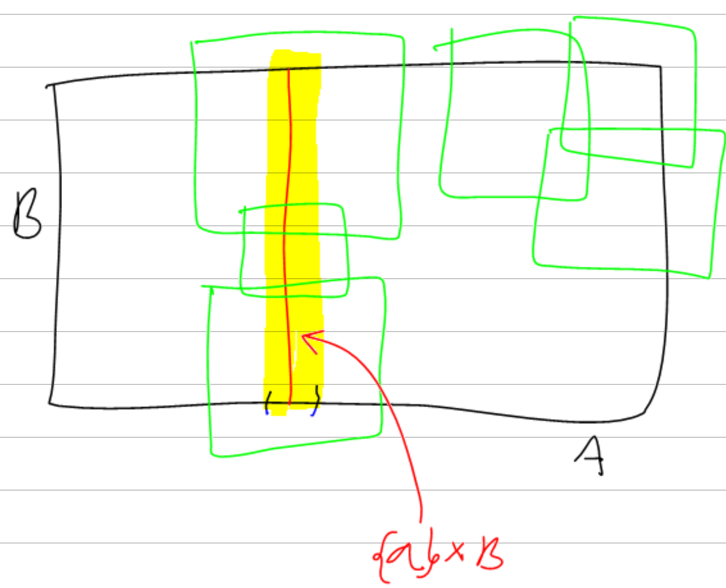


$a+b = x+y+z$ $\qquad a = x + \frac{1}{2}y$

$a-b = x-z$ $\qquad b = z + \frac{1}{2}y$

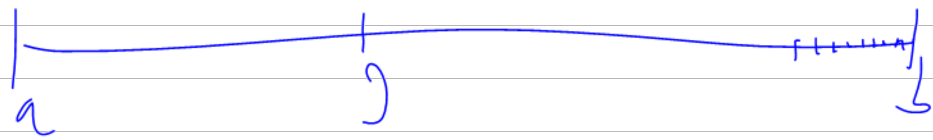
circle: $\{x : |x-a| = r\}$

$a \in \mathbb{R}^2 \qquad r > 0$



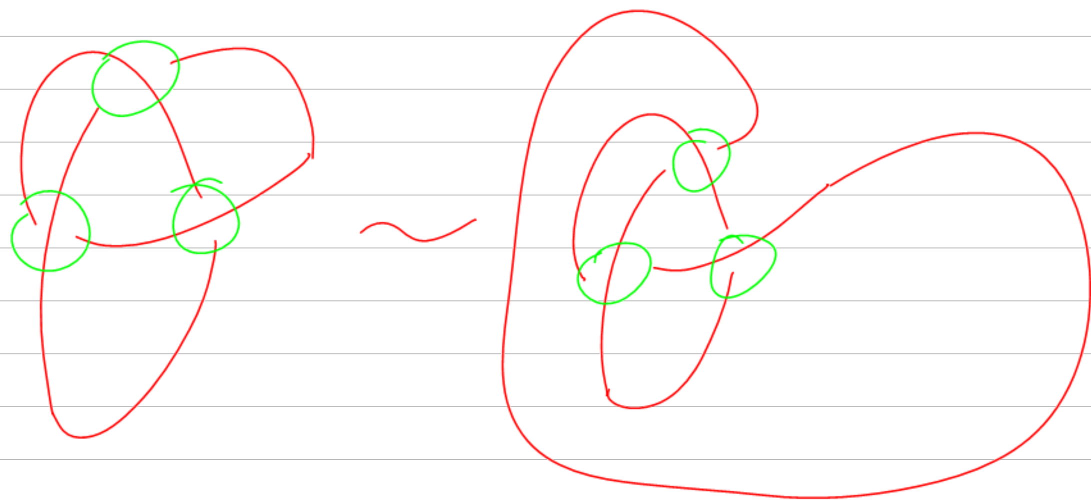
2 2
5 5
3.14159265358979323846264338327950288419716939937510582097494459230 7
7 7

$G = \{ g \in [a,b] : [a,g]$ can be covered by finitely many elements of $\mathcal{U} \}$

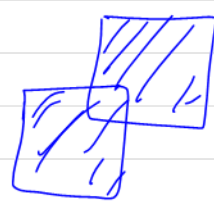


$\gamma = \sup G$

1. $\gamma > a$
2. $\gamma = b$
3. $b \in G$.

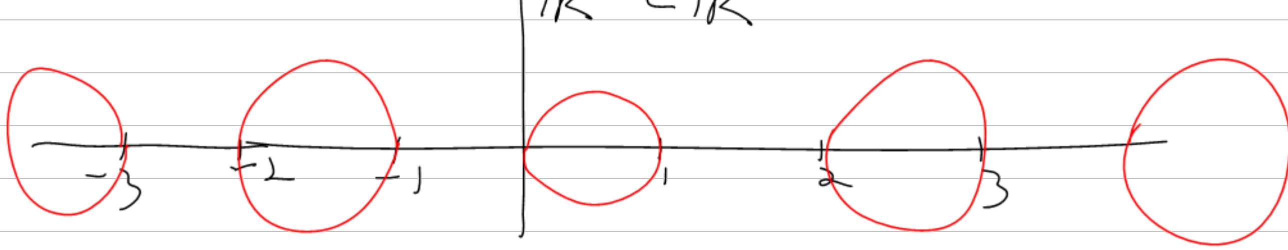


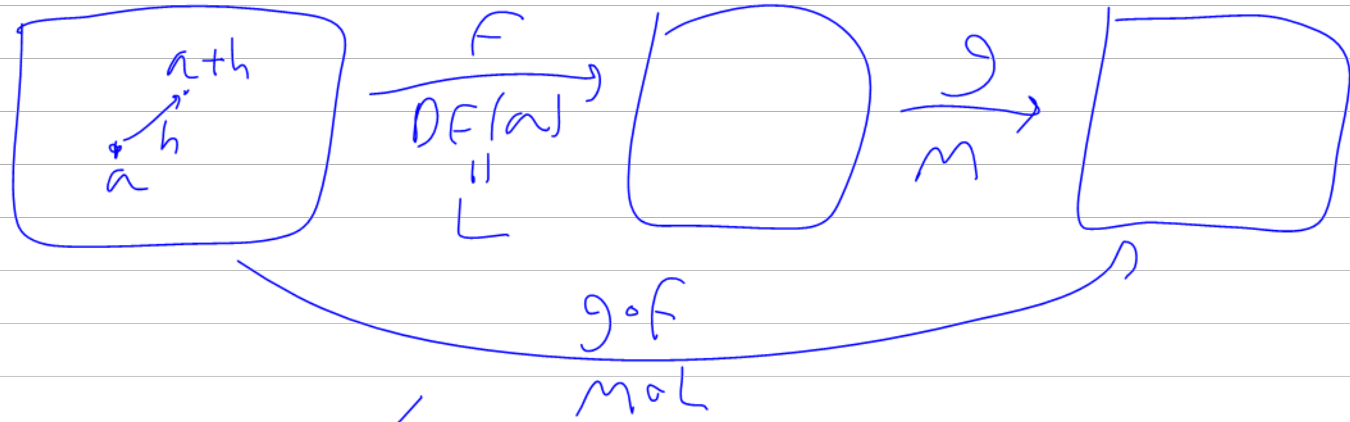
$A_{1,2}$ $B_{1,2}$

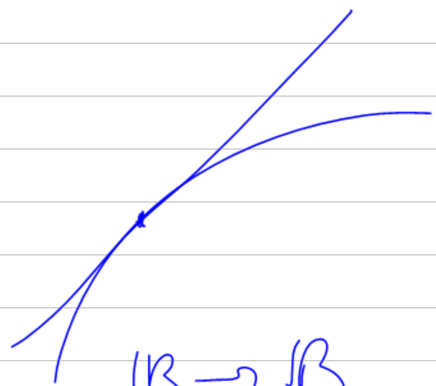


$A_1 \times B_1 \cup A_2 \times B_2$

$\mathbb{R}^2 \subset \mathbb{R}^3$







$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$

$Df = L$

$\mathbb{R} \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$

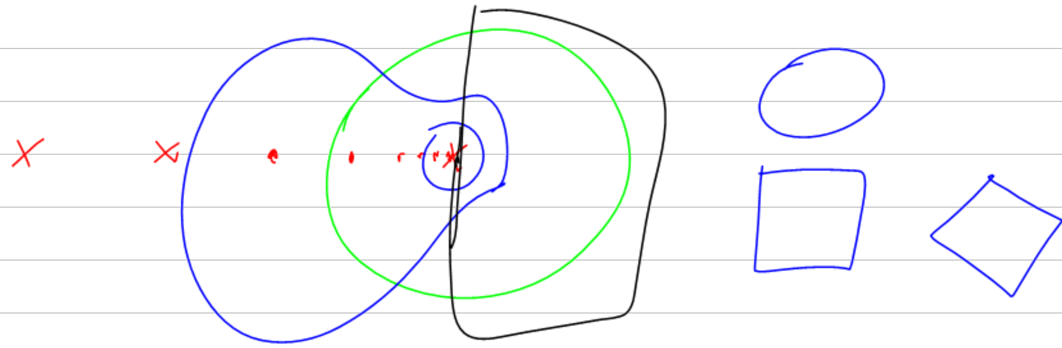
$$f(a+h) = f(a) + f'(a)\cdot h + e(h)$$

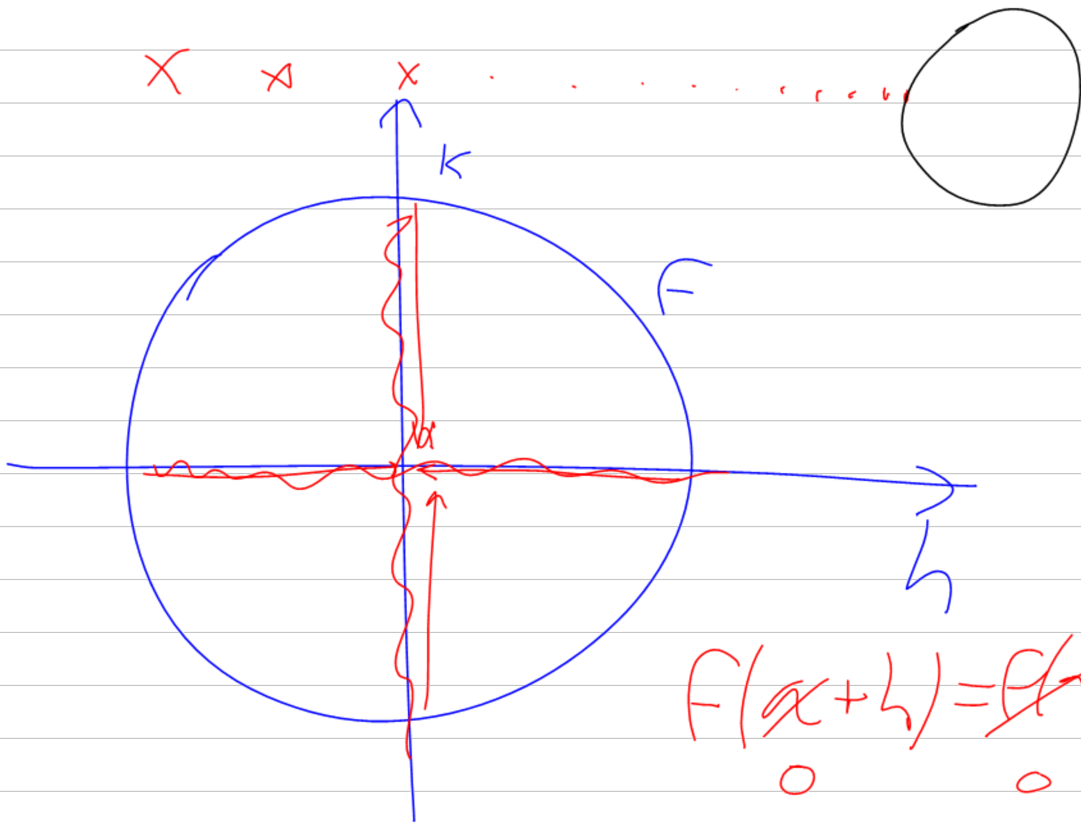
$\dfrac{f}{g} \rightsquigarrow \dfrac{f'}{g'}$ $\qquad \dfrac{Df(a)}{Dg(a)}$

$f : \mathbb{R} \to \mathbb{R}$

$\forall x \;\; \forall \epsilon \;\; \exists \delta \;\; \forall y \;\; |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

$\forall \epsilon \;\; \exists \delta \;\; \forall x \;\; |x-y| < \delta \Rightarrow$ _ _ _ .





$$F(\underset{0}{\cancel{x}}+h)=\underset{0}{\cancel{F(a)}}+DF(a)\cdot h+o(h)$$

$$0 = DF(0)\cdot\begin{pmatrix}h\\k\end{pmatrix}+o\begin{pmatrix}h\\k\end{pmatrix}$$

$$f(a+h)=f(a)+\overset{DF(a)}{L}\cdot h+e(h)$$

$$e(h)=f(a+h)-f(a)-\underline{\underline{L}}\cdot h$$

$$F : \underset{\substack{\Downarrow \\ a}}{\mathbb{R}^n} \times \underset{\substack{\Downarrow \\ b}}{\mathbb{R}^m} \longrightarrow \mathbb{R}^p$$

$$(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$$

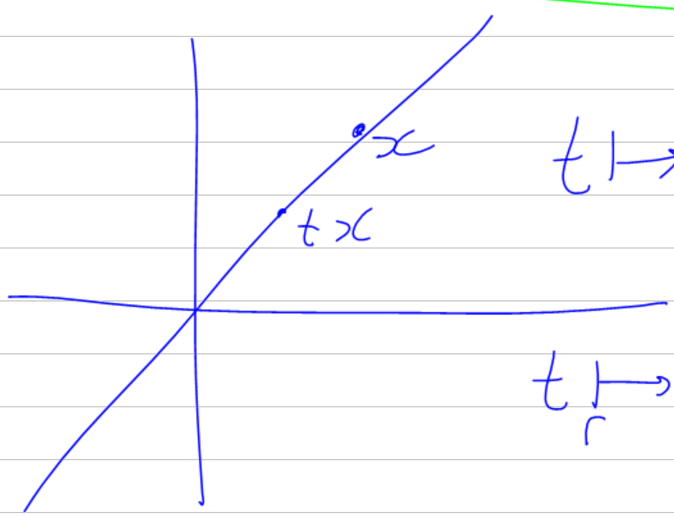
$$Df(a,b) : \underset{\substack{\Downarrow \\ x}}{\mathbb{R}^n} \times \underset{\substack{\Downarrow \\ y}}{\mathbb{R}^m} \longrightarrow \mathbb{R}^p$$

$$Df(a,b)(x,y) \in \mathbb{R}^p$$

$$(x,y) \longmapsto f(x,b) + f(a,y)$$

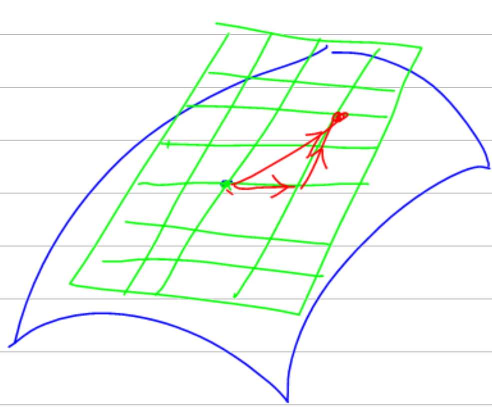
$$(a,b) + (h,k) = (a+h, b+k)$$

$$f(a+h, b+k) = f(a,b) + \underset{\text{Diff}}{\underset{\uparrow}{\boxed{}}} \cdot (h,k) + o(\dots)$$



$$t \mapsto f(tx)$$

$$t \mapsto t \cdot x \longmapsto f(t \cdot x)$$



$$f(a) + L \cdot h$$

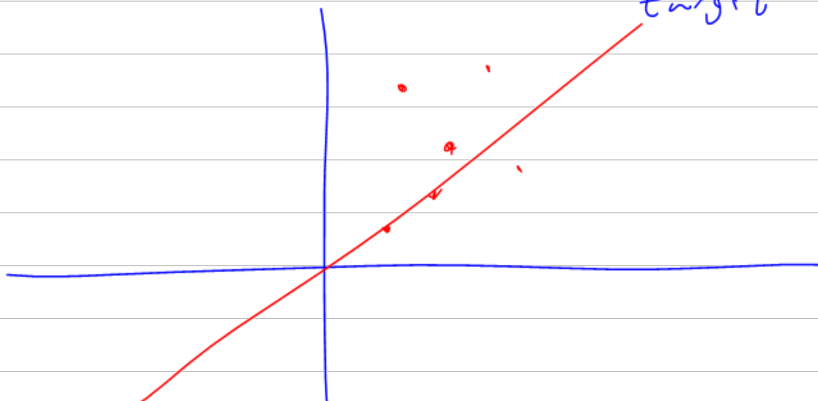
$$\|$$

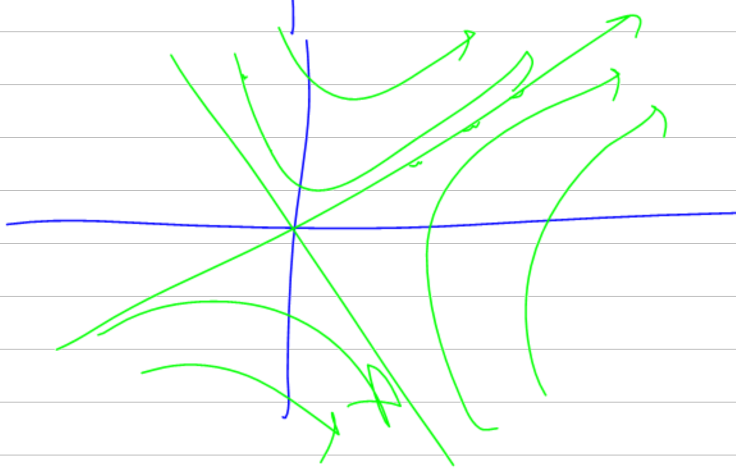
$$D_1 f \cdot h_1 + D_2 f \cdot h_2$$

$D_1(\sin xy) = y\cos(xy)$

$F : \mathbb{R}^n \to \mathbb{R}^m$ $\quad DF(a) \in M_{m\times n}$

$F : \mathbb{R}^n \to \mathbb{R}^n$ $\quad \underbrace{L : V}_{\text{target}} = \underbrace{V}_{\text{domain}}$

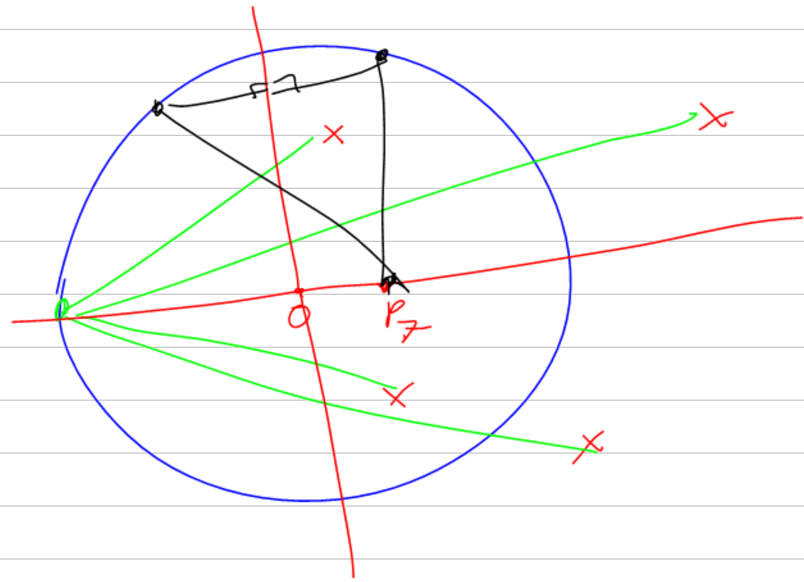


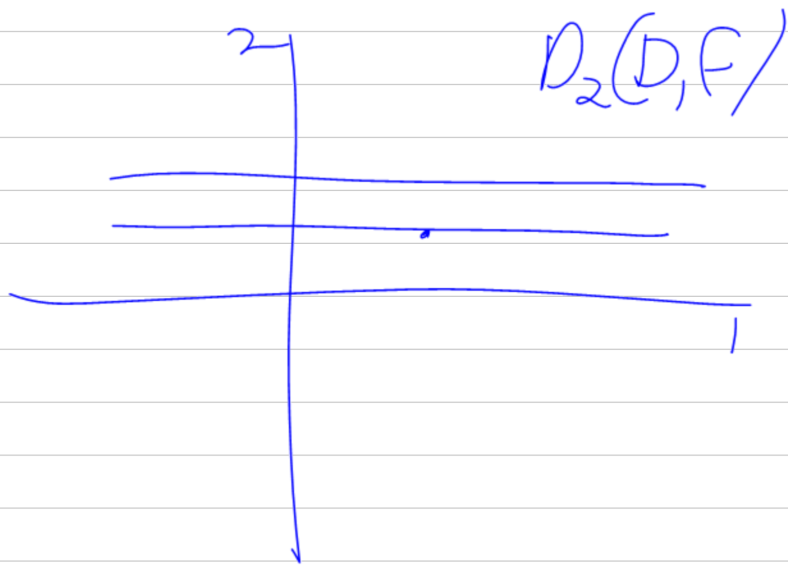


$f(x) = \sum x_i g_i(x)$

$g_2 \dots g_n = 0 \quad g_1(x) = \begin{cases} \dfrac{f(x)}{x_1} & x_1 \neq 0 \\ 0 & \text{otherwise} \end{cases}$

$x_1 = 0,\ x \neq 0 \quad ($



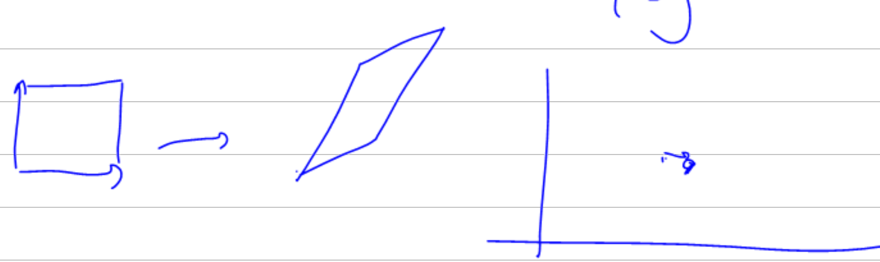


If $F \circ g$ & g is invertible

then $F^{-1} = g \circ (F \circ g)^{-1}$

$\bar{F} = F \circ (\lambda)^{-1}$

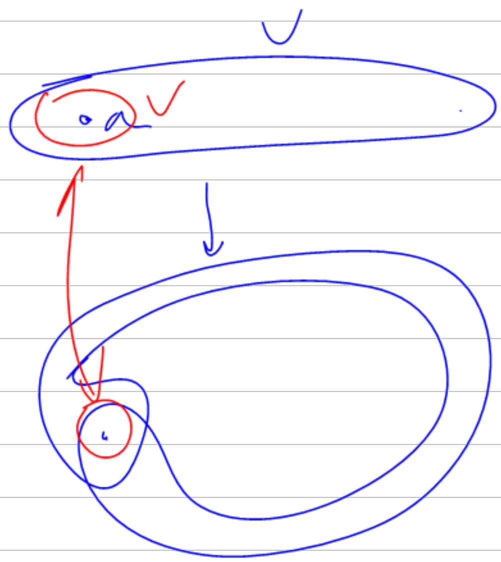
$$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[rr, bend right, "F \circ g"'] & \bullet \arrow[r, "F"] & \bullet \end{tikzcd}$$



$a_1 \ldots \ldots \ldots a_n \in \mathbb{R}$

Compute how many increasing subsequences of length 5 are there?

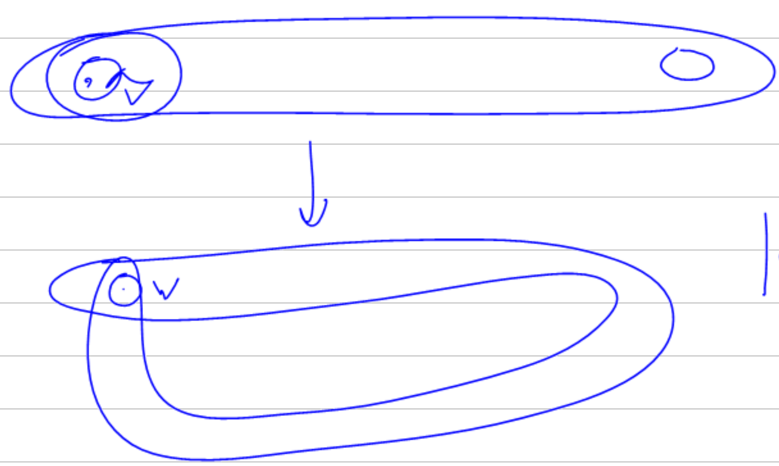
time $\sim n^5$ naively



$A \subset B$

257^{n^2}

$257 \cdot n^2$



$$|\alpha-\beta| \geq \big||\alpha|-|\beta|\big|$$
$$\geq |\alpha|-|\beta|$$

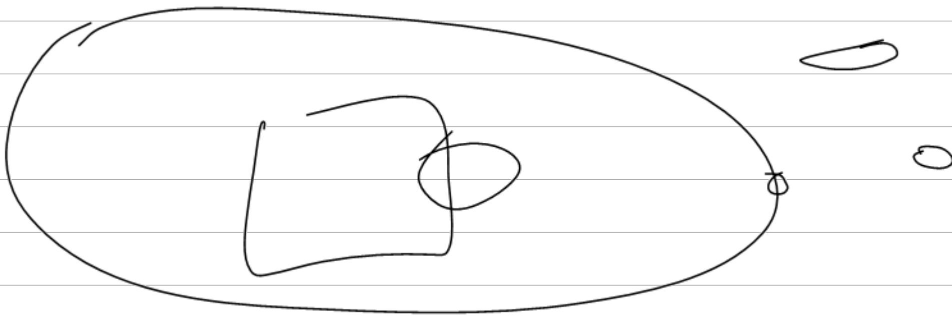
$$\big|(f(x_1)-f(x_2))+(x_1-x_2)\big| < \frac{1}{257}|x_1-x_2|$$

$$\geq$$

$$-|f(x_1)-f(x_2)|+|x_1-x_2|$$

Weak IFT: Given F w/ $F'(a)$ invertible, $\exists F^{-1}$ ... ~~F^{-1} is cont diffable~~ F^{-1} is diffable at a.

Lemma: WIFT = IFT.



$$\begin{pmatrix} \left.\begin{matrix}1&&&\\&1&&\\&&1&\\&&&1\end{matrix}\right\}k & 0 \\ 0 & 0 \end{pmatrix} = L_k$$

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad F(0) = 0$

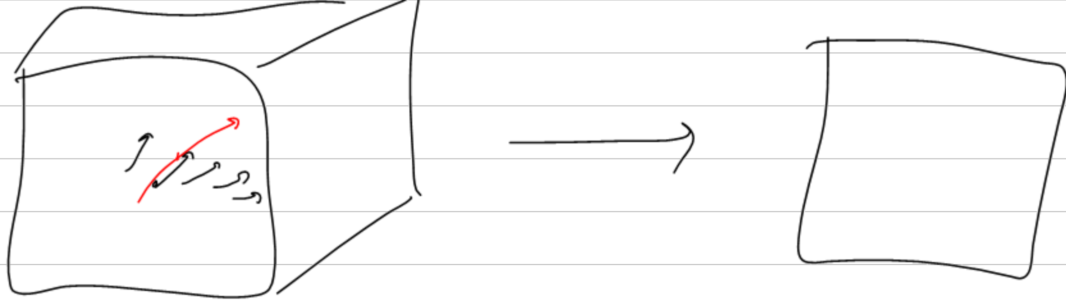
rank $F' = k$ near 0

$\exists$ invertible $\phi : \mathbb{R}^n \circlearrowleft$
$\qquad\qquad\qquad \psi : \mathbb{R}^m \circlearrowleft$

s.t. $\quad \psi \circ F \circ \phi = L_k$

T

(x)



$F: \mathbb{R}^n \longrightarrow V \qquad \int_R F$

$F(x,y) = 0 \Rightarrow$

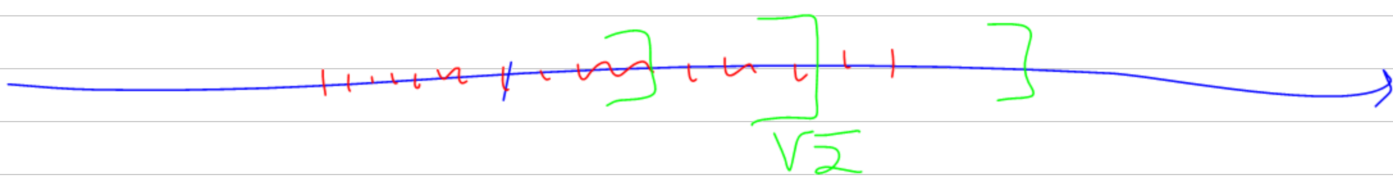
$F(x, y(x)) = 0$

$\frac{\partial}{\partial x} \qquad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0$

$\mathbb{R} = \{$ Cauchy sequences of rationals $\} / \sim$

3, 3.1, 3.14, 3.141

"Dedekind cuts"



$T: V \longrightarrow W$

$T^*: W^* \longrightarrow V^*$

given $\langle , \rangle : W \longrightarrow V$

$$\mathbb{C} = \mathbb{R}^2$$

$$\mathbb{C}^n = \mathbb{R}^{2n}$$

$F : \mathbb{C} \to \mathbb{C}$ $\quad L : \mathbb{C} \to \mathbb{C}$ $\quad \begin{pmatrix} a & -b \\ +b & a \end{pmatrix}$

$F : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad L : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$F : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ s.t. $F' \in \mathrm{Im}\, \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m)$

$z \quad (z) : \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \| & & \| \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \end{array} \longrightarrow \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$

$S_1, S_2 : \{\text{edges}\} \to \mathbb{Z}/2$

$S = S_1 + S_2$ is "even around faces"

$\Rightarrow \exists\, \sigma : \{\text{verts}\} \to \mathbb{Z}/2$

s.t. $\underbrace{(S_1 + S_2)(\xi)}_{S(\xi)} = \sigma(\xi_1) - \sigma(\xi_0)$

$\text{Face} \xrightarrow{\partial} \text{edges} \xrightarrow{\partial} \text{vertices}$

$$\text{verts} \xrightarrow{d} \text{edges} \xrightarrow{d} \text{faces}$$

$$\sigma \in C^0 \longrightarrow s \in C^1 \longrightarrow ds = 0$$

$$d\sigma = s$$

$$H^1\left(\begin{matrix}\text{2-skeleton} \\ \text{of a cube}\end{matrix}, \mathbb{Z}/2\right)$$

$\dfrac{\partial f}{\partial y}$ invertible

$\forall x \ \exists y$ s.t. $f(x,y) = 0$

$$f(x,y) = xy \quad (a,b) = (0,0)$$

$$\frac{\partial f}{\partial y}(a,b) = 0$$

$$x^2 + y^2 = 1$$

$$2x + 2yy' = 0$$

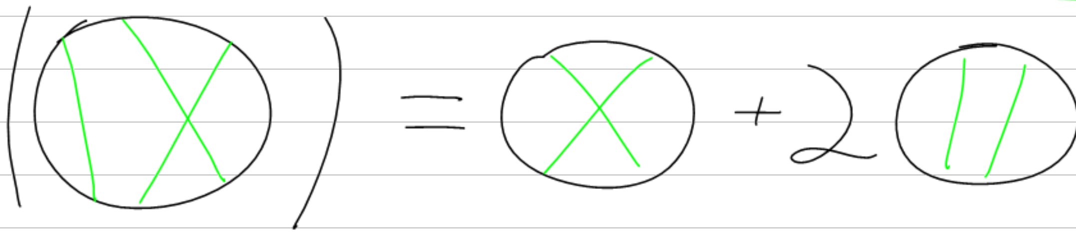
$$y' = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}$$

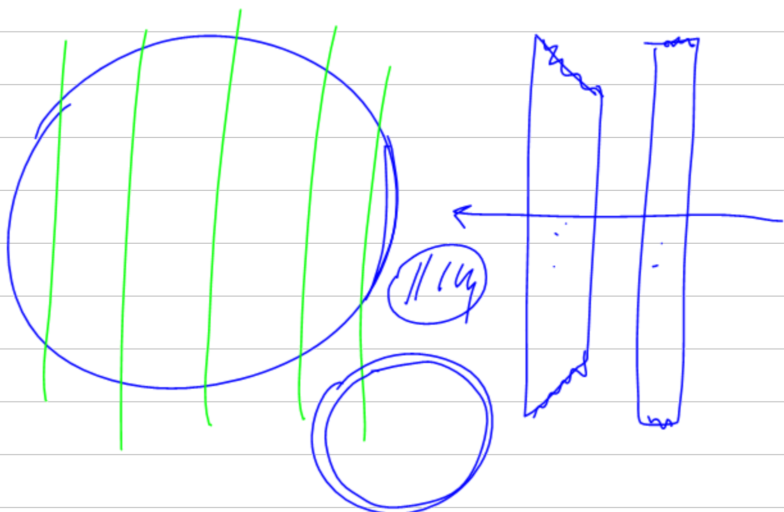
Suppose g: s.t. $y(x)$

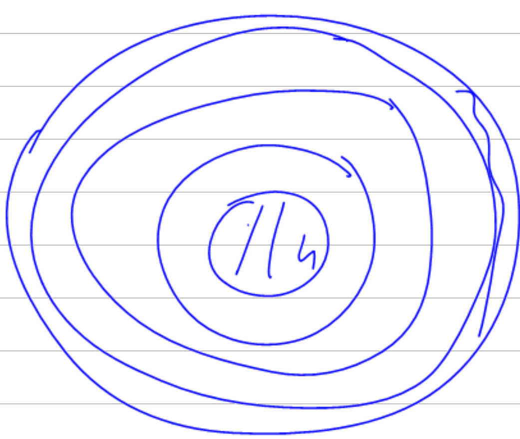
$$f(x, g(x)) = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x} = 0$$

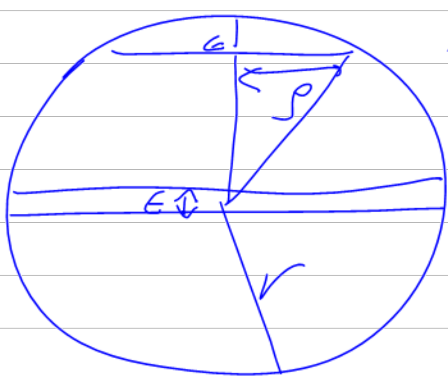
$$F : \underbrace{\mathbb{R}^n}_{x} \times \underbrace{\mathbb{R}^k}_{y} \to \underline{\underline{\mathbb{R}^{k+1}}}$$

∂_0 





$$\int_M dw = \int_{\partial M} w$$

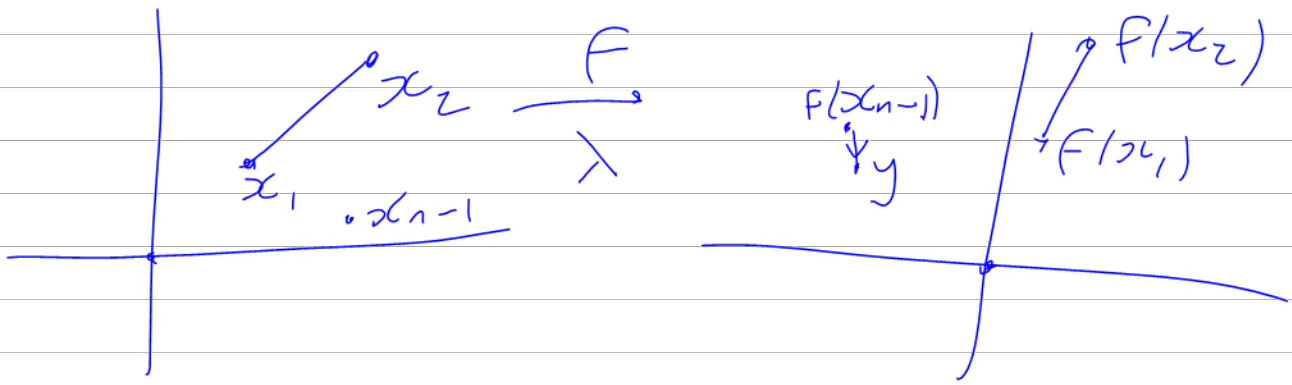


$$A \sim \pi\rho^2 = 2\pi r\epsilon$$

$$A \sim 2\pi r\cdot\epsilon$$

$$r^2 = \rho^2 + (r-\epsilon)^2$$

$$\rho = \sqrt{r^2-(r-\epsilon)^2} = \sqrt{2r\epsilon-\epsilon^2}$$
$$\sim \sqrt{2r\epsilon}$$



$$\left|\lambda(x_1-x_2) - \big(F(x_1)-F(x_2)\big)\right| < \epsilon\,|x_1-x_2|$$

$x_0 = 0$ assume $x_0 \ldots x_{n-1}$ are def,

$$x_n = x_{n-1} + \lambda^{-1}(y - f(x_{n-1}))$$
$$x_{n+1} = x_n + \lambda^{-1}(y - F(x_n))$$

$$x_n - x_{n+1} = (x_{n-1}-x_n) - \lambda^{-1}(F(x_{n-1})-f(x_n))$$
$$= \lambda^{-1}\Big(\underbrace{\lambda(x_{n-1}-x_n) - (F(x_{n-1})-f(x_n))}_{\leq \epsilon|x_{n-1}-x_n|}\Big)$$

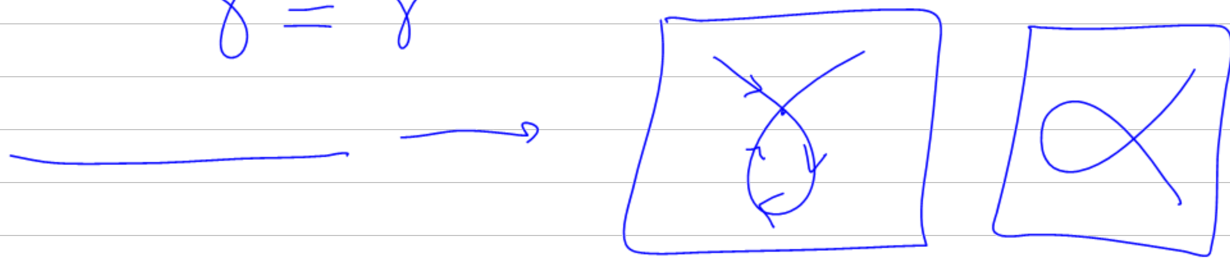
$\exists M$ s.t.

$$|x_n - x_{n+1}| \leq \overbrace{M\epsilon}^{<1}|x_{n-1}-x_n|$$

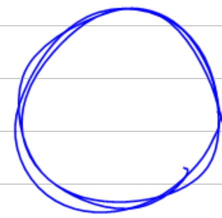
<u>Q11</u> $\gamma : \mathbb{R} \to \mathbb{R}^2$ $\quad D\gamma$ is 1-1

γ is not 1-1

$\gamma = \gamma$



$$\gamma(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$\gamma' = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Q7

$$x + y + z = \sin(xyz)$$

$F(\bar{x}, \bar{y}) = 0$ $\quad \frac{\partial F}{\partial y}$ invertible near $x = y = z = 0$

$$F = x + y + z - \sin(xyz) = 0$$

$$\frac{\partial F}{\partial z} = 1 - xy\cos(xyz)\Big|_{x=y=z=0} = 1$$

$z = g(x,y)$ Find $\frac{\partial g}{\partial x}$ & $\frac{\partial g}{\partial y}$

$$F(x, y, g(x,y)) = 0$$

$$\Bigg(\underset{\displaystyle 0}{\underset{\shortparallel}{\frac{\partial F}{\partial x}(x,y,g(x,y)) + \frac{\partial F}{\partial z}(\cdot)\frac{\partial g}{\partial x}}} \;,\; \underset{\displaystyle 0}{\underset{\shortparallel}{\underline{\qquad\qquad\qquad}}} \Bigg)$$

$$|\lambda(x_1-x_2)-(f(x_1)-f(x_2))| \leq \epsilon|x_1-x_2|$$

$$g(x)=f(\lambda^{-1}(x))$$
$$f(x)=g(\lambda(x))$$

$\cancel{\text{Let } g(x)=\lambda^{-1}(f(x)),\ f(x)=\lambda(g(x))=y}$

$\cancel{|\lambda(x_1-x_2)-\lambda(g(x_1)-g(x_2))| \leq \epsilon|x_1-x_2|}$

$\cancel{|\lambda((x_1-x_2)-(g(x_1)-g(x_2)))| \leq \epsilon|x_1-x_2|}$

$\leq M|x_1-x_2-(g(x_1)-g(x_2))|$

$$|\lambda(x_1-x_2)-(g(\lambda x_1)-g(\lambda x_2))| \leq \epsilon|x_1-x_2|$$

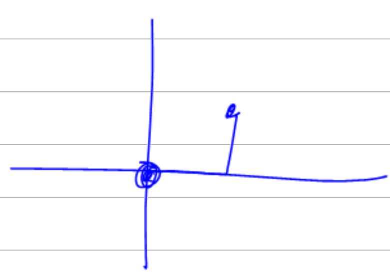
$$\lambda x_1 \to y_1 \qquad \lambda x_2 \to y_2$$

$$|(y_1-y_2)-(g(y_1)-g(y_2))| \leq \epsilon|\lambda^{-1}(y_1-y_2)|$$
$$\leq \underbrace{M\epsilon}|y_1-y_2|$$

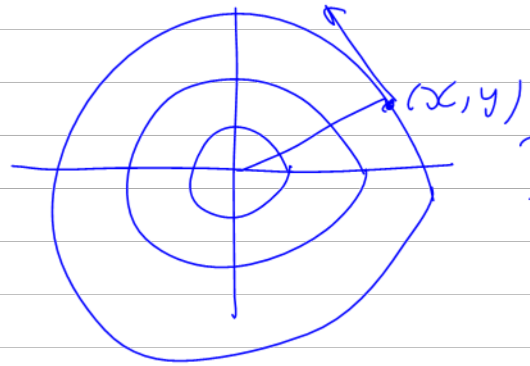
$\Rightarrow g$ is invertible $\Rightarrow f$ is.

If $f'(a)=0$

f is cont. diffable

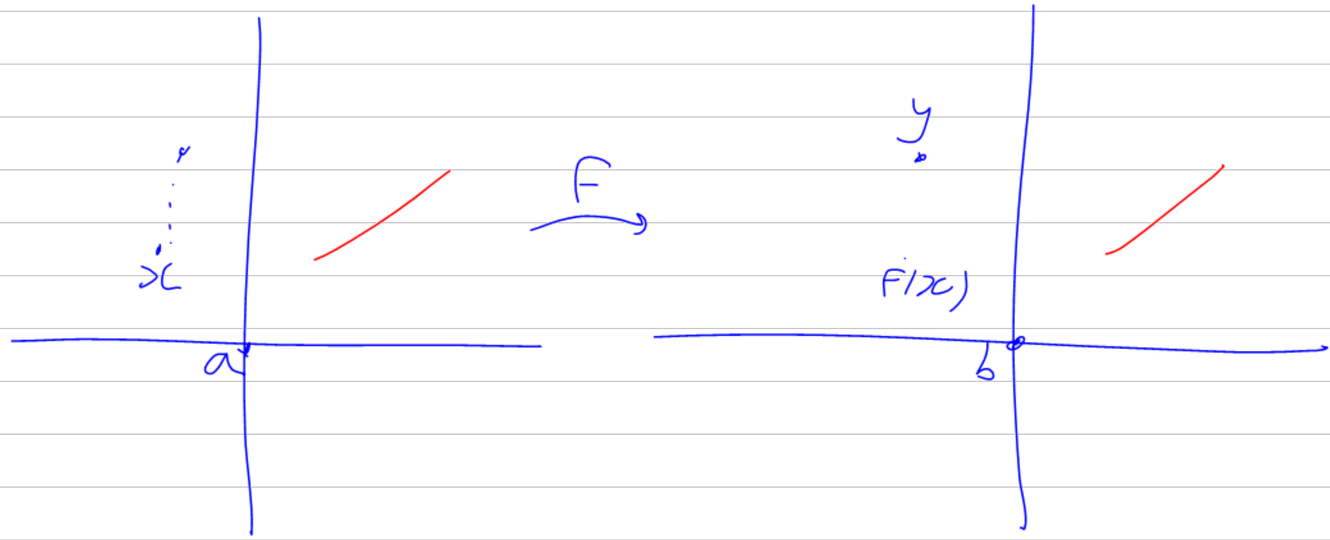


Q13



$$DF(x,y)(-y,x)$$

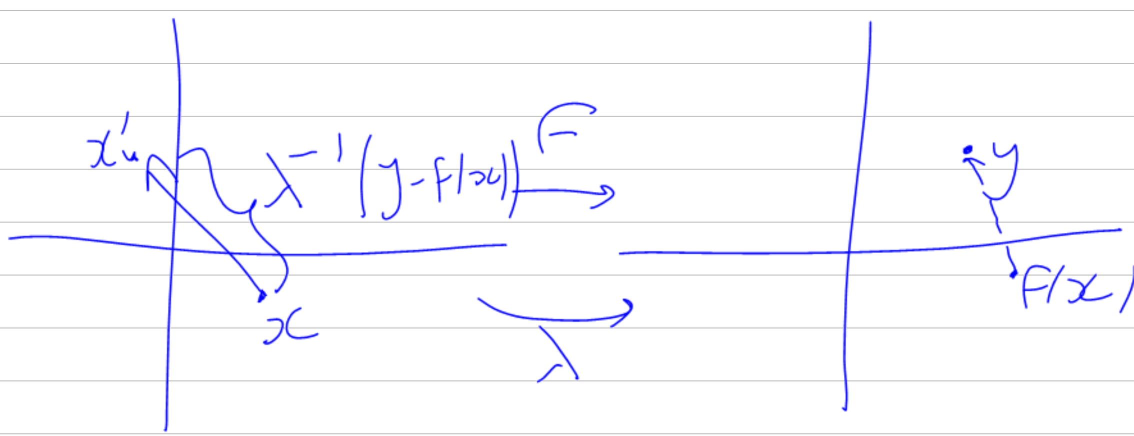
$$\frac{\partial}{\partial\theta}\left(F\circ T_\theta\right) = \cancel{x}\ 0$$



$$x' = x + y - f(x)$$

$$\left|(x_1 - x_2) - (f(x_1) - f(x_2))\right| < \epsilon \ldots$$

$$\Big|\,|x_1 - x_2| - |f(x_1) - f(x_2)|\,\Big| < \epsilon \ldots$$



$$\left|\lambda(x_1 - x_2) - (f(x_1) - f(x_2))\right| \leqslant \epsilon\,|x_1 - x_2|$$

$$x' = x + \lambda^{-1}(y - f(x))$$

$$x_0 = 257^{2020}$$

$$x_n = x_{n-1} + \lambda^{-1}(y - f(x_{n-1}))$$

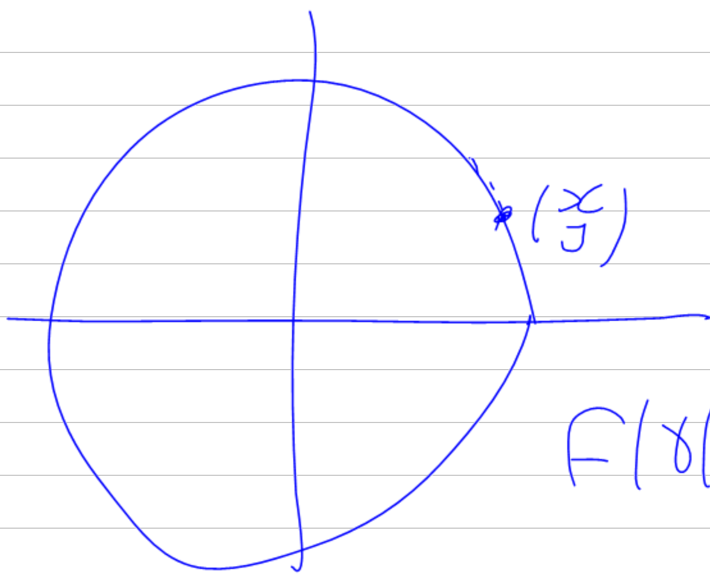
$$x_{n+1} = x_n + \lambda^{-1}(y - f(x_n))$$

$$x_n - x_{n+1} = (x_{n-1} - x_n) - \lambda^{-1}(f(x_{n-1}) - f(x_n))$$

$$= \lambda^{-1}\Big(\lambda(x_{n-1} - x_n) - (f(x_{n-1}) - f(x_n))\Big)$$

$$\Rightarrow \exists M \text{ s.t. } |\lambda^{-1} z| \leq M|z| :$$

$$|x_n - x_{n+1}| \leq M \epsilon \cdot |x_{n-1} - x_n|$$



$$f \circ T_\theta = f$$

$$\gamma(\theta) = T_\theta\binom{x}{y}$$

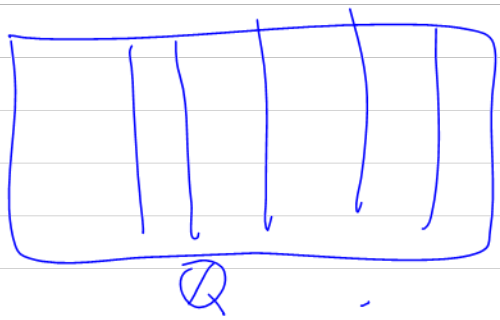
$$f(\gamma(\theta)) = f\left(T_\theta\binom{x}{y}\right)$$

$$= f\binom{x}{y} \text{ indep of } \theta$$

$$0 = \frac{d}{d\theta} f(\gamma(\theta))\Big|_{\theta=0} = Df(x,y)\underline{\underline{\gamma'(0)}}$$

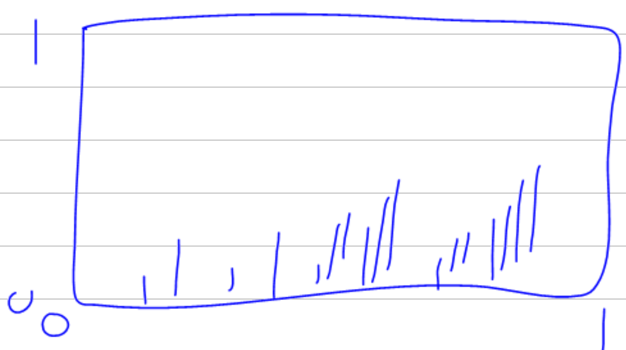
$$\gamma(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta\end{pmatrix}$$

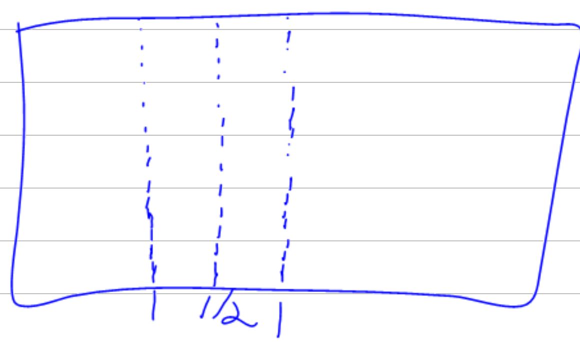
$$\int dx \int dy\, f(x,y) = \quad - \quad - \quad -$$



$$g(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$$f(x,y) = 1 + g(x)\chi_{\mathbb{Q}}(y)$$





$$\left(\int f(x,y)\,dy\right)\chi_{\mathbb{Q}^c}(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

A cover A with N of balls of size ϵ.

Suppose you wanted to cover A w/ balls of size $\frac{\epsilon}{n}$. How many do you need?

$$\text{Ans} \sim n^d N \qquad \log \text{Ans} = d \log n + \log N$$

So $$d = \lim_{n \to \infty} \frac{\log(\#\text{ of } \epsilon/n \text{ balls needed})}{\log n}$$

$$\downarrow \quad \lim_{k \to \infty} \frac{\log 2^k}{\log 3^k}$$

To cover C with balls of size 3^{-k}, you need 2^k of them.

$n = 3^k$

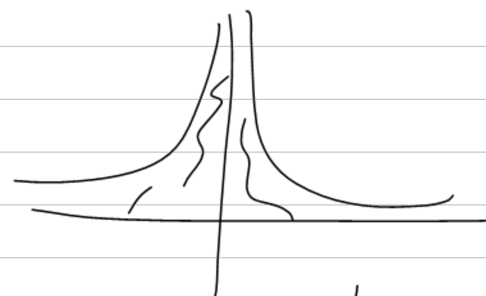
Hausdorff Dimension

A_i each is meas-0.

Given ϵ, find U_{ij} that cover A_i & s.t. $\sum_{j=1}^{\infty} v(U_{ij}) < \frac{\epsilon}{2^i}$

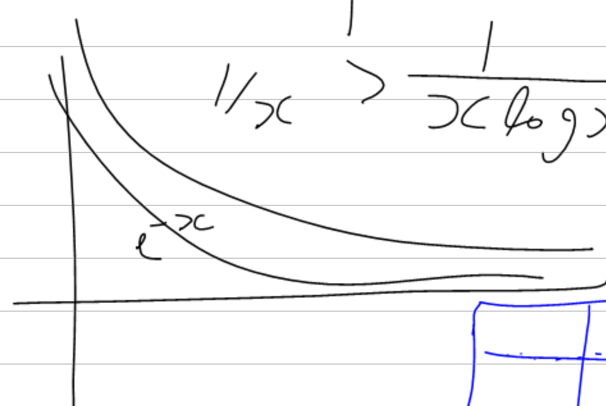
$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v(U_{ij}) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$

$\frac{1}{x^2}$ $\quad$ $\frac{1}{\sqrt{|x|}}$



$\int_0^\infty$

$1/x > \frac{1}{x \log x} > \frac{1}{x^{1.01}}$



$\lambda \in \mathbb{Q}^c$

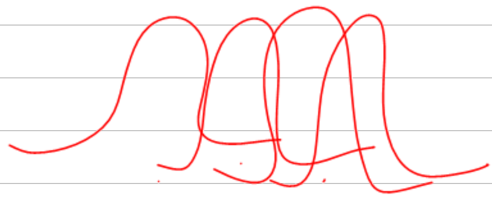
$$D = \{(a + \lambda b, b) : a, b \in \mathbb{Q}\}$$

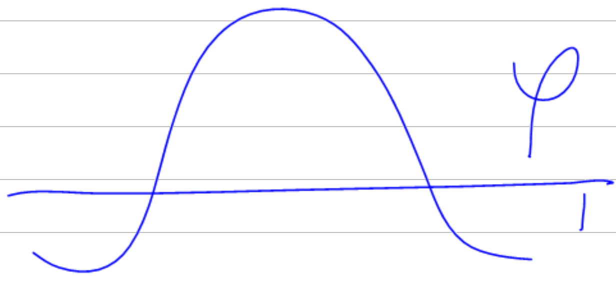
$F = 1_D$ $\quad$ 1. If $y \in \mathbb{Q}$, $D \cap (\mathbb{R} \times \{y\})$ is dense

2. $\forall x \; |D \cap (\{x\} \times \mathbb{R})| \leq 1$ $\quad$ If $y \notin \mathbb{Q}$ $\quad D \cap (\mathbb{R} \times \{y\}) = \emptyset$

$a_1 + \lambda b_1 = a_2 + \lambda b_2 \Rightarrow \lambda = \frac{a_1 - a_2}{b_2 - b_1} \in \mathbb{Q}$

$\Rightarrow \in$

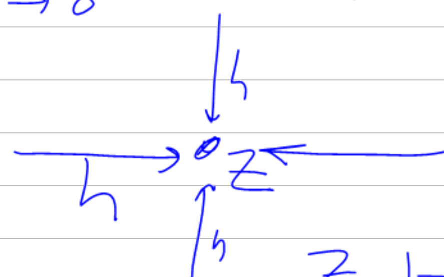




$\theta \circ \varphi$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$z, h \in \mathbb{C}$



$z \longmapsto \bar{z}$

$(x, y) \to (x, -y)$

$$\begin{array}{ccc} f: \mathbb{C} & \longrightarrow & \mathbb{C} \\ \parallel & & \parallel \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \end{array}$$

$f(x) \longmapsto f(\phi(t))$

$\int f(x, y) \longmapsto f(\underline{\phi(t)}, \underline{\lambda(s)})$

$f(x, y) \longmapsto f(\underline{r}\cos\underline{\theta}, \underline{r}\sin\underline{\theta})$

$$\int_{\underline{\underline{M}}} d\underline{\underline{w}} = \int_{\partial M} \underline{\underline{w}}$$

$A = \mathbb{Q} \cap [0,1] \quad \bar{A} = [0,1]$

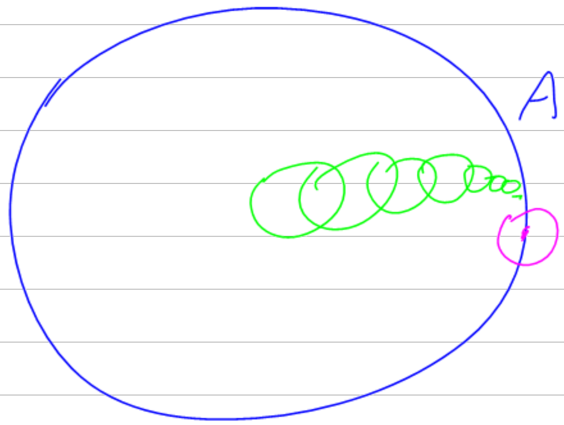
$\sum a_i$ is abs conv.

means $\sum |a_i|$ in conv.

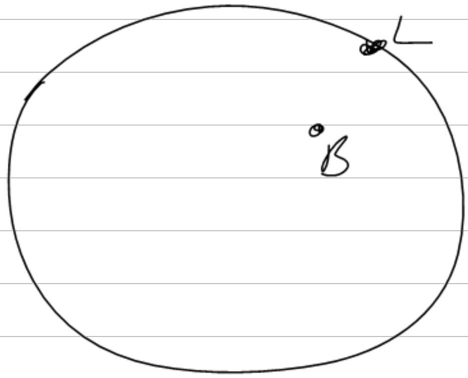
$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \frac{1}{8},$

$\left(-1 + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{4}\right) + \left(-\frac{1}{5} + \frac{1}{6}\right) + \ldots$

$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} - 1\right) + \left(\frac{1}{10} + \frac{1}{12} + \ldots - \frac{1}{3}\right) + \ldots$



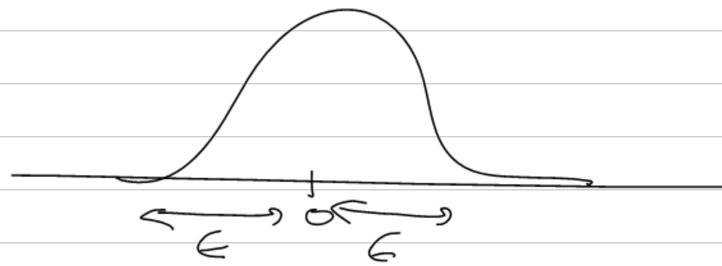
A is open



$V_L = 4V_B$

$V_L \quad V_V$

β_ϵ :



$$\beta_{a,\epsilon} = \beta_{\epsilon^2}(|x-a|^2)$$

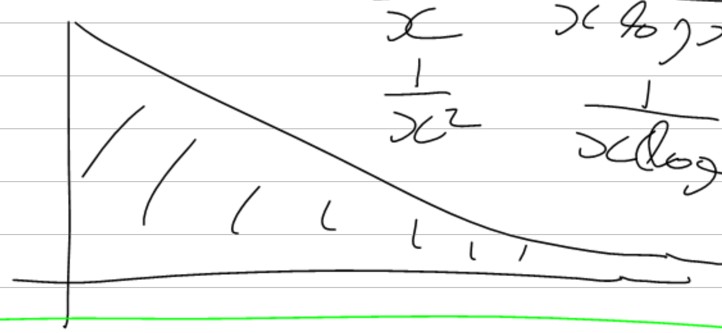
$$\int_0^\infty f(x)\,dx$$

$$\int f < \infty \Rightarrow \exists g \quad \int g < \infty \quad \frac{g}{f} \to \infty$$

$$\int f = \infty \Rightarrow \exists g \quad \int g = \infty \quad \frac{g}{f} \to 0$$

$\frac{1}{x}$ $\qquad$ $\frac{1}{x \log x}$

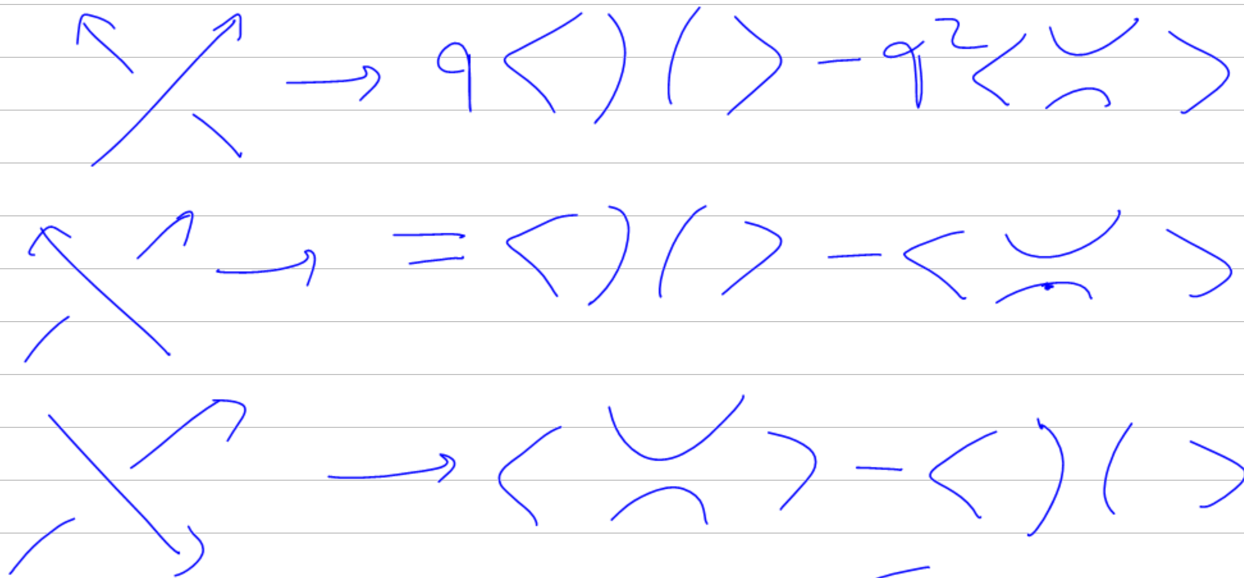
$\frac{1}{x^2}$ $\qquad$ $\frac{1}{x(\log x)^2}$

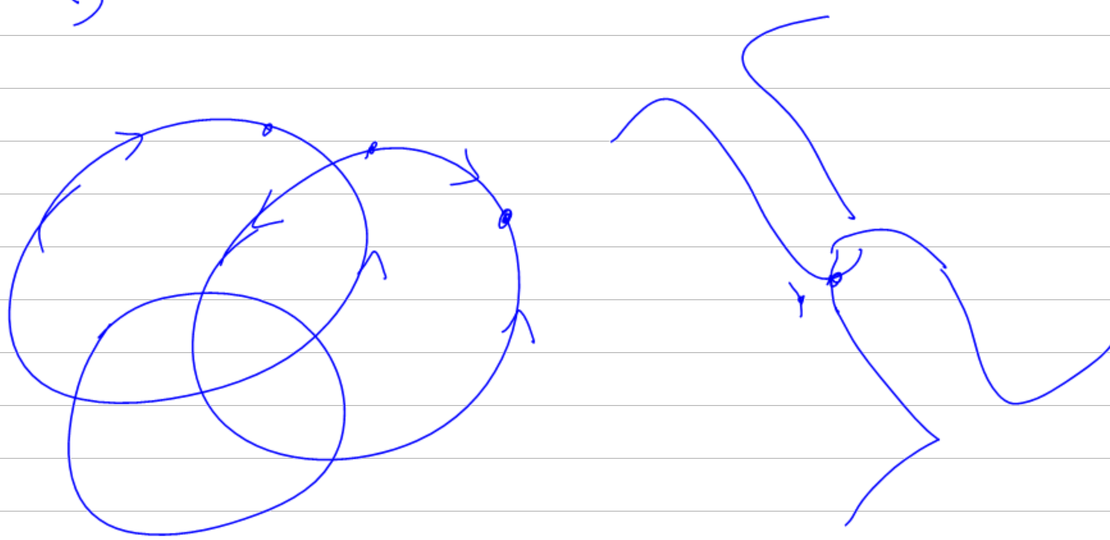


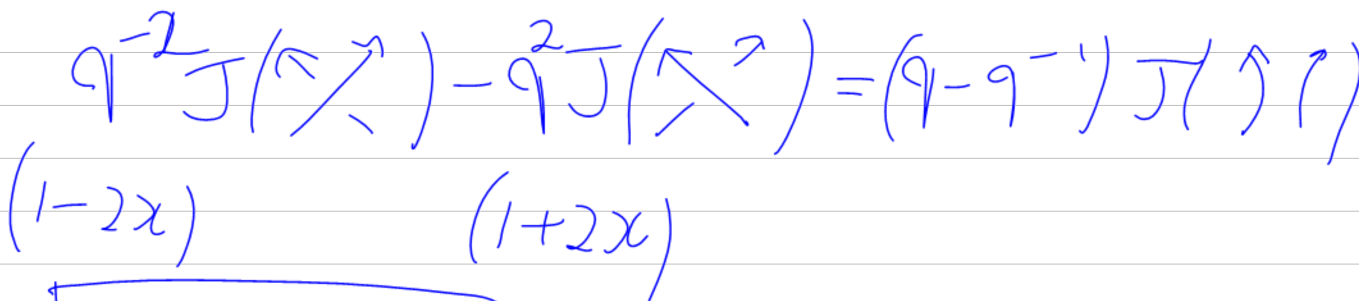
$$0 = h(x,y) = f(x,y,g(x,y))$$

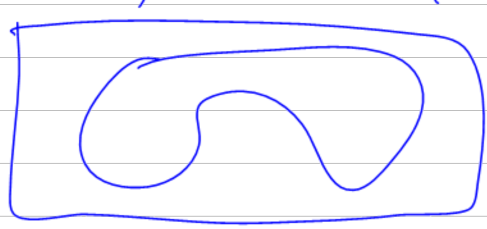
$$\partial_x : 0 = D_1 f(x,y,g(x,y)) + D_3 f(-)\frac{\partial g}{\partial x}$$

$$\frac{\partial g}{\partial x} = -\frac{D_1 f}{D_3 f}$$









For every $x \in A$

$\exists\, U_x^{\text{open}}$ $x \in U_x$ $\exists\, g_x : U_x \to \mathbb{R}$ smooth

s.t. $g_x = f$ on U_x Extend g_x arbitrarily

$\mathcal{U} = \{U_x\}$ is an open cover beyond U_x of A, so find a POI for A subordinate to $\mathcal{U}$, $\Phi = \{\varphi_i\}$

Then for each i there is some x_i s.t. $\operatorname{supp} \varphi_i \subset U_{x_i}$

set $\quad \bar{F} = \sum_i \varphi_i \, g_{x_i}$

loc-finiteness $\Longrightarrow$ smooth.

$\bar{F}(a) = \sum_i \varphi_i(a) \, g_{x_i}(a)$

$a \in A$

$$= \sum_{i : a \in U_{x_i}} \varphi_i(a) \, \cancel{g_{x_i}(a)} \; f(a)$$

$$\cancel{+ \sum_{i : a \notin U_{x_i}} \varphi_i(a) \, g_{x_i}(a)}$$

$$= \sum_{i : a \in U_{x_i}} \varphi_i(a) f(a)$$

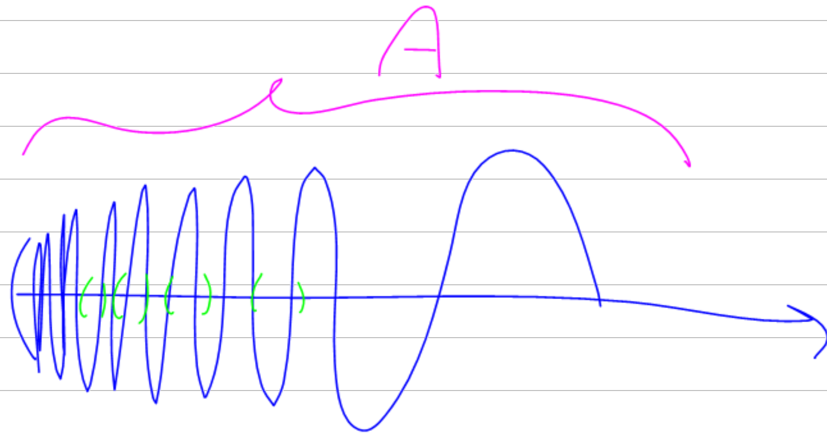
$$= f(a) \Big(\sum_{i : a \in U_{x_i}} \varphi_i(a) + \sum_{i : a \notin U_{x_i}} \underbrace{\varphi_i(a)}_{} \Big)$$

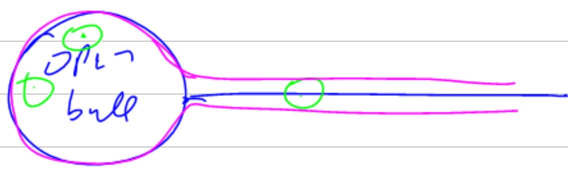
(where $\varphi_i(a) = 0$ in the last sum)

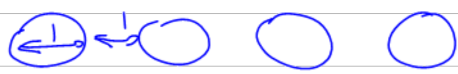
for all a
$\varphi_i(a) g_{x_i}(a)$
$= \varphi_i(a) f(a)$

$$= f(x)\sum_i \varphi_i(x) = f(x)$$

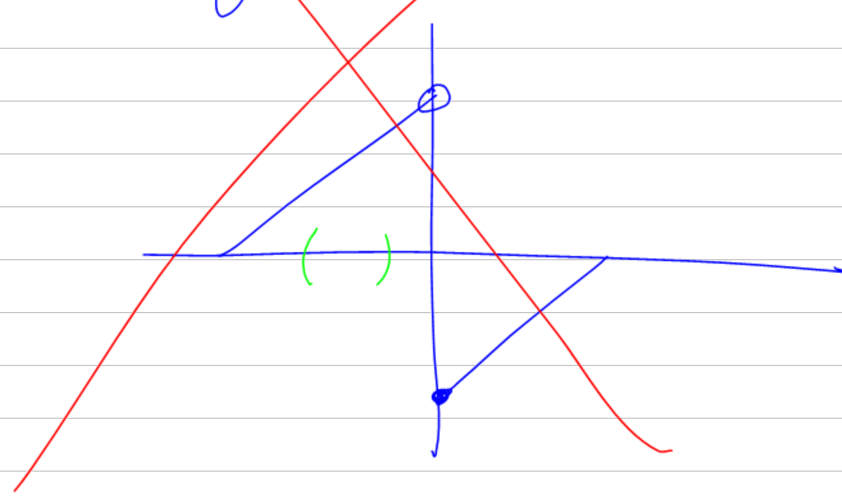
$\sin\frac{1}{x}$





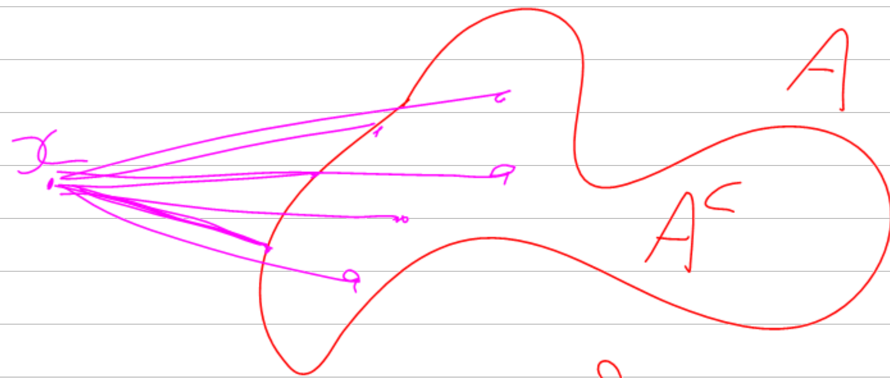


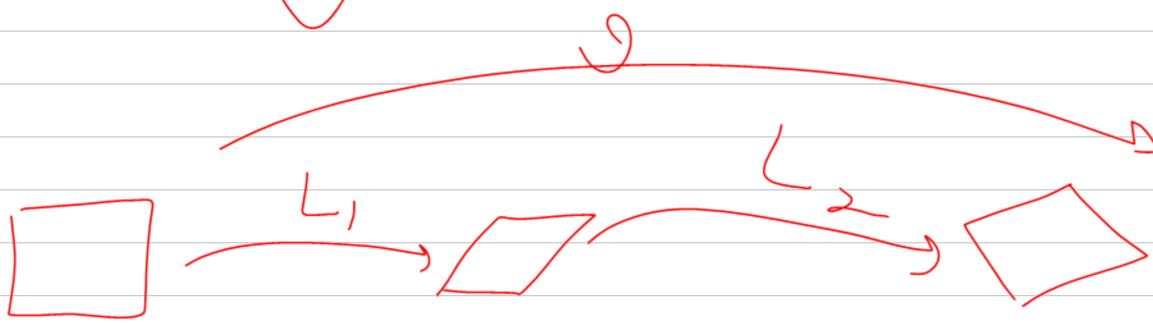
$$f(x) = \begin{cases} x-1 & x \in [0,1) \\ x+1 & x \in (-1,0) \end{cases}$$

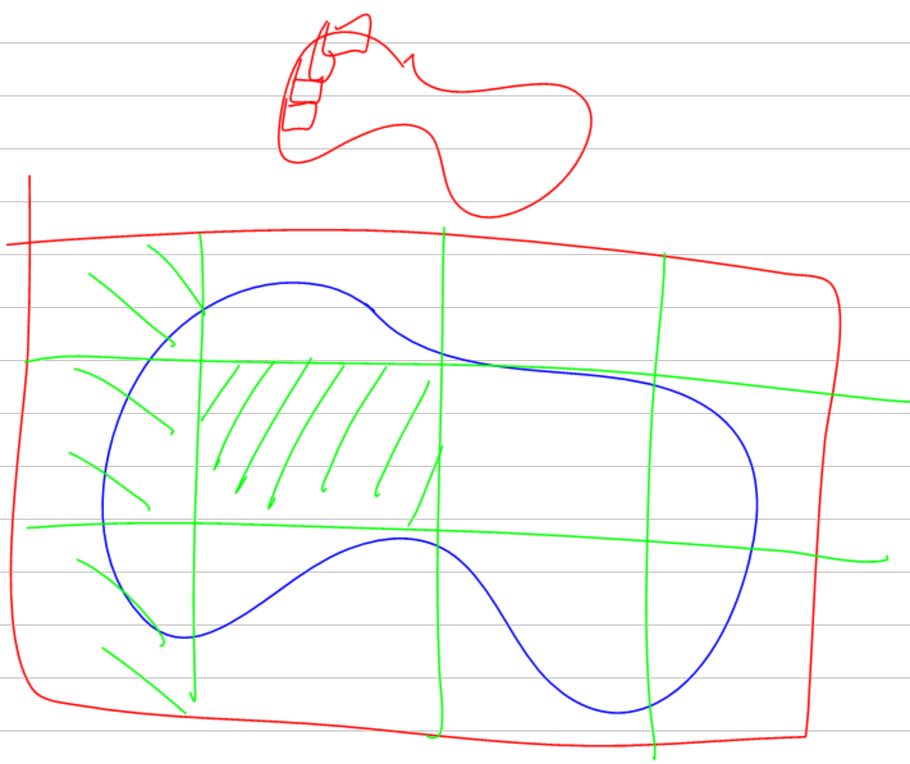


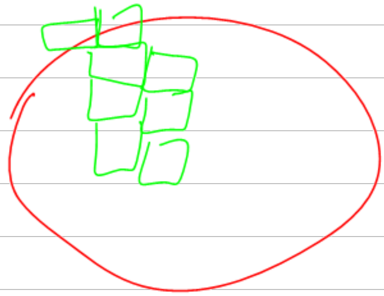
Winter Semester.

"Classification of simple groups"

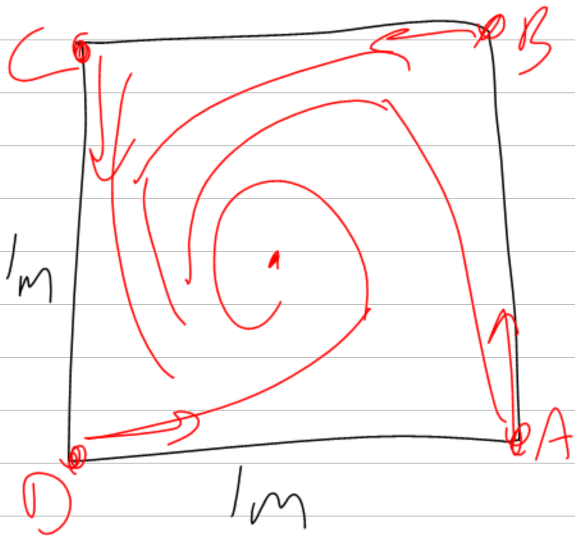




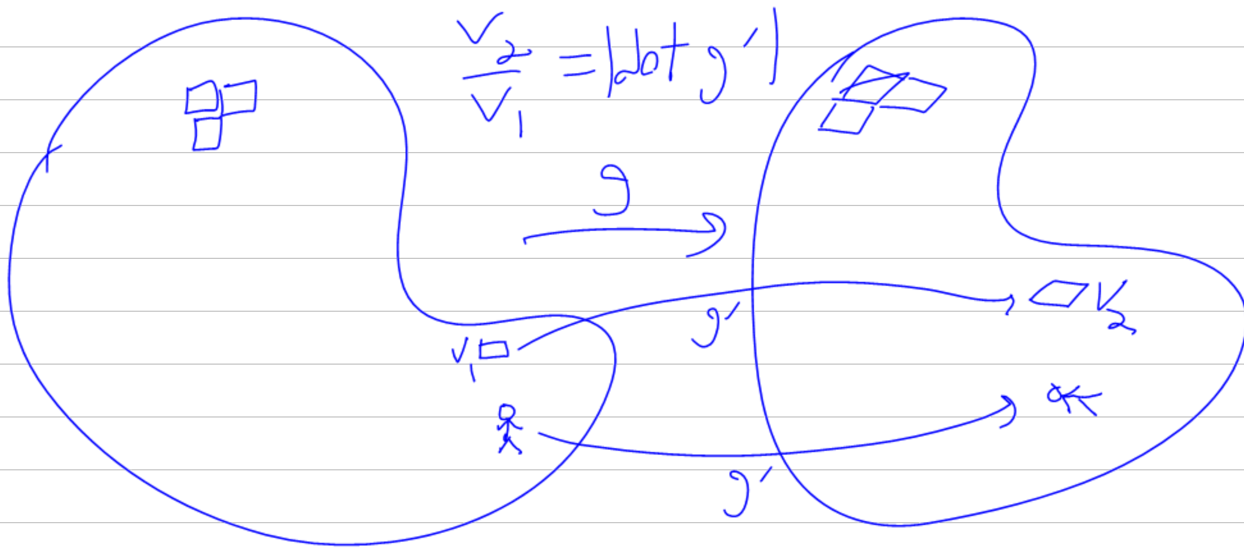


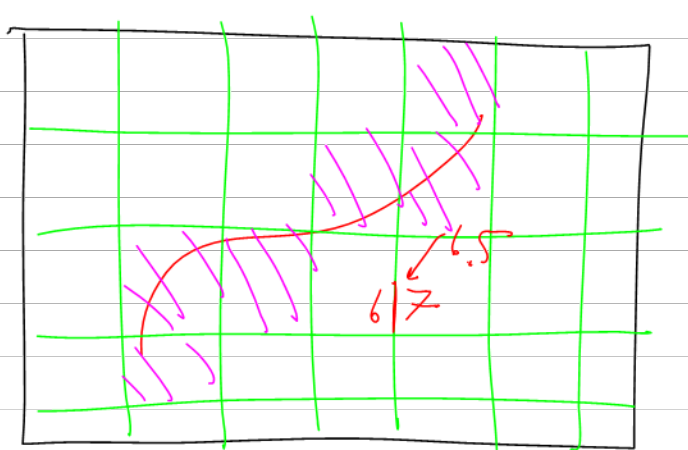


$\frac{L}{\det(L)} = 7$



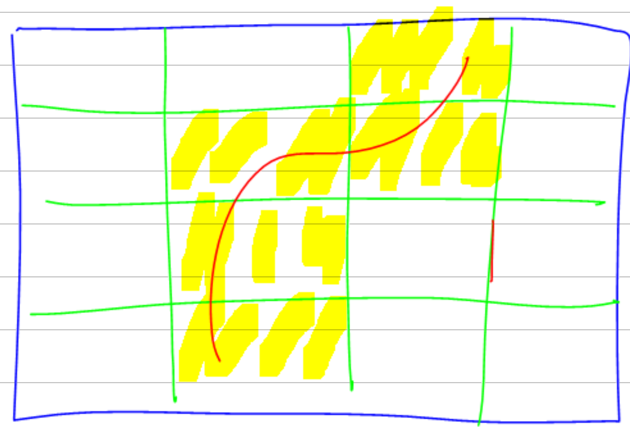
$V = 1\ m/s$





$$\int \psi_i f$$

$$A_n = \{x : O(f, x) \geq \tfrac{1}{n}\}$$



integ $\Rightarrow$ meas-0

meas-0 $\Rightarrow$ integ:

9. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth and has an invertible differential at 0, then near 0 it can be written as a composition $T_n \circ g_n \circ \cdots \circ T_2 \circ g_2 \circ T_1 \circ g_1 \circ T_0$, where each T_i is a "permutation map" that merely permutes the coordinates of $x = (x_1\ x_2\ \ldots\ x_n) \in \mathbb{R}^n$, and each g_i changes the value of only the last coordinate; precisely, $g_i(x_1\ x_2\ \ldots\ x_n) = (x_1\ \ldots\ x_{n-1}\ h_i)$, where the $h_i : \mathbb{R}^n \to \mathbb{R}$ are smooth.

$$\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_k \end{pmatrix} \longrightarrow \begin{pmatrix} y_1 \\ \vdots \\ \cancel{y_k} \\ y_n \\ y_k \end{pmatrix}$$

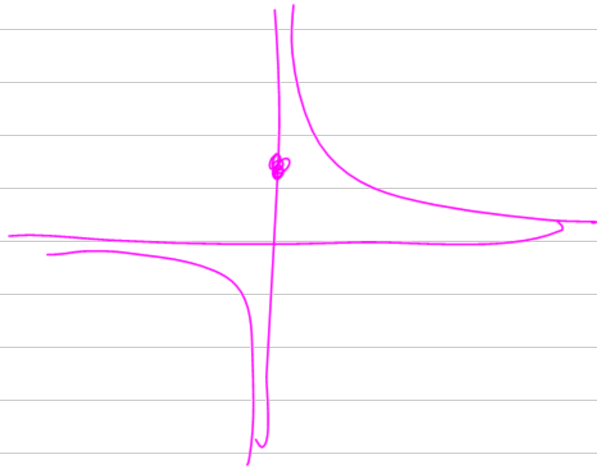
$$\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_{k_1} \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ y_{k_2} \\ y_{k_1} \end{pmatrix} \longmapsto$$

Lemma
Suppose $g: \mathbb{R}^n \to \mathbb{R}^n$ $g'(0)$ invertible

& $g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ g_{k+1} \\ \vdots \\ g_n \end{pmatrix}$ then $\exists\ j > k$

st $\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \overset{\alpha}{\longmapsto} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ g_j \\ x_{k+1} \\ \vdots \\ x_n \end{pmatrix}$ is invertible.

$$\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \overset{\alpha_1}{\longmapsto} \begin{pmatrix} g_{j_1} \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \overset{\alpha_2}{\longmapsto} \begin{pmatrix} g_{j_1} \\ g_{j_2} \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} g_{j_1} \\ \text{and} \\ \text{other} \\ g\text{'s} \end{pmatrix}$$



φ_i subordinate to a cover $\mathcal{U}$ of A

$$\#_1 = \sum \int \varphi_i |F| < \infty$$

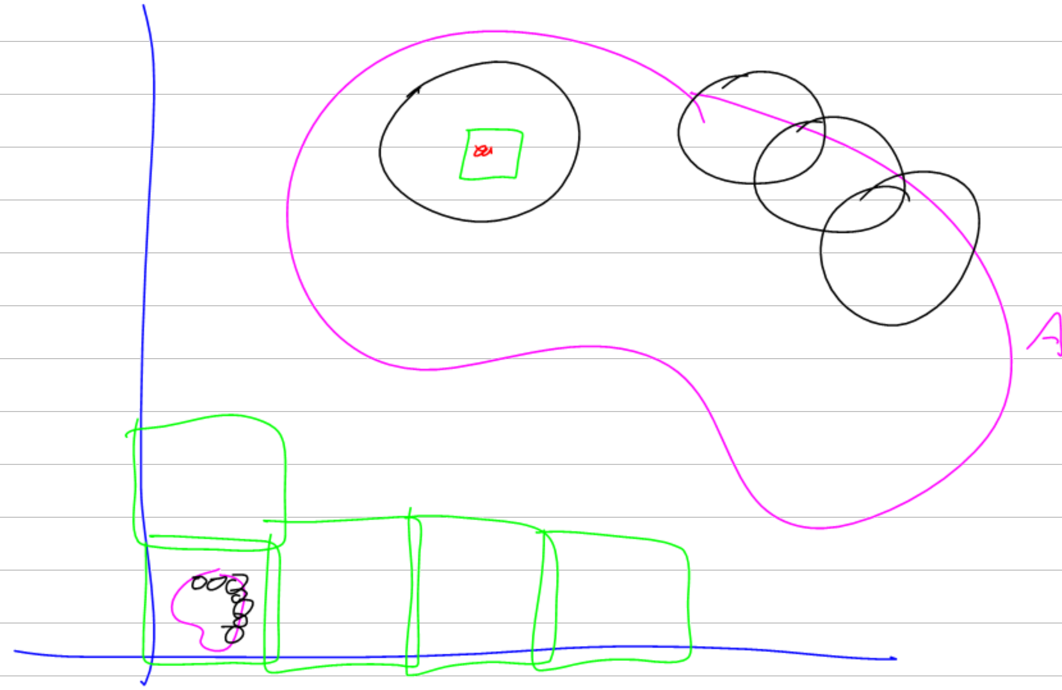
$\psi_i = \varphi_i \cdot \chi_B$ subordinate to $\mathcal{V} = \{U \cap B : U \in \mathcal{U}\}$

$$\sum \int \psi_i |F| = \sum \int \varphi_i \cdot \chi_B |F| \leq \#_1 < \infty$$

$\mathrm{disc}(F) \cap \mathrm{supp}\, \varphi_i$ is meas-0.

$\mathrm{disc}(F) \cap \left(\bigcup \mathrm{supp}\, \varphi_i\right)$ is meas-0

$\mathrm{disc}(F) \cap A$ is meas-0.



$$G_1 = \left\{(x,y) : \begin{array}{l} 0 \le x \le a \\ 0 \le y \le f(x) \end{array}\right\}$$

$$G_2 = \left\{(x,y) : \begin{array}{l} 0 \le y \le b \\ 0 \le x \le f^{-1}(y) \end{array}\right\}$$

then $G_1 \cup G_2 \supset R$

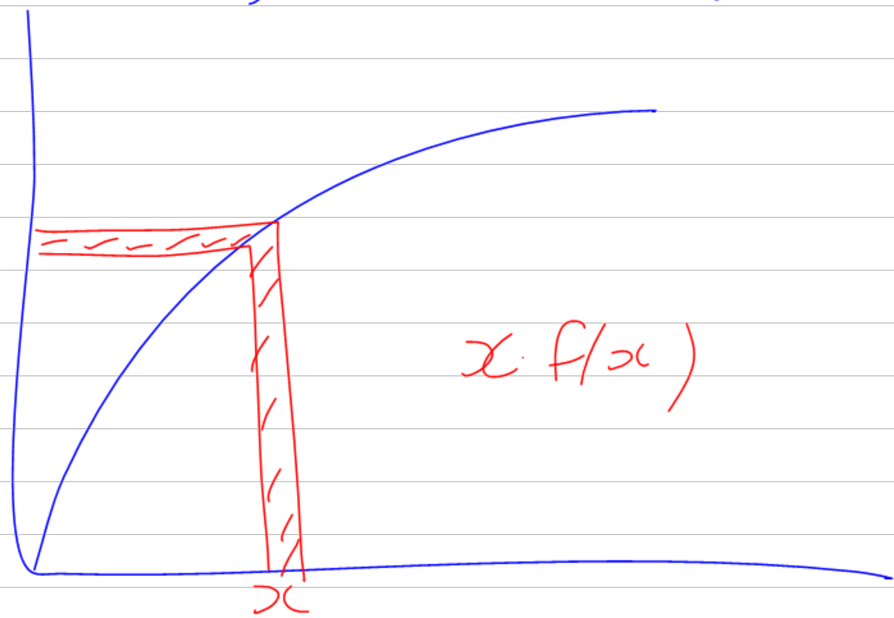
$$\underset{\displaystyle \int \chi_{G_1}}{\underset{\shortparallel}{Vol(G_1)}} + \underset{\shortparallel}{Vol(G_2)} \ge Vol(R) = ab$$

$$\int \chi_{G_1} = \int dx \underbrace{\int dy\, \chi_{G_1}(x,y)}_{f(x)} = \int dx\, f(x)$$

$$\int_0^b f^{-1}(y)\,dy = \int_0^c f^{-1}\circ f\,(dt f')$$

$$= \int_0^c x f' \, dx =$$

$$\int (xf' + f) = \int (xf)'$$

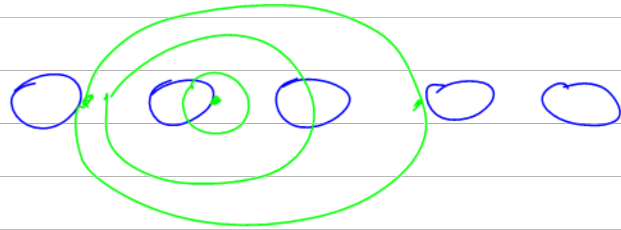


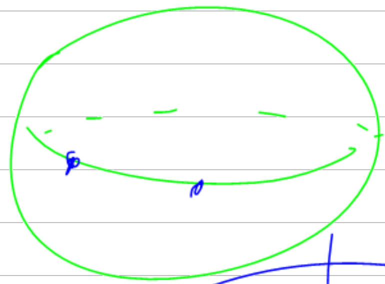
$$\underline{\int_U f} = \underline{\sum_i \int_{\mathbb{R}} \varphi_i f}$$

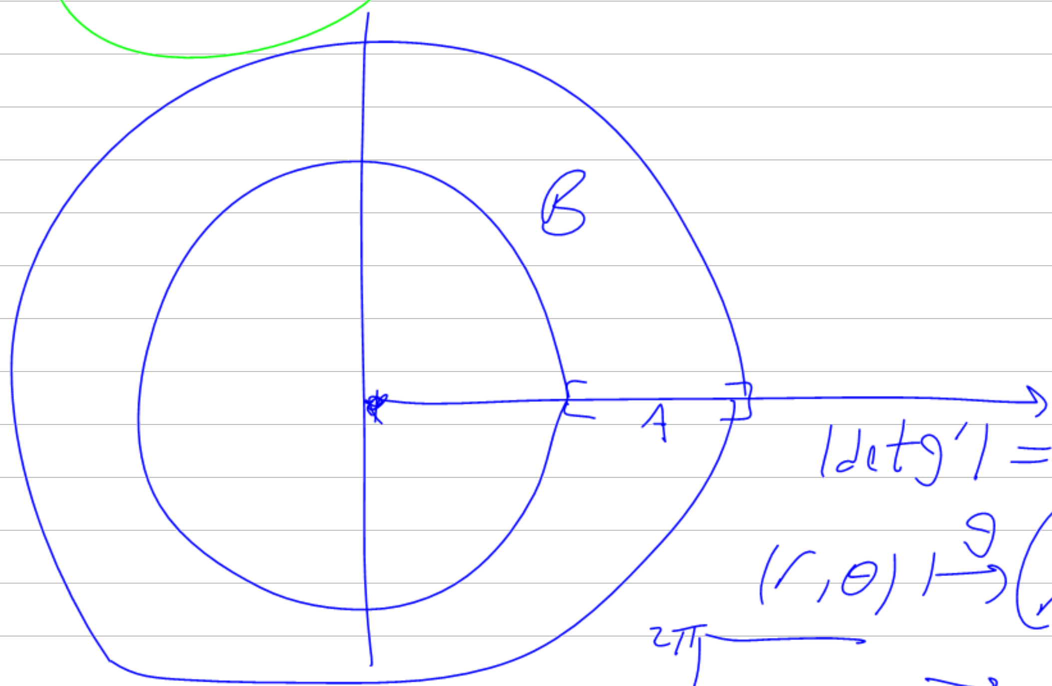
$$\mathbb{R} \subset \mathbb{R}^2$$

$$\bar{y} = g(\bar{x})$$

$$\begin{tikzcd}
\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \arrow[r, "\alpha_1"] \arrow[rrr, bend left, "g"] & \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_{j_1} \end{pmatrix} \arrow[r, "\alpha_{j_2}"] \arrow[rr, bend left] & \begin{pmatrix} x_1 \\ \vdots \\ x_{n-2} \\ y_{j_2} \\ y_{j_1} \end{pmatrix} \arrow[r] \cdots & \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}
\end{tikzcd}$$

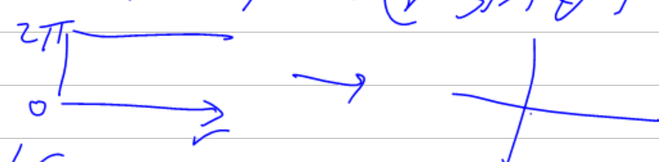






$$|\det g'| = r$$

$$(r,\theta) \overset{g}{\longmapsto} \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

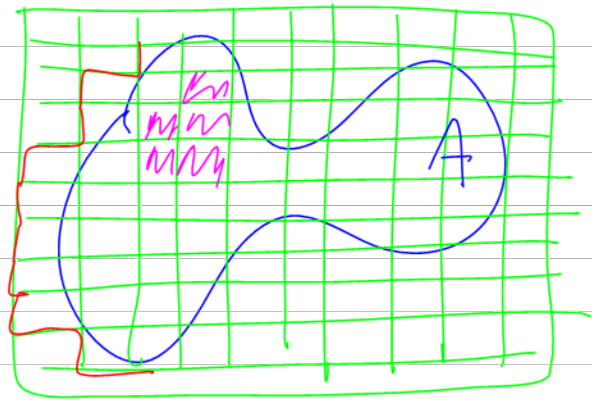


$$\mathrm{Vol}(B) = \int_{\mathbb{R}^2} \chi_B = \int \underbrace{\chi_B\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}}_{= \chi_A(r)} \cdot r \, dr \, d\theta$$

$$= \int r \chi_A(r) \, dr \, d\theta = 2\pi \int r \chi_A(r) \, dr$$

$$g(r,\theta,z) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

Q4



$\int \chi_A$ makes sense

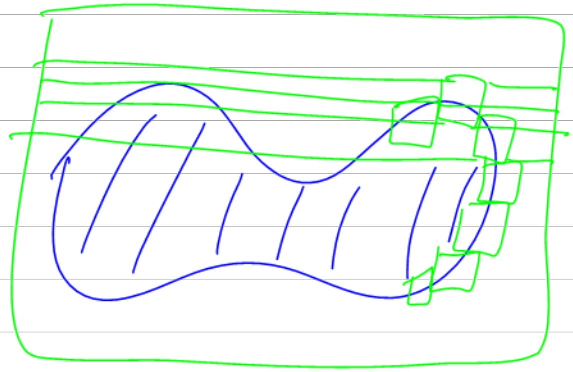
$\Updownarrow$

disc χ_A is meas-0

$\Updownarrow$

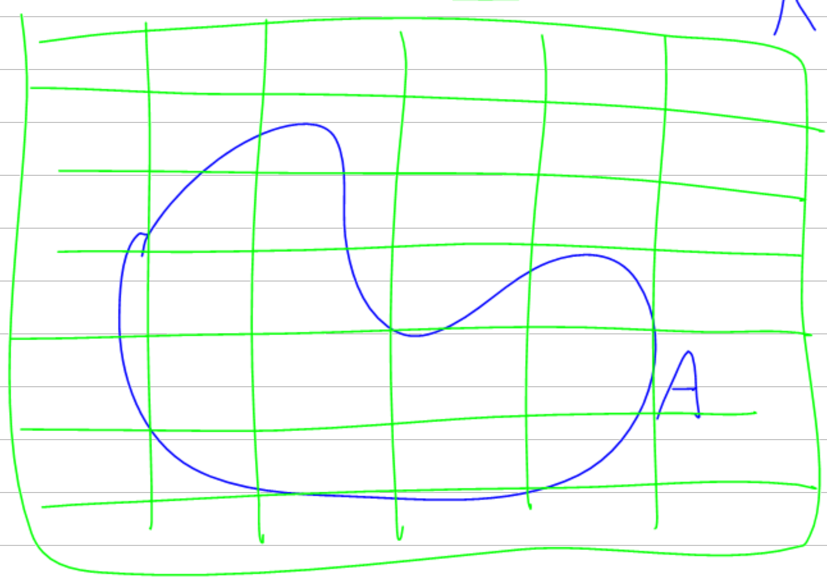
Bd A is meas-0.

Bd A is content-0



$\mathcal{R}$ = all rectangles in P that touch A

$\mathcal{R}'$ = all rectangles that are contained in A.



$$U(\chi_A, P) - L(\chi_A, P) < \frac{1}{257}$$

$$\| $$

$$\sum_{\substack{S \in P \\ S \not\subseteq A \\ S \not\subseteq A^c}} \mathrm{vol}(S) \cdot 1$$

$$V \oplus W = \left\{ (v, w) : \begin{array}{l} v \in V \\ w \in W \end{array} \right\}$$

$$(0_V, 0_W)$$

$$0_{\sigma T^*(V)} = \left(0_{\sigma^0(V)}, 0_{\sigma^1(V)}, \dots \right)$$

$$T(v_i, v_j) = a_{ij}$$

R_{abcd}

$$\bigoplus_{k=0}^{\infty} T^k(V) = \left\{ \left(\overset{T^0}{\overset{\Downarrow}{T_0}}, \overset{T^1}{\overset{\Downarrow}{T_1}}, \overset{T^2}{\overset{\Downarrow}{T_2}} \ldots \right) : \right\}$$

$$(T_0, T_1 \ldots) \cdot (T_0', T_1', \ldots) = (\qquad)$$

$$V^* = \langle a, b, c, d, \ldots, z \rangle$$

$$T^k(V) = \langle \varphi_{i_1} \otimes \varphi_{i_2} \otimes \ldots \varphi_{i_k} \rangle$$
$$= \langle \underbrace{aabc}_{k}, \underbrace{dror}_{k=4}, \ldots$$

$$T^*(V) = \bigoplus T^k(V)$$
$$= \text{l.c. of words of all lengths.}$$
$$= \left\{ 35ab + 2dror - \tfrac{1}{2}vid + \tfrac{22}{7}u \right\}$$

dror bar natan =

drorbarnatan

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "f /\!/ g"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$\cancel{f\circ g}$ $\quad g\circ f$

$x /\!/ f \qquad \sqrt{}(a+b) = \sqrt{a+b}$

$(a+b)(c+d) = \overline{a+b}\;\overline{c+d}$

$\{ \quad \}$

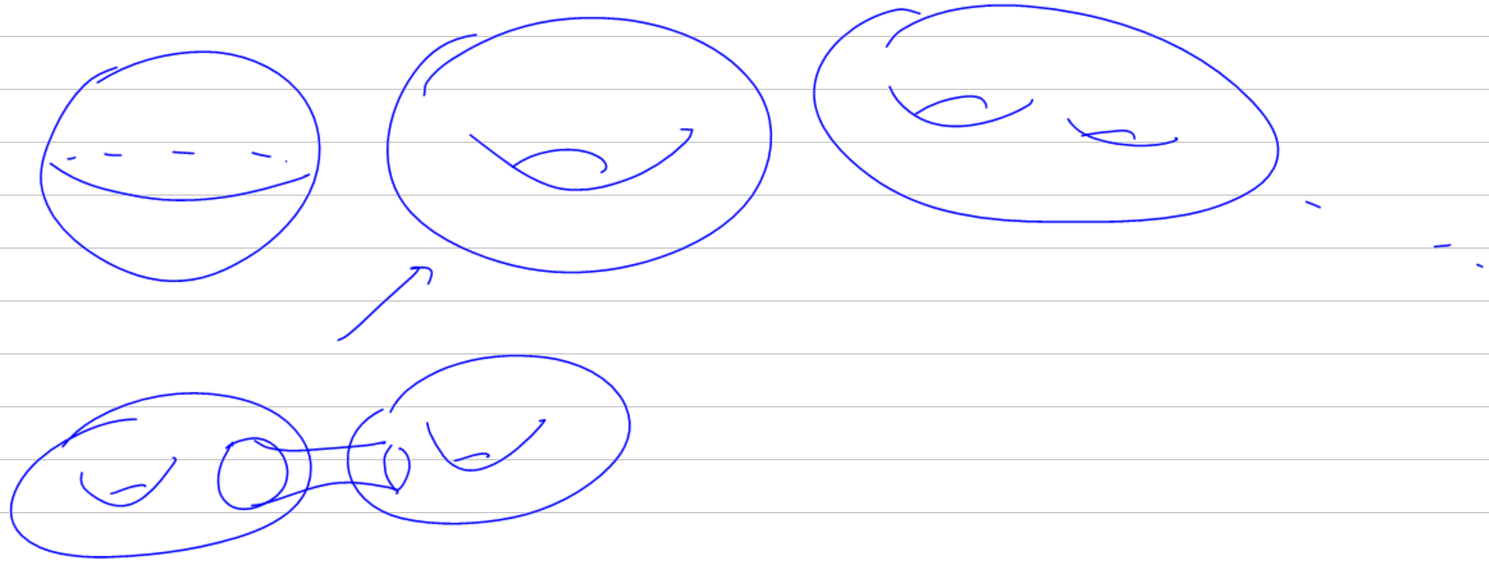
$(((\quad)) \quad (\quad))$

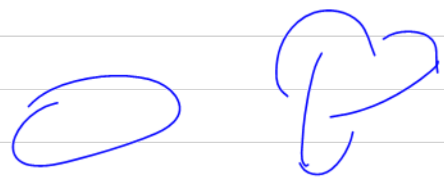
$\underline{v \in V} \qquad v = \sum a_i v_i$

(a_i)

$$\int f = \int f\,dx = \int f$$

$$\int \mathcal{D}q\, e^{-\frac{i}{\hbar}\int (\frac{1}{2} m (\dot{q})^2 + V(q))}$$



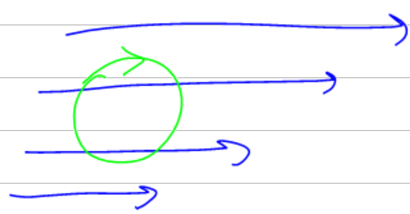


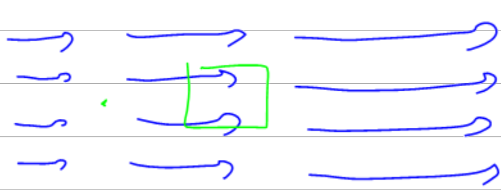
$\subset \mathbb{R}^3$

$F : \mathbb{R}^3 \to \mathbb{R}^3$

$$\int_{\partial\Sigma} F\cdot T\,ds = \int_{\Sigma} \underline{\mathrm{curl}(F)}\cdot \underline{\vec{n}}\,dA$$

$$(\nabla \times F)\cdot \vec{n}\,dA$$

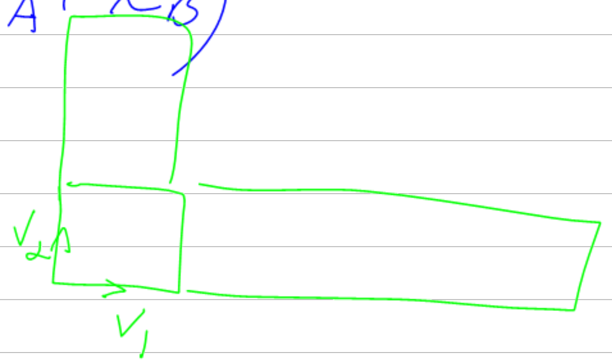


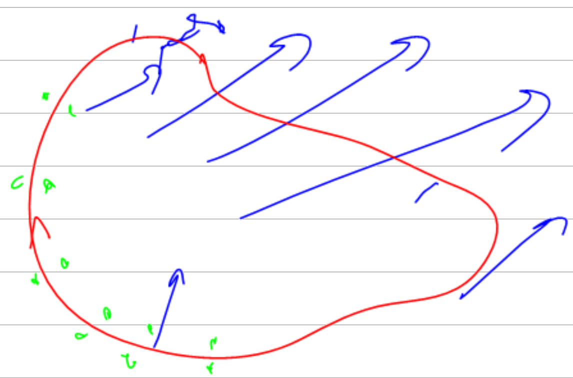


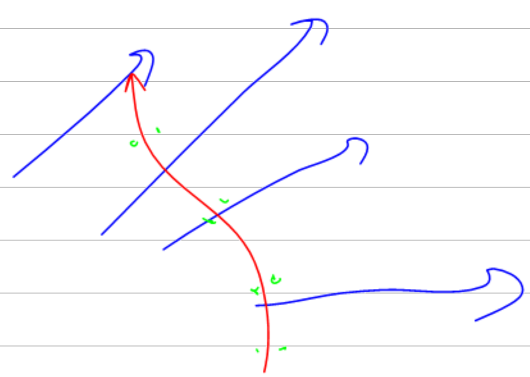
$A \cap B = \emptyset$

$$\int_{A \cup B} F = \int_A F + \int_B F$$

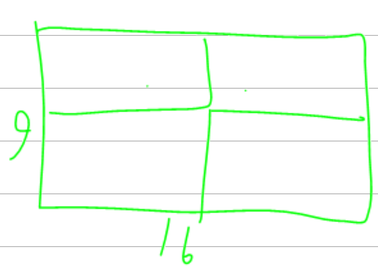
$$\int F \cdot \chi_{A \cup B} = \int F(\chi_A + \chi_B)$$







1. Check if it is possible to also record the whiteboard.



2. Try harder not to assign HW on material not yet covered.

3 change HW cycle to Mon → Mon.

$$(\lambda\wedge\eta)\underbrace{(u_1\ldots u_{k+l})} = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} (-1)^{\sigma}\,\lambda(u_{\sigma 1},\ldots u_{\sigma k})\cdot\eta(\qquad)$$

$k=2 \quad l=1$

123 $\quad +\lambda(u_1,u_2)\eta(u_3)$

132

~~213~~ $\quad -\lambda(u_2 u_1)\eta(u_3)$

231

~~312~~

~~321~~

Aside $\quad \underline{n}^k_{nd} = \{1\le i_1\le i_2\le i_3\le\ldots\le i_k\le n\}$

$n=5 \quad k=7$

$$\overbrace{\boxed{*2**3*45***}}^{7+5-1}$$

$1223555 \longleftrightarrow 1*2**3*45***$

$$|\underline{n}^k_{nd}| = \binom{n+k-1}{k}$$

Chars in green box: $n+k-1$ of those k are $*$'s.

$$W_I = \varphi_{i_1}\wedge\varphi_{i_2}\wedge\ldots\wedge\varphi_{i_k}\in\Lambda^k(V)$$

$$\mathrm{Alt}\circ\mathrm{Alt}=\mathrm{Alt}$$

$$\mathrm{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma \underbrace{T \circ \sigma^*}_{(-1)^\sigma T}$$

$$\omega_I = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$$

$$(\varphi_{i_1} + \varphi_{i_2}) \wedge \varphi_{i_2}$$

$$(\varphi_{i_1} + \varphi_j)$$

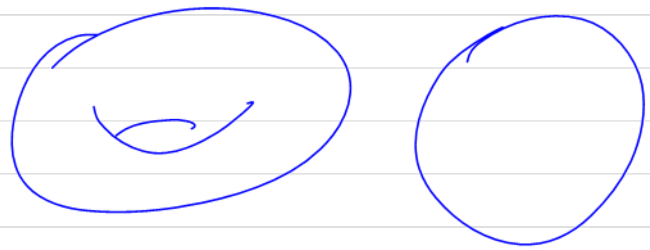
$$(\omega_{12} + \omega_{34}) \wedge (\omega_{12} + \omega_{34})$$
$$= 2\omega_{1234} \qquad \gamma = \omega_I$$

F a v.f.

$D_F : \{\text{functions}\} \circlearrowleft$

$$D_F \circ D_G - D_G \circ D_F =$$

$$Fg = D_F g$$



$A \vee \neg A$

0.0000000

```
float x;
if (x=0) { - - - }
      else {     }
```

$\sin(x)$

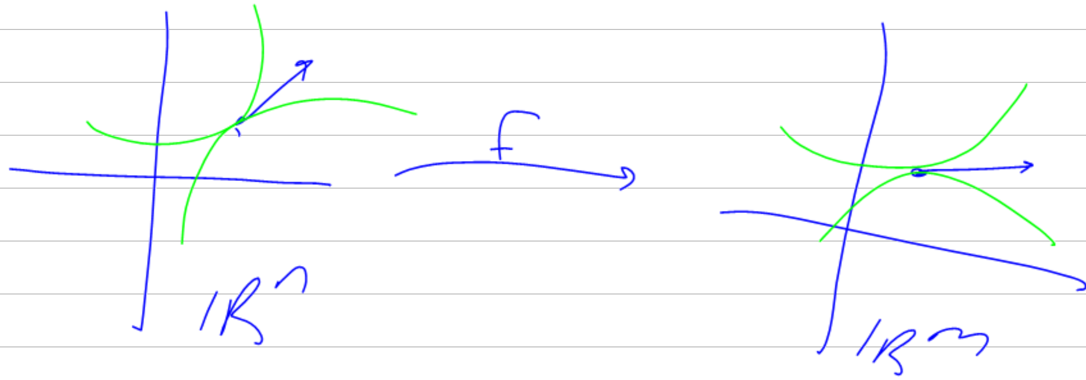
$x_n \in A_n$

$$\mathcal{A} = \{A \subset \mathbb{R} : A \neq \emptyset\}$$

$f : \mathcal{A} \to \mathbb{R}$ s.t. $f(A) \in A$

1.4142135 _ _ _

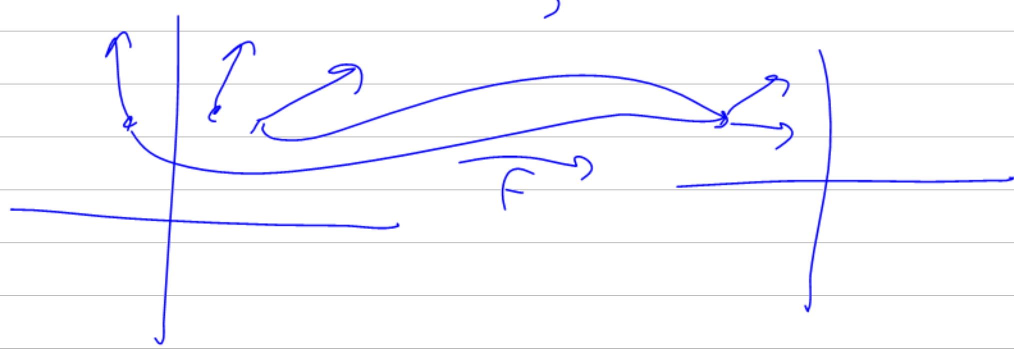
$$f : \mathbb{R}^n \to \mathbb{R}^m$$

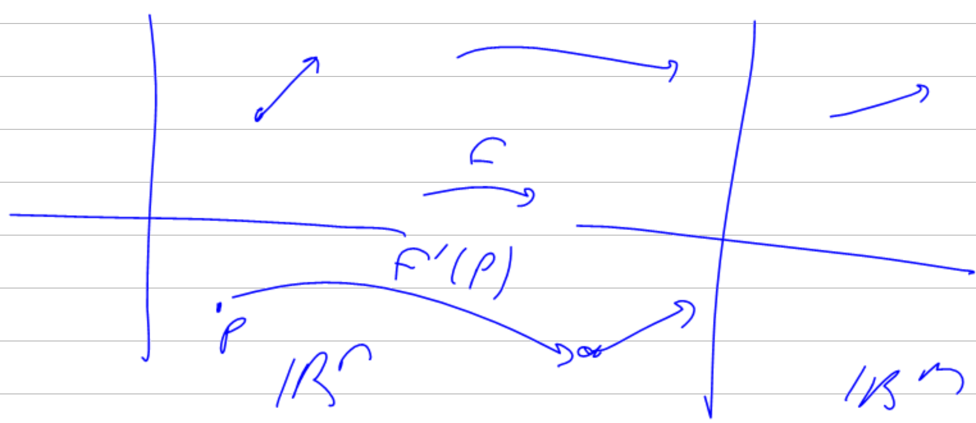


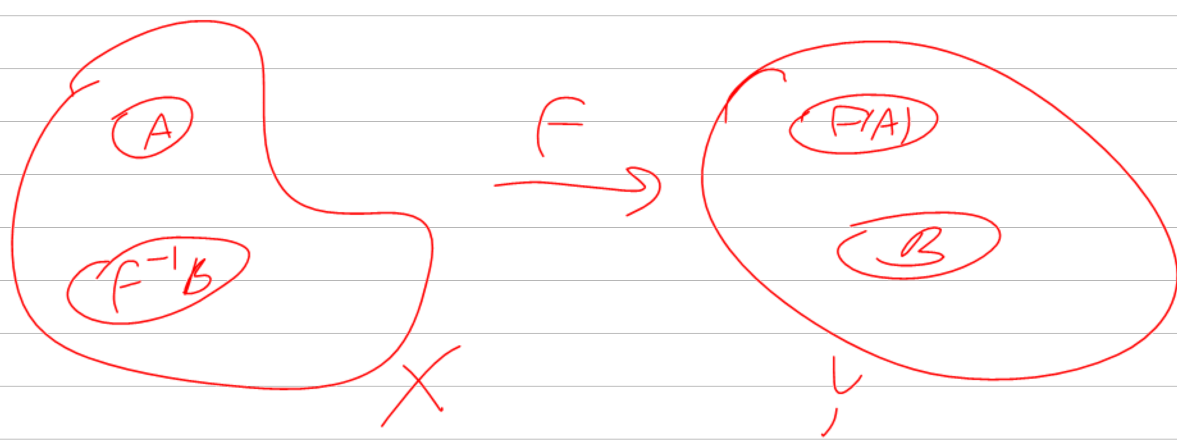
$$F_*(p,v) \neq (F(p), v)$$

$$F_*(p,v) = (F(p), F'(p)v)$$

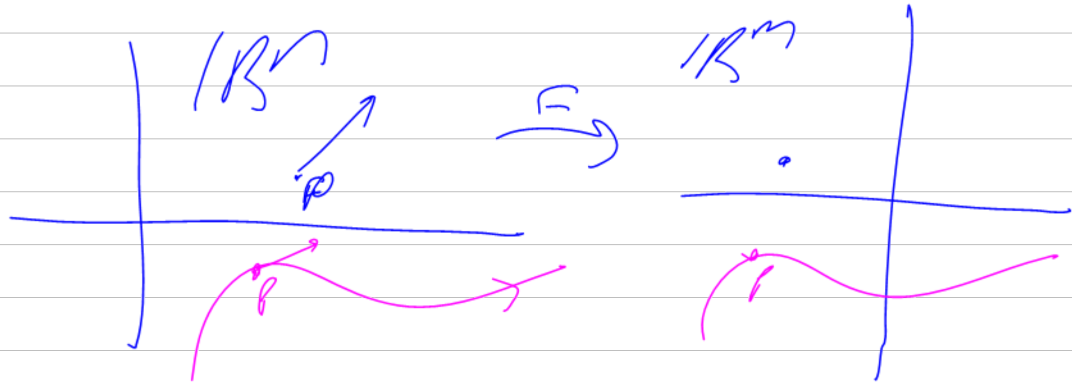
$$F \neq F \qquad D_\xi F$$

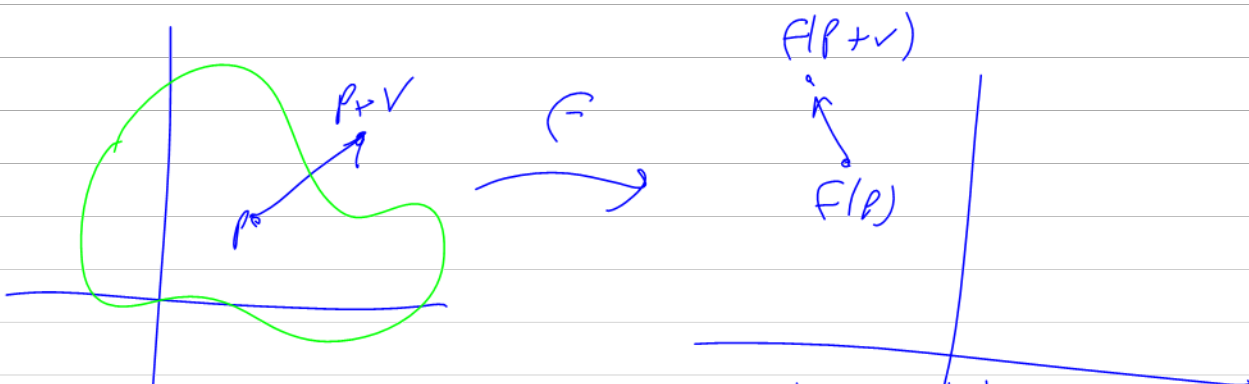






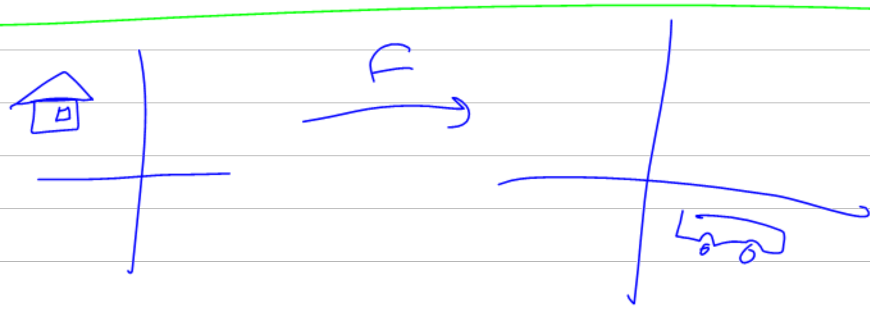
$$\int_0^{2\pi} d\theta \int_0^{\infty} dr \cdot r \cdot f(r) = 2\pi \int_0^{\infty} dr\; r \cdot f(r)$$

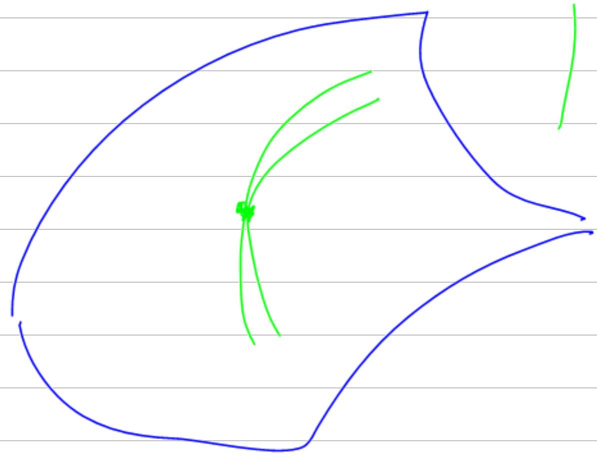




$$F_* \xi = (F(p), F(p+v) - F(p))$$

$$f(p+tv) = f(p) + t\,\underbrace{f'(p)\cdot v} + o(t)$$





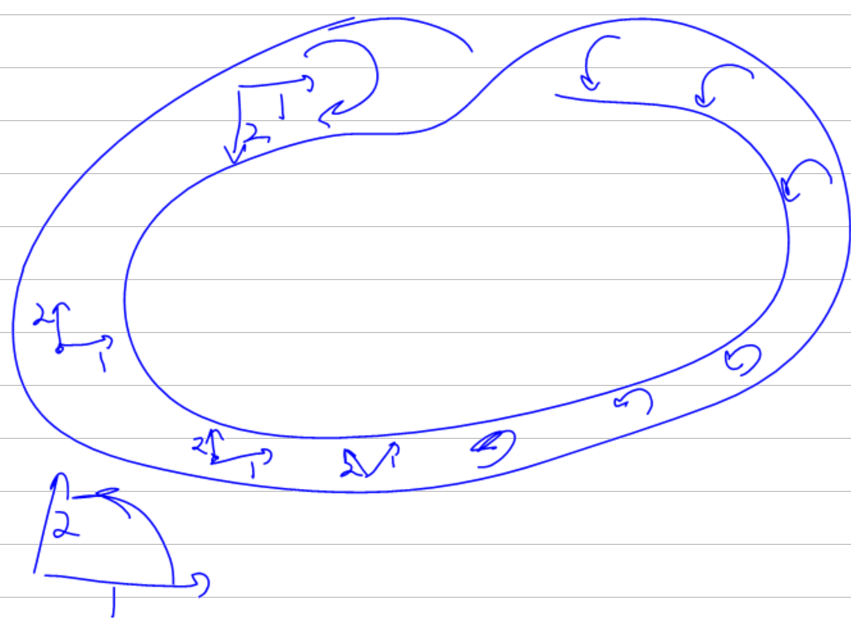
$$|\gamma_1(t)-\gamma_2(t)| \in o(t)$$

$$\sigma : \underline{n} \longrightarrow \underline{n}$$

$$(v_1 \dots v_n) \longmapsto (v_{\sigma 1} \dots v_{\sigma n})$$

$$\begin{tikzcd} \underline{n} \arrow[r, "\sigma"] \arrow[dr, "\sigma^* V"'] & \underline{n} \arrow[d, "V=(v_1 \dots v_n)"] \\ & V \end{tikzcd}$$

$$\sigma^* V = (v_{\sigma 1} \dots v_{\sigma n})$$



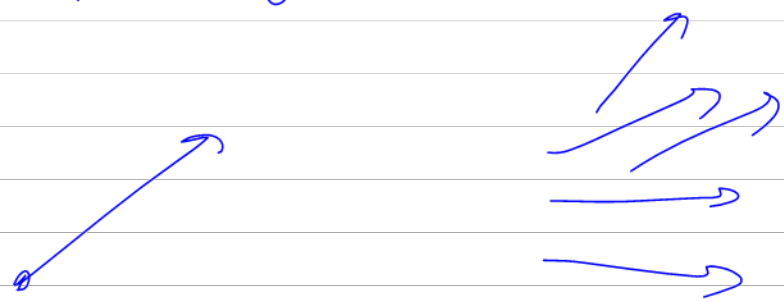
ω 2-tensor

η 1-tensor

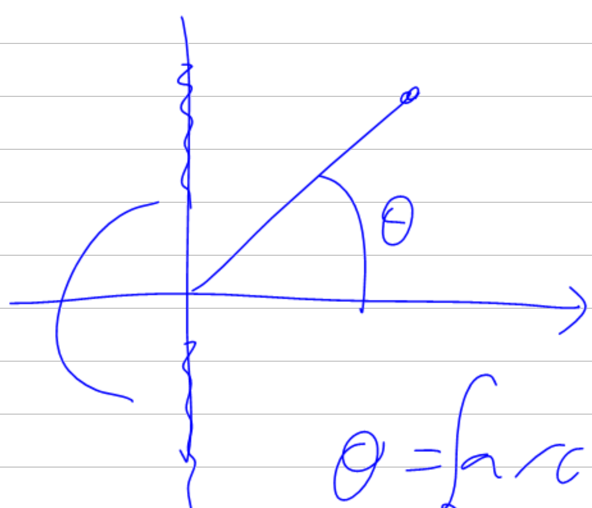
$$(\omega\cdot\eta)(x,y,z)=\omega(x,y)\eta(z)$$

$$(\omega\cdot\eta)(x,z,y)=\omega(x,z)\cdot\eta(y)$$

$$(\omega\wedge\eta)=\sum_{\sigma}(-1)^{\sigma}\omega(\ldots)\eta(\ldots)$$



$$(\omega\wedge\eta)(p)=\omega(p)\wedge\eta(p)$$



$\theta\in(-\pi,\pi)$

$\arctan z\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\theta=\begin{cases}\arctan\frac{y}{x} & x>0\\ \pi/2 & x=0,\ y>0\\ -\pi/2 & x=0\ y<0\\ & y<0\end{cases}$$

on $\mathbb{R}^3$

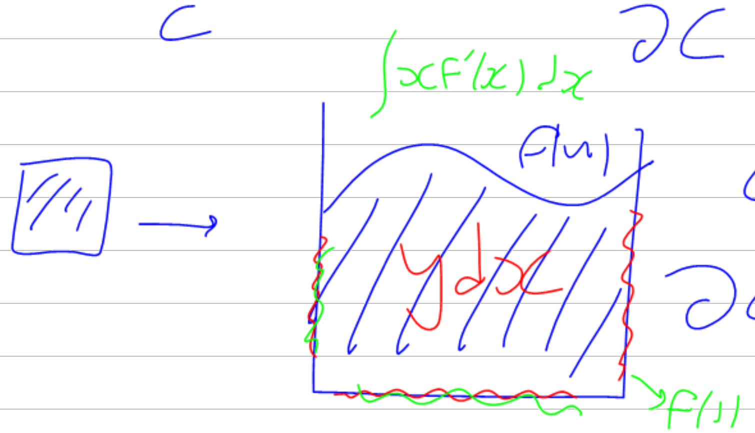
$$\begin{array}{ccccccc} \Lambda^0 & \xrightarrow[\text{grad}]{d} & \Lambda^1 & \xrightarrow[\text{curl}]{d} & \Lambda^2 & \xrightarrow[\text{div}]{d} & \Lambda^3 \\ \parallel & & \parallel & & \parallel & & \parallel \\ \text{Fnctns} & & \text{V.F.} & & \text{V.F.} & & \text{Fnctns} \\ & & & & & & g\,dx\wedge dy\wedge dz \end{array}$$

$$1 \qquad 3 \qquad 3 \qquad 1$$

$$\int_C dx\wedge dy = \int_{[0,1]^2} C^*(\underset{\uparrow\, u}{dx}\wedge \underset{\uparrow\, d(f(u)v)}{dy}) = \int_{[0,1]^2} f(u)\,du\wedge dv$$

$$d(f(u)v) = \underline{f'(u)v\,du} + f(u)\,dv$$

$$\int_C dx\wedge dy = \int_C d(-y\,dx) = -\int_{\partial C} y\,dx$$



$C : (u,v) \longmapsto (u, f(u)v)$

$\partial C = -C_{(1,0)} + C_{(1,1)} + C_{(2,0)} - C_{(2,1)}$

$C_{(1,0)} : I_t \longrightarrow \mathbb{R}^2$

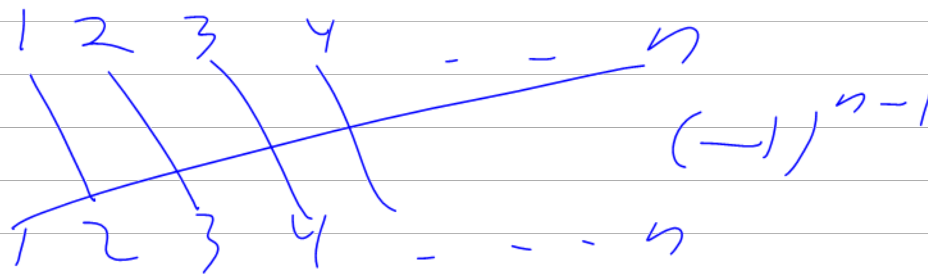
$t \longmapsto (0,t) \overset{C}{\longmapsto} (0, f(0)t) \xrightarrow{\int} 0$

$C_{(1,1)} : t \longmapsto (1,t) \longmapsto (1, f(1)t) \xrightarrow{\int} 0$

$$C_{(2,0)}: t \longmapsto (t,0) \longmapsto (t,0) \xrightarrow{\int} 0$$

$$C_{(2,1)}: t \longmapsto (t,1) \longmapsto (t, f(t)) \xrightarrow{\int} -\int f(t)\,dt$$

$$\int_{C_{(1,0)}} -y\,dx = -\int_{C_{(1,0)}} f(0)\cdot t \; d\cdot 0 = 0$$



$$\begin{tikzcd}
\Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3) \arrow[d, "\beta"] \\
\text{V.F} \arrow[u, "\alpha"] \arrow[r, "\operatorname{div}\checkmark"] & \text{Functions}
\end{tikzcd}$$

1. There is a bijection $\alpha : \left\{\begin{matrix}\text{smooth} \\ \text{V.F} \\ \text{on } \mathbb{R}^3\end{matrix}\right\} \to \Omega^2(\mathbb{R}^3)$

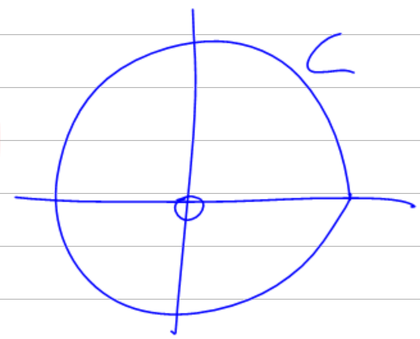
defined by $\alpha\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = F_1\, dx_2 \wedge dx_3 + F_2\, dx_3 \wedge dx_1 + F_3\, dx_1 \wedge dx_2$

2. . . . . $\beta : \Omega^3(\mathbb{R}^3) \to \left\{\begin{matrix}\text{smooth} \\ \text{Functions}\end{matrix}\right\}$

$\beta : f\, dx_1 \wedge dx_2 \wedge dx_3 \longrightarrow f$

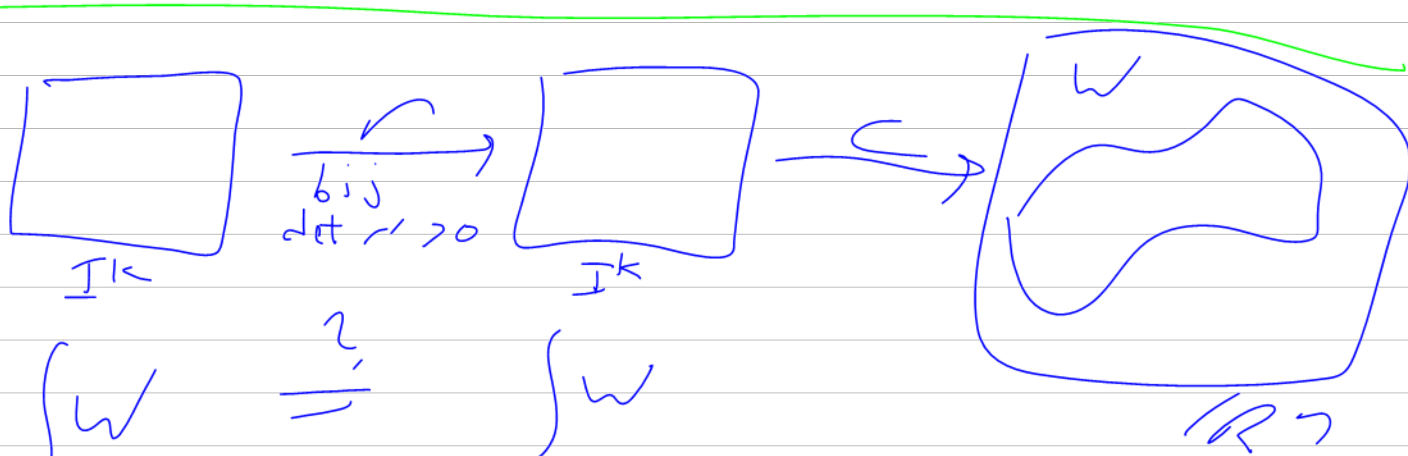
<u>Claim</u> $\beta \circ d \circ \alpha = \operatorname{div}$ <u>Pf</u> . . . .

ω exact, $\partial c = 0 \Rightarrow \int_c \omega = \int_c d\lambda = \int_{\partial c} \lambda = 0$



$c: t \mapsto \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}$

$\int_c \omega = 2\pi \neq 0$



$\int_{c \circ r} \omega \overset{?}{=} \int_c \omega = \int_{I^k} c^* \omega = \int_{I^k} f$

$c^*\omega = f\, dx_1 \ldots dx_k$

$\| $

$\int_{I^k} (c \circ r)^* \omega = \int_{I^k} r^*(c^* \omega) = \int_{I^k} r^*(f\, dx_1 \ldots dx_k)$

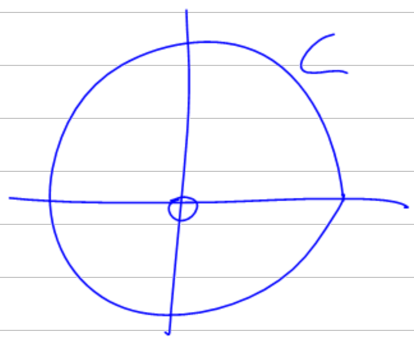
$= \int \underbrace{(f \circ r)(\det r')}\, dx_1 \ldots dx_k$

$L^* \phi = \phi A$

$$\begin{array}{ccc} \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) \\ \updownarrow\ \uparrow & & \updownarrow\ \downarrow \\ \text{functions} & \xrightarrow{\text{grad}} & \text{vector fields} \end{array}$$

8d

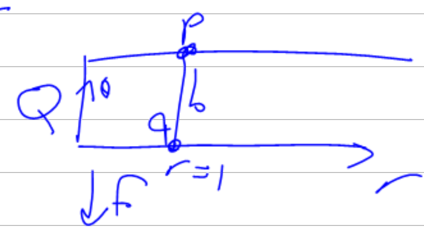
ω exact, $\partial C = 0 \Rightarrow \int_C \omega = \int_C d\lambda = \int_{\partial C} \lambda = 0$

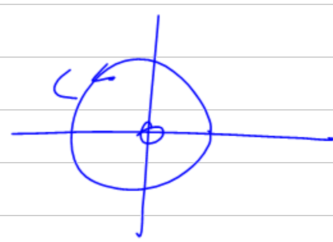


$C : t \mapsto \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}$

$\int_C \omega = 2\pi \neq 0$

$$\int_C \omega = \int_{f_* b} \omega$$

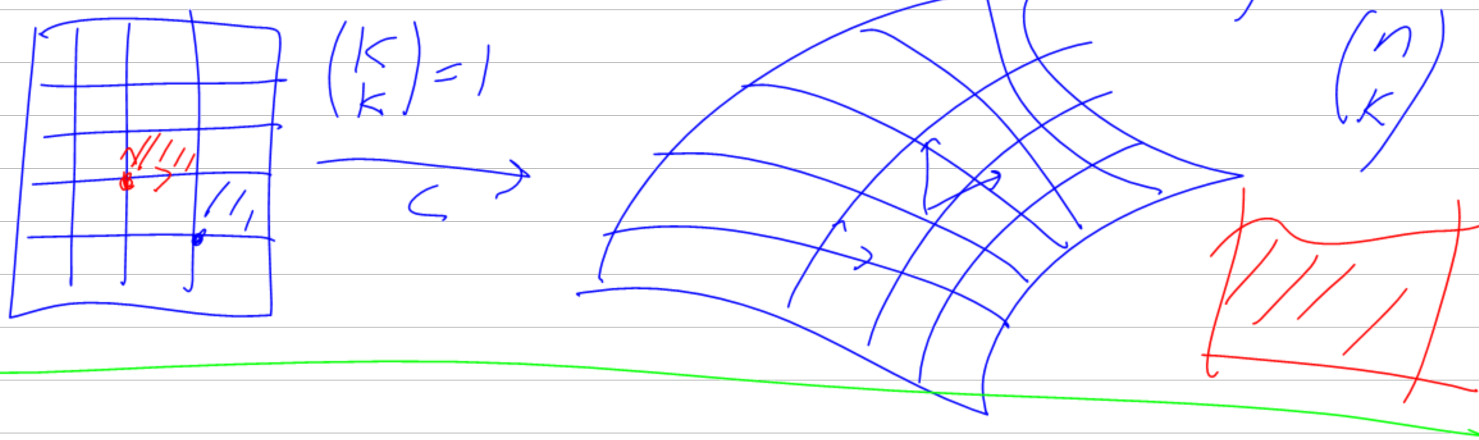




$$= \int_b f^* \omega = \int_b d\theta = \int_{\partial b} \theta = \theta(p) - \theta(q) = 2\pi$$

$c^* \omega = f\, dx_1 \ldots dx_k$ $\qquad \omega \in \Omega^k(\mathbb{R}^n)$

$\binom{k}{k} = 1 \qquad \xrightarrow{\;c\;} \qquad \binom{n}{k}$



$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "L"] \arrow[rrd, bend right, "L^*\phi = \phi \circ L"'] & \mathbb{R}^m \arrow[rd, "\phi"] & \\
 & & \mathbb{R}
\end{tikzcd}$$

$() = v \mapsto Av$

$() = v$ $\qquad$ $() = w \mapsto \phi \cdot w$

$() = v \mapsto \underbrace{(\phi \cdot A)}_{(\_\_\_\_)} v$

$$d\omega(\xi_1, \xi_2) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left(\int_{\partial \square} \omega \right)$$

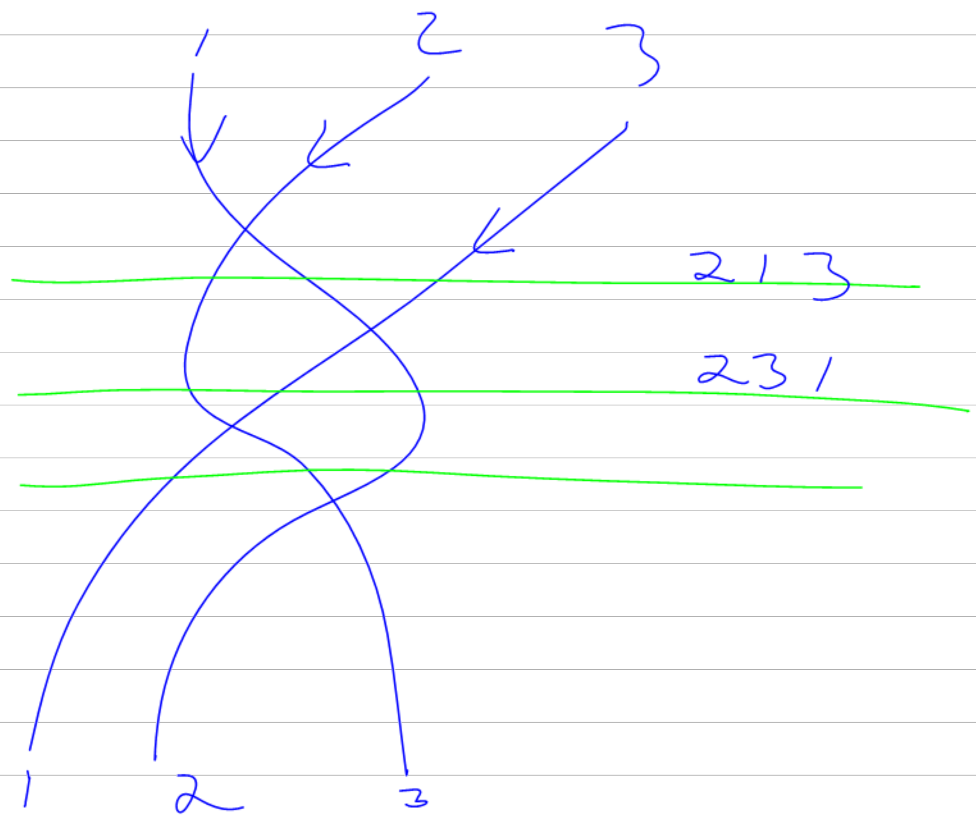
$\Lambda^k(V) : \quad (v_1, \ldots, v_k) \longmapsto \mathbb{R}$

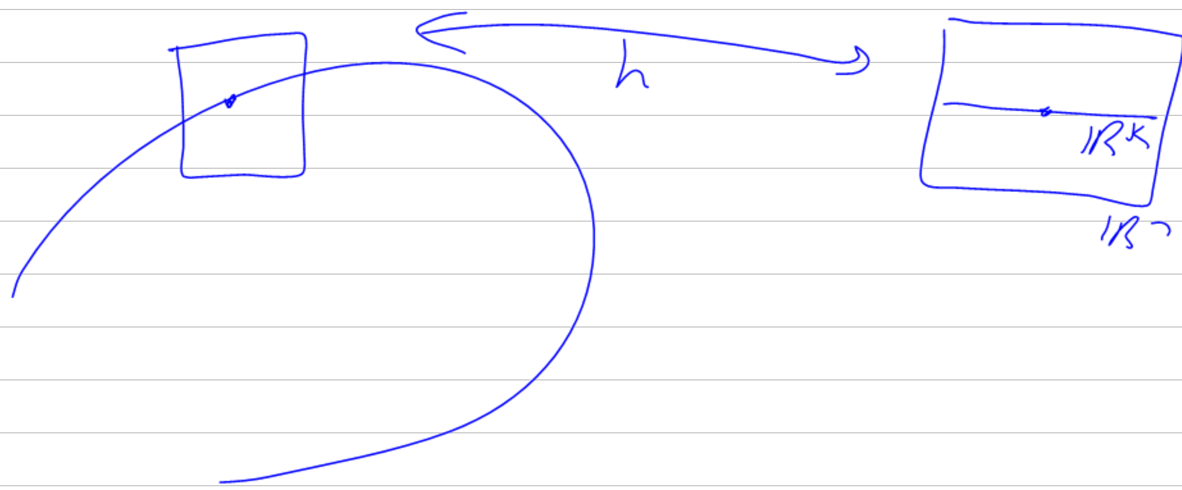
$k = 0 \qquad () \longmapsto \mathbb{R}$

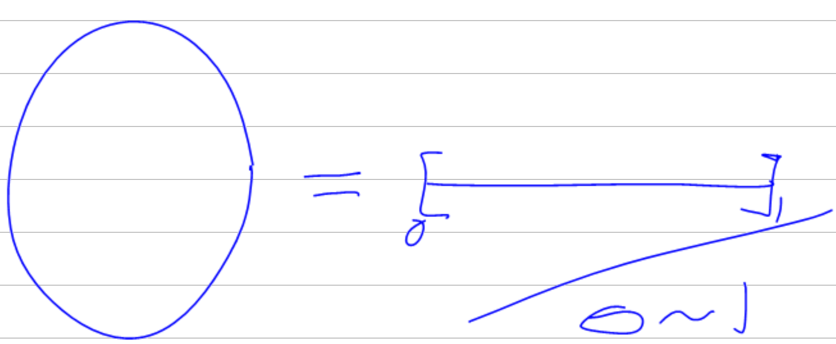
$\Lambda^{\overset{0}{\overset{\parallel}{k}}}(V) \sim \mathbb{R}$

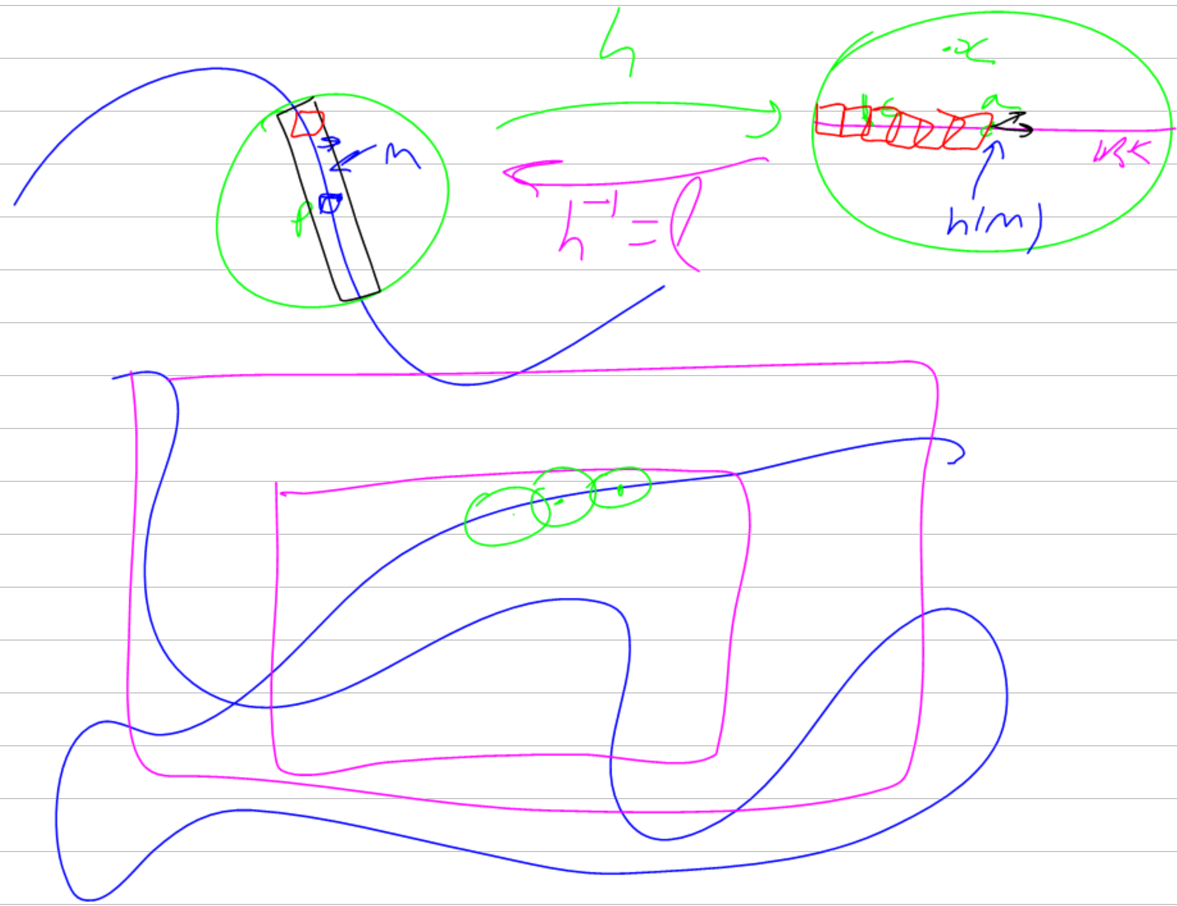
$\Lambda^k(\mathbb{R}^n)$ $\quad \mathbb{R}^n \ni p \longrightarrow \Lambda^k(T_p\mathbb{R}^n)$

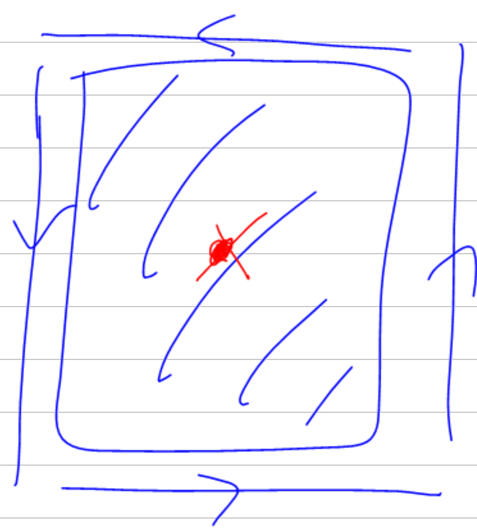
$k=0 \quad \mathbb{R}^n \ni p \to \Lambda^0(\_) \sim \mathbb{R}$





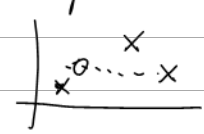




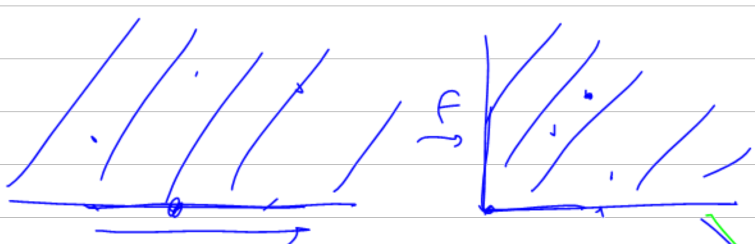


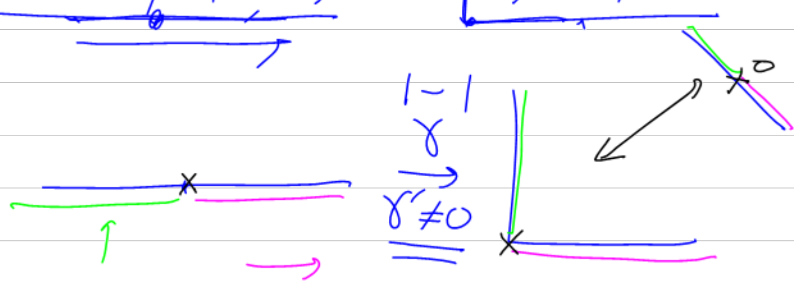
Claim A 1-1 cont

$\beta : \mathbb{R} \to \mathbb{R}$ is monotone



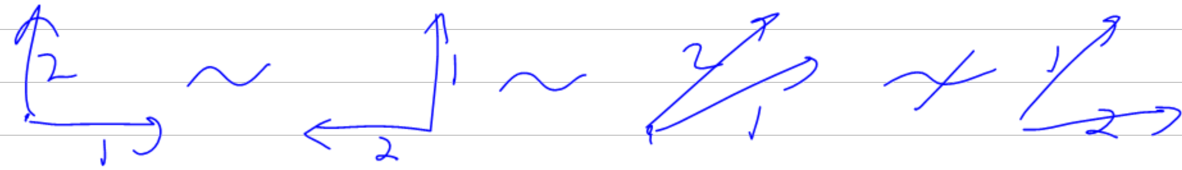
$f(\bar{A}) \subset \overline{f(A)}$



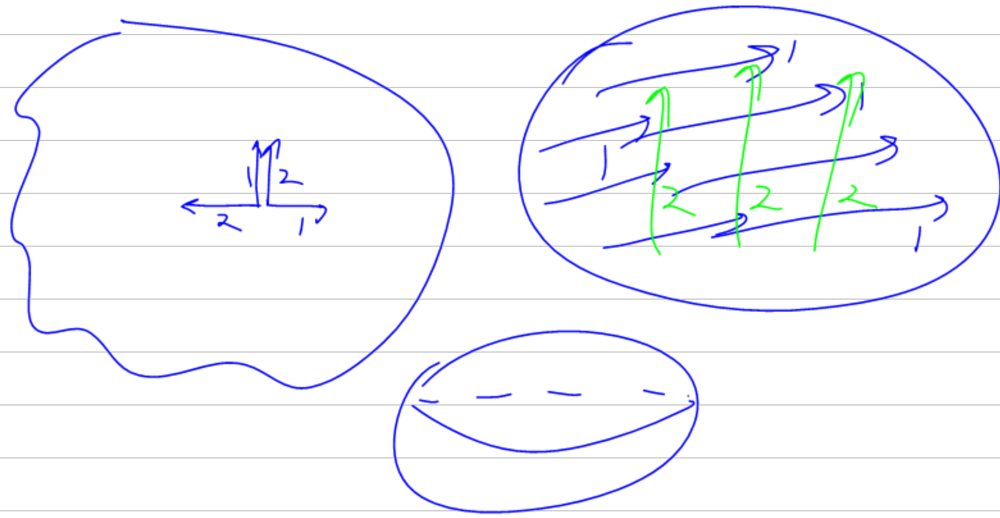


$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h} = (*, 0)$$

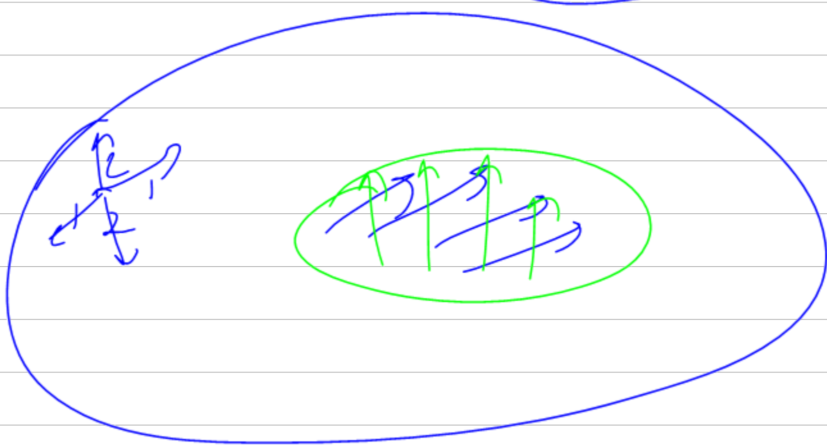
$(0, *)$



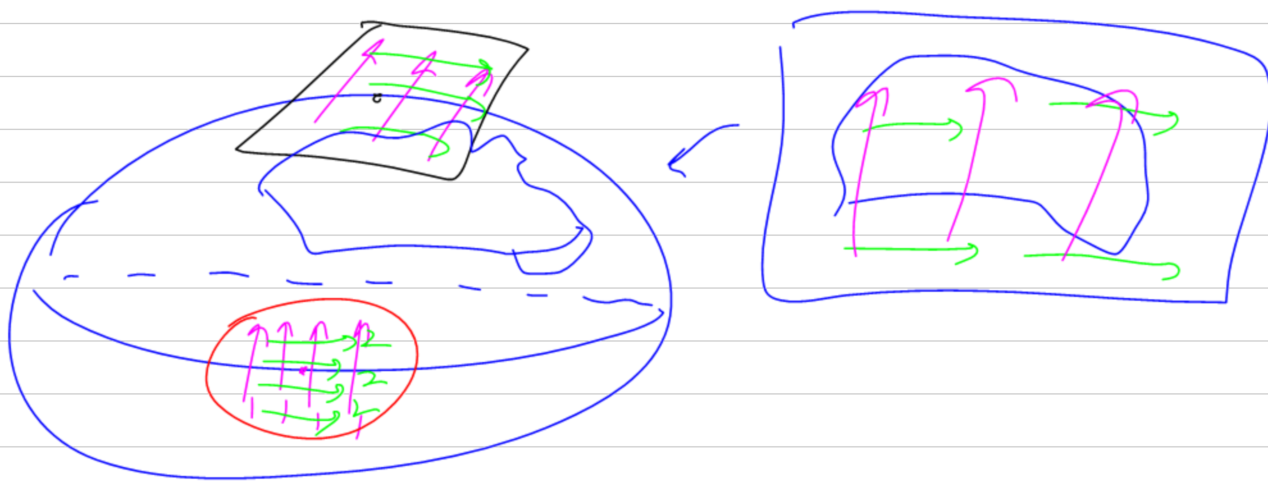
2
1
~
1
2
~
2
1
1
2



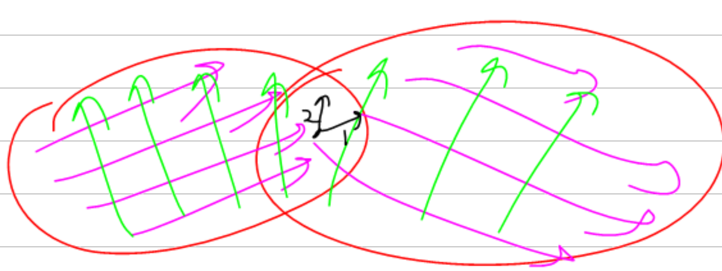
1
2



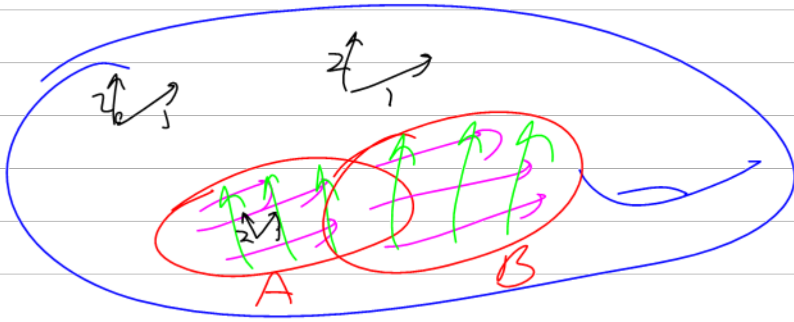
2
1

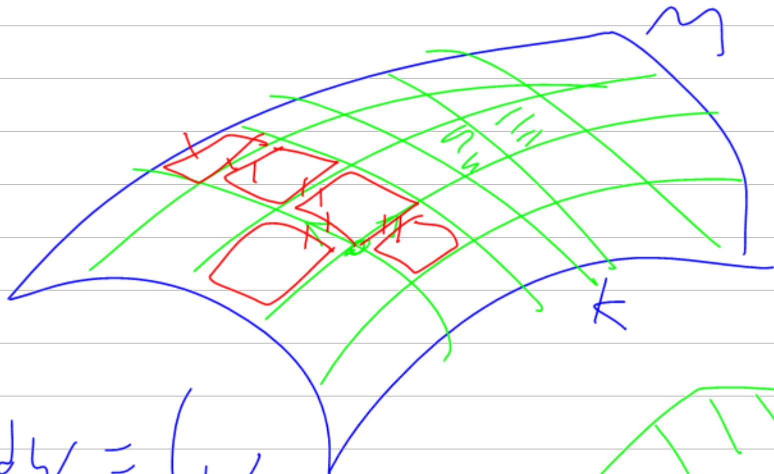


1
2



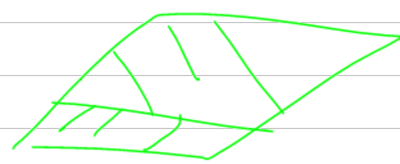
2
1



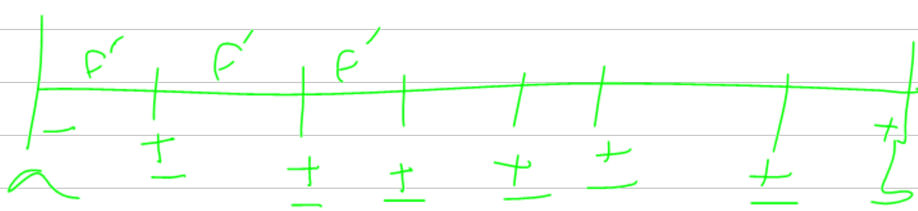


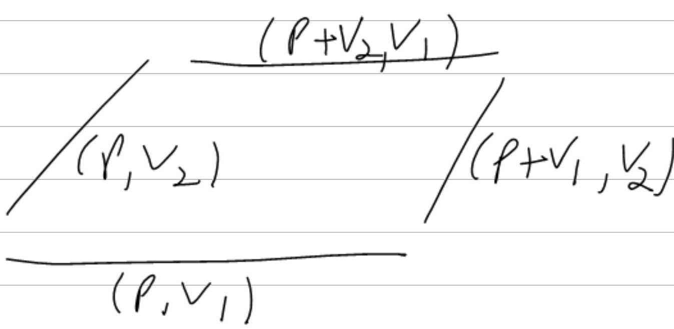
$\subset \mathbb{R}^n$

$$\int_M d\omega = \int_{\partial M} \omega$$



$$\int_{[a,b]} F' = \int_{\partial[a,b]} F = F\Big|_a^b$$





$\xi_i = (P, V_i)$

$$\int_Q d\omega = \int_{\partial Q} \omega$$

$$d\omega(\xi_1, \xi_2) \sim \begin{array}{l} \pm \omega_p(v_1) \\ \pm \omega_p(v_2) \\ \pm \omega_{p+v_1}(v_2) \\ \pm \omega_{p+v_2}(v_1) \end{array}$$

$$\begin{array}{ccc} \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) \\ \wr & & \wr \\ & \xrightarrow{\text{curl}} & \end{array}$$

$F'(x) : T_x M \to T_x N$

$$M^k \xrightarrow{F} N^k$$

ω $\qquad \alpha\omega = F^* \lambda$ $\qquad \lambda$

$\alpha : M \to \mathbb{R}$

$(u_1 \ldots u_k)$ $\qquad\qquad (w_1 \ldots w_k)$

the change of basis from $(F_* u_1 \ldots F_* u_k)$ to $(w_1 \ldots w_k)$

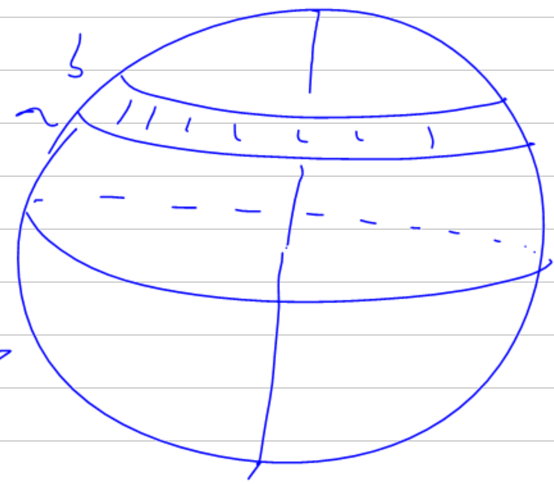
has a pos. det.

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "z = a+ib"] \arrow[d, equal] & \mathbb{C} \arrow[d, equal] \\
\mathbb{R}^2 \arrow[r, "\begin{pmatrix} a & -b \\ b & a \end{pmatrix}"] & \mathbb{R}^2
\end{tikzcd}
$$

$$\det\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2+b^2 \geq 0$$

$$w = \underbrace{\left(\frac{x\,dy - y\,dx}{x^2+y^2}\right)}_{} \wedge dz$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{1-z^2}\cos\theta \\ \sqrt{1-z^2}\sin\theta \\ z \end{pmatrix}$$

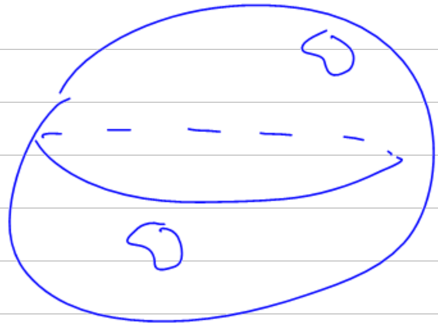


$$[0,2\pi]_\theta \times [-1,1]_z \xrightarrow{c}$$

$$c^* w = d\theta \wedge dz$$

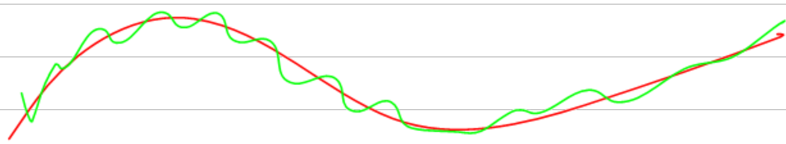
$$w = d\theta \wedge dz$$

$$\int_{[0,2\pi]_\theta \times [a,b]_z} w = 2\pi(b-a)$$

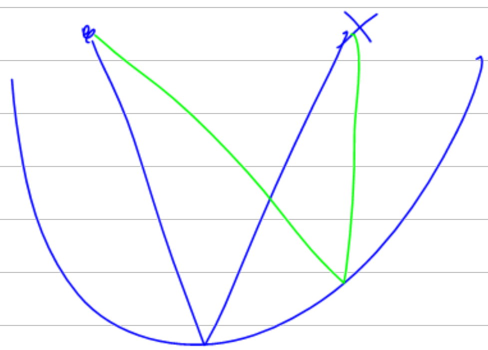


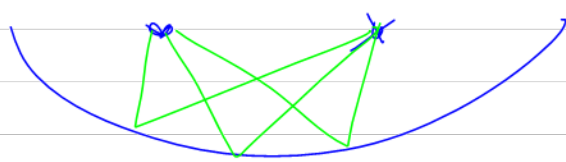
$f: \mathbb{R}^n \to \mathbb{R}^m$

$(\partial_i f_j) = f'$



Huygens principle





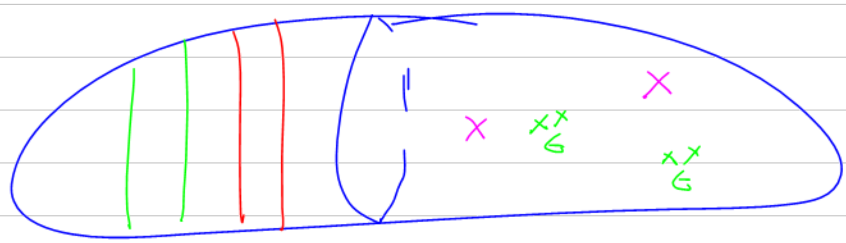
$$x\,dy\,dz + y\,dz\,dx + \underline{\underline{z\,dx\,dy}}$$

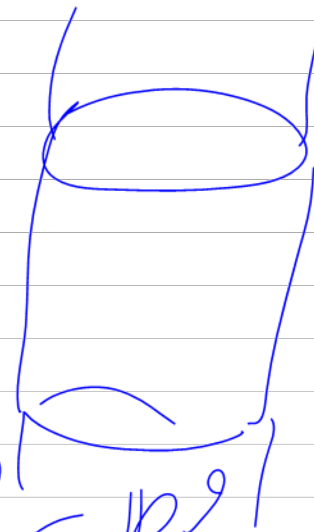
$$= \left(\frac{x^2}{z} + \frac{y^2}{z} + z\right) dx\,dy$$

$$= \left(\frac{1 - \cancel{z^2} + \cancel{z^2}}{z}\right) dx\,dy$$

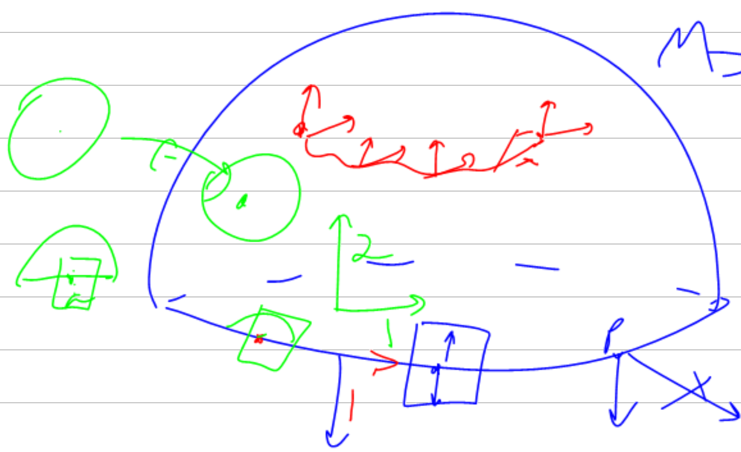
$$\frac{x\,dy - y\,dx}{x^2+y^2}\wedge dz = \frac{x^2/z + y^2/z}{x^2+y^2}\,dx\,dy$$

$$= \frac{1}{z}\,dx\,dy$$



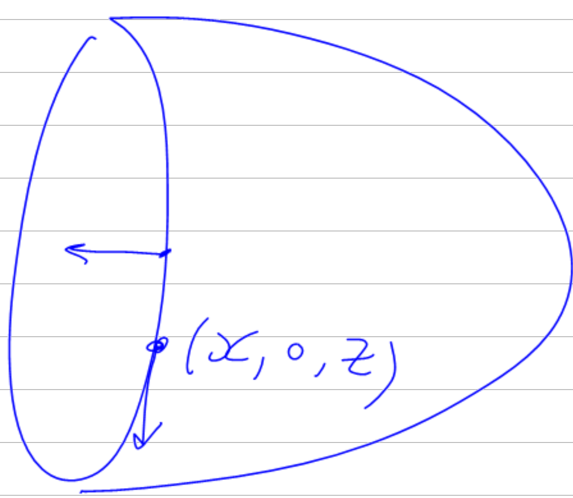


$$SO(3) = \left\{ A : \begin{array}{c} A^T A = I \\ \det A = 1 \end{array} \right\} \subset \mathbb{R}^9$$



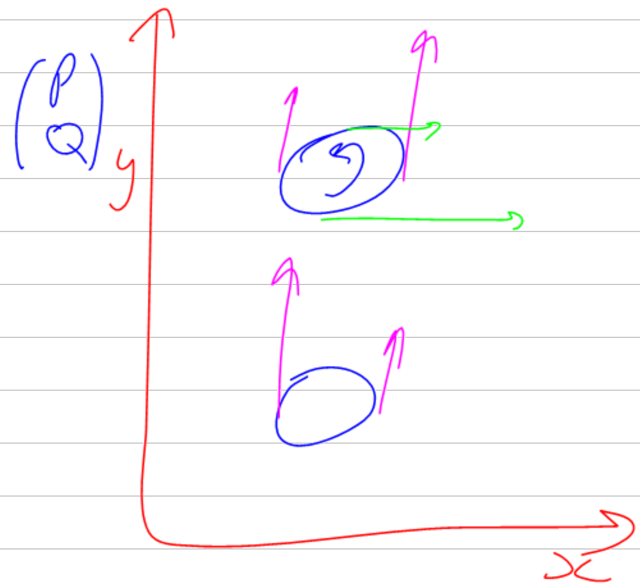
$$M = \left\{ (x,y,z) : \begin{array}{c} x^2+y^2+z=1 \\ z \geq 0 \end{array} \right\}$$

$T_p M = \_ \quad \_ \; .$

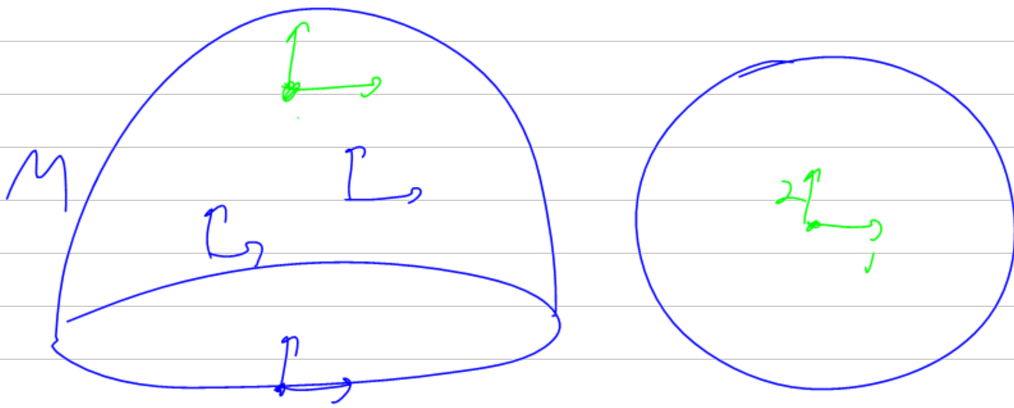


$$\xi = (p, v)$$

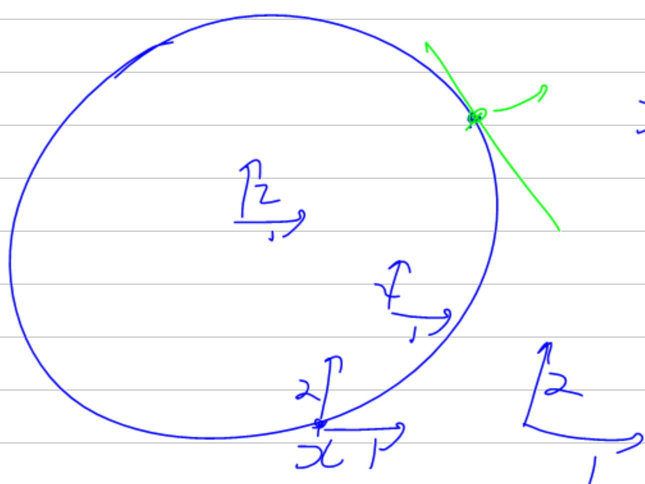
$$= \left((x,0,z), (y, z, -x) \right)$$



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$



$$(e_1, e_2, e_3)$$



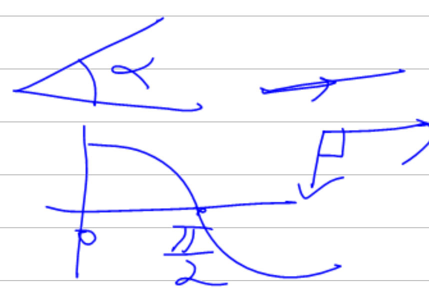
induce 1

$$x \in D^2 \subset \mathbb{R}^2$$

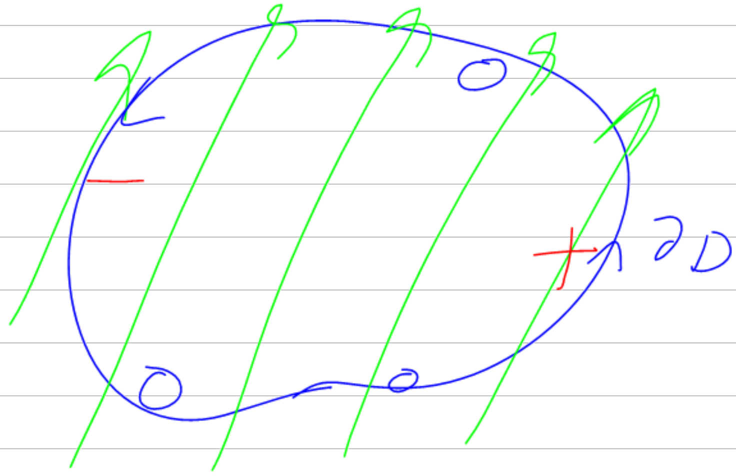
induce 2

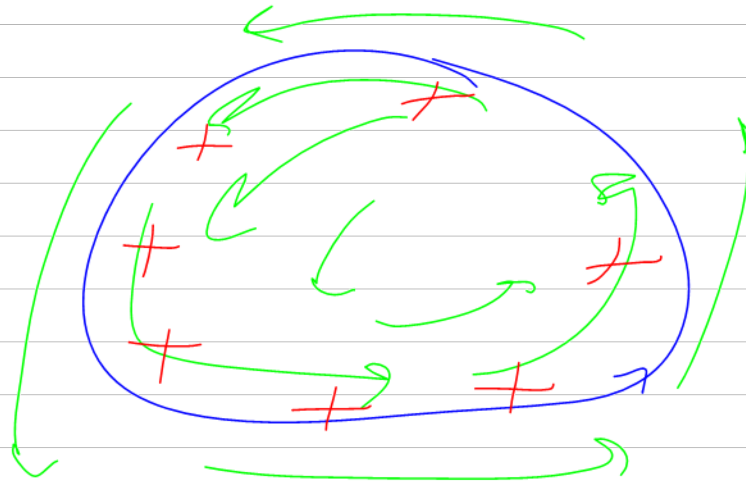
$$x \in \partial D^2$$

$$V \cdot W = |V||W| \cos\alpha$$

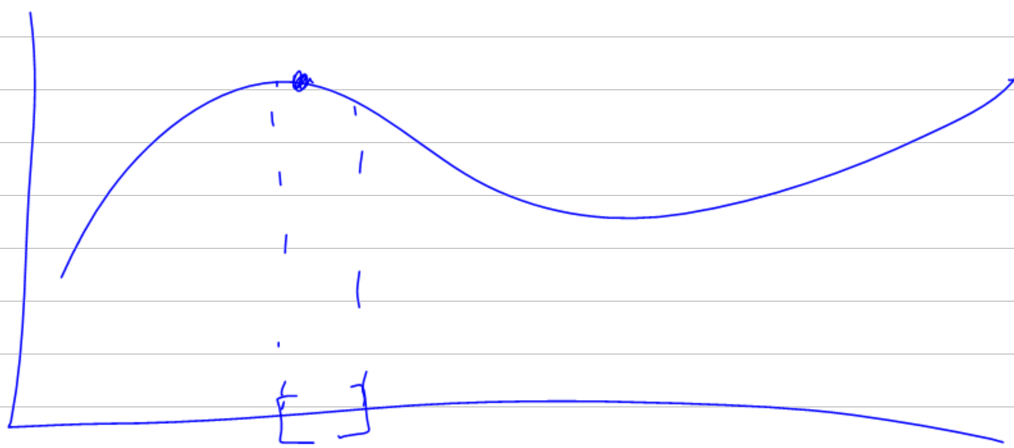


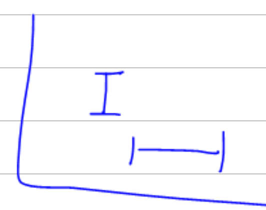
$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D} \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \dot{\gamma}\, dt$$





$$F = \underline{B_x}\, dy\, dz + \dots$$
$$+ E_x\, dx\, dt$$





$$C_{I,a,\epsilon} : (t_1 \dots t_k) \longmapsto a + \epsilon(0\ t_1\ 0\ t_2\ t_3 \dots\dots)$$
$$\qquad\qquad\qquad\qquad\qquad 1\ 2\ 3\ 4\ 5 \dots\dots n$$
$$\qquad\qquad\qquad\qquad\qquad\quad i_1 \quad i_2 \quad i_3$$

side

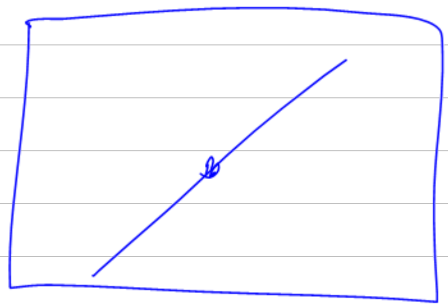
lowest corner

which coords are non-constant

$$= (i_1 \dots i_k) \in \underline{n}^k_a \sim \binom{n}{k}$$

$\{1, \ldots, n\} = \underline{n}$ $\quad X^k = (x_1 \ldots x_k)$

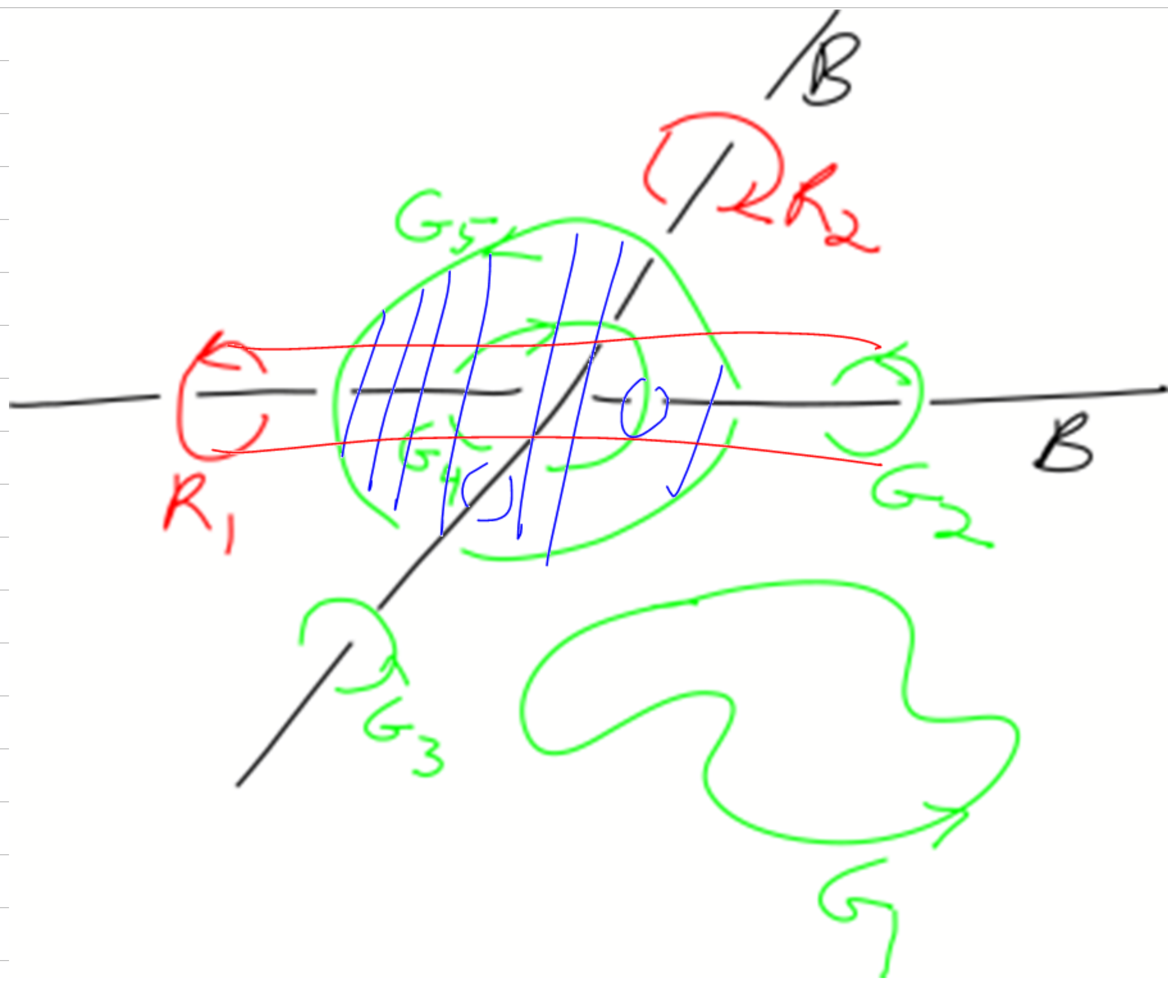
$\underline{n}^k = \{(i_1 \ldots i_k) : 1 \le i_\alpha \le n\}$

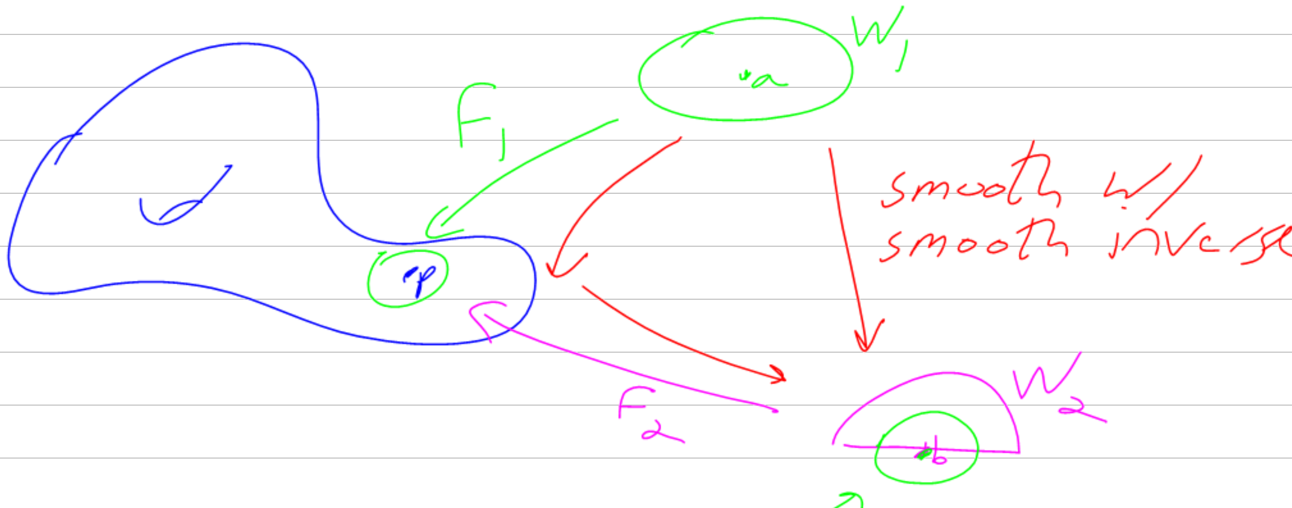


$|xy|^{1/2}$

$t \mapsto (t,t)$

$|tt|^{1/2} = |t|$





$b \in F_2^{-1}(F_1(W_1)) \subset H_k$
open

$$F(a+h) = F(a) + \underline{\underline{L}} \cdot h + \underline{o(h)}$$

Prob 9 $\quad \mathbb{R}^n = \mathbb{R}^k_x \times \mathbb{R}^{n-k}_y \qquad F : \mathbb{R}^n \to \mathbb{R}^k$

wlog $\dfrac{\partial F}{\partial x}$ is invertible, $\quad \underbrace{\left(\begin{array}{c|c} k\times k & k\times(n-k) \end{array} \right)}_{n} \Big\} k$

$$h(x,y) = \begin{pmatrix} F(x,y) \\ y \end{pmatrix}$$

$$h : \mathbb{R}^n \to \mathbb{R}^n$$

$$h' = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ 0 & I \end{pmatrix} \text{ is invertible near } 0$$

Set $g = h^{-1}$ (near 0)

$F(g(x,y))$
$\quad \|$
$F(z,y)$
$\quad \|$
$\quad x$

z s.t. $F(z,y) = x$
$g(x,y) = (z,y)$

$M^k \subset \mathbb{R}^n \qquad \nu_1 \dots \nu_{n-k}$

$$dV(u_1 \dots u_k) = \begin{vmatrix} - & u_1 & - \\ & \vdots & \\ - & u_k & - \\ - & \nu_1 & - \\ & \vdots & \\ - & \nu_{n-k} & - \end{vmatrix}$$

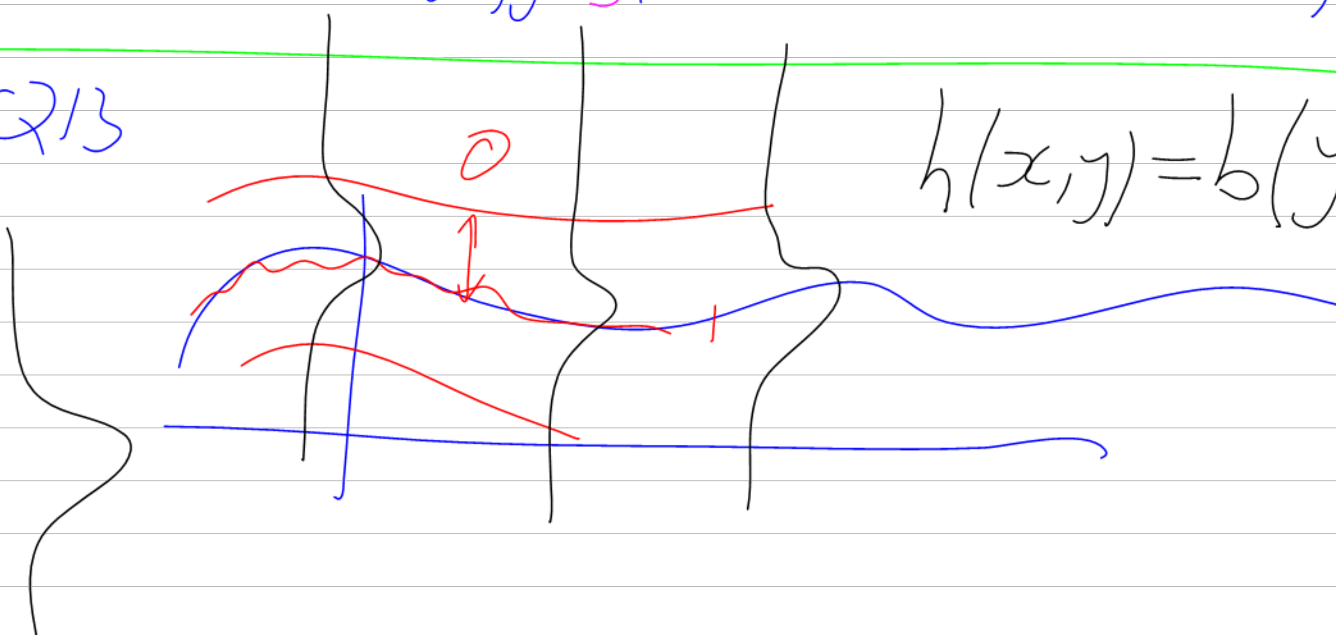
$\mathbb{R}^4_{t,xyz}$

Ω^0	Ω^1	Ω^2	Ω^3	Ω^4
$f = w^0_f$	$g\,dt + F_x\,dx + F_y\,dy + F_z\,dz = w^1_{g,F}$	$G_x\,dt\,dx + G_y\,dt\,dy + G_z\,dt\,dz + H_x\,dy\,dz + H_y\,dz\,dx + H_z\,dx\,dy = w^2_{G,H}$	$h\,dx\,dy\,dz + L_x\,dt\,dy\,dz \mp L_y\,dt\,dz\,dx + L_z\,dt\,dx\,dy = w^3_{h,L}$	$\ell\,dt\,dx\,dy\,dz = w^4_\ell$

$$dw^0_f = w^1_{\partial_t f,\, \mathrm{grad}_3 f} \qquad dw^2_{G,H} = w^3_{\pm \mathrm{div} H,\ \pm \mathrm{curl}\, G \pm \partial_t H}$$

Q13

$h(x,y) = b(y - f(x))$



$b : \mathbb{R} \to \mathbb{R}$; $b(0) = 1$ $\quad b(x) = 0$ if $|x| \geq 1$

$b(x) \geq 0$

Q7 $\quad |(x-y) - (f(x) - f(y))| \leq \frac{1}{3}|x-y| \quad f : \mathbb{R}^n \to \mathbb{R}^n \ni z$

$x_0 = 0$

$\vdots$

$x_n = x_{n-1} + (z - f(x_{n-1}))$

$x_{n+1} = x_n + (z - f(x_n))$

$$|x_{n+1} - x_n| = |(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1}))|$$
$$\leq \frac{1}{3}|x_n - x_{n-1}| \leq \frac{1}{9}|x_{n-1} - x_{n-2}|$$
$$\leq \cdots \frac{1}{3^n}|x_1 - x_0|$$

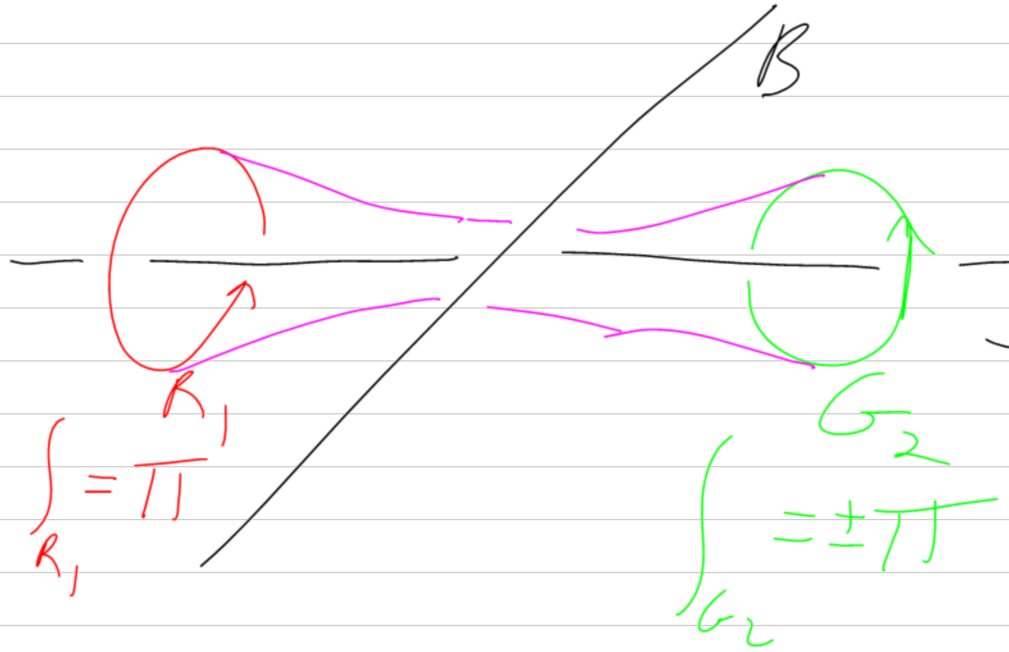
$$|f(x_n) - z| = |x_{n+1} - x_n| \leq \frac{1}{3^n}|x_1 - x_0| \to 0$$

$\cancel{\langle x, x \rangle \geq 0 \quad = \iff x = 0}$

"$\langle , \rangle$ is non-degenerate"

meaning $\quad \forall y \ \langle x, y \rangle = 0 \implies x = 0$

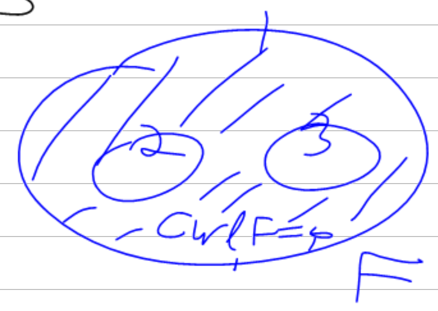
Greg Egan "orthogonal"

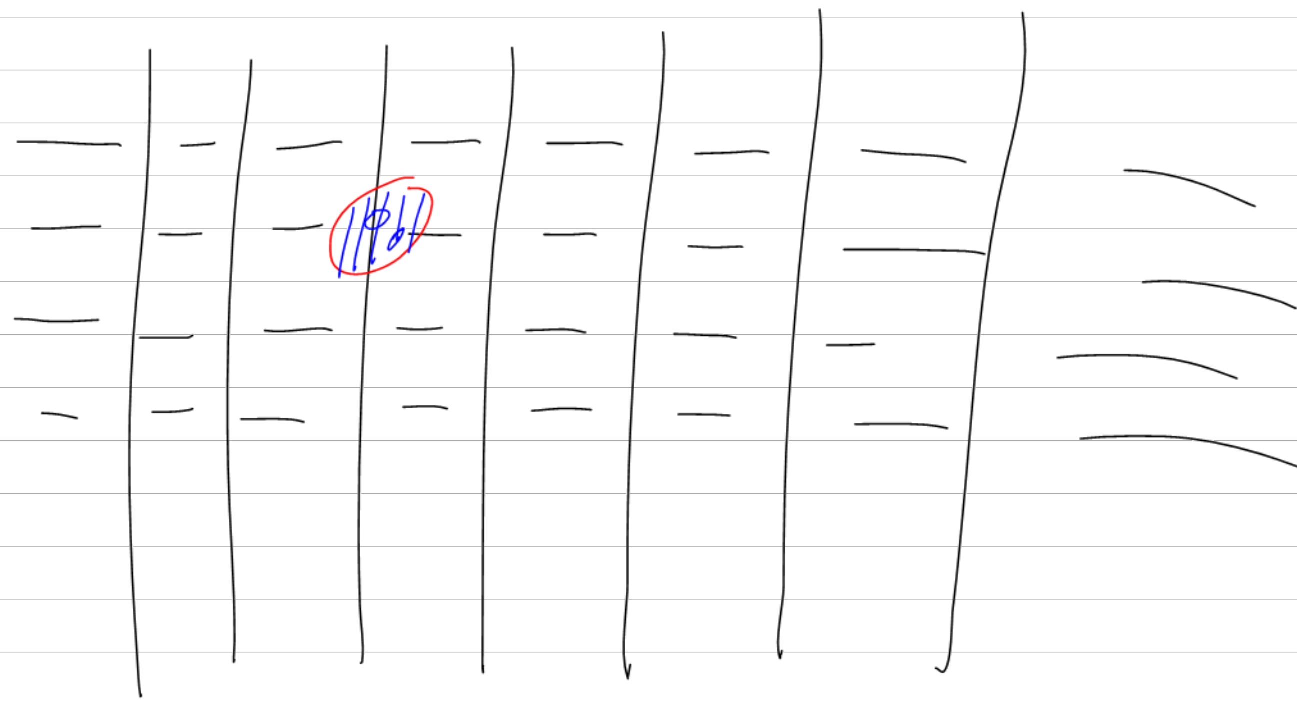


$a\pi + b\ell$

$a, b \in \{0, \pm 1\}$

$1 = 2 + 3$





Q19 Precise: $\int_C w$ depends only on ∂C

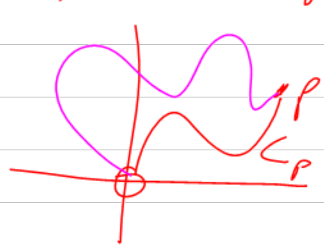
$w \in \Omega^1$ C: 1-cube.

Exact: $\exists \lambda$ s.t. $w = d\lambda$ $\lambda \in \Omega^0$

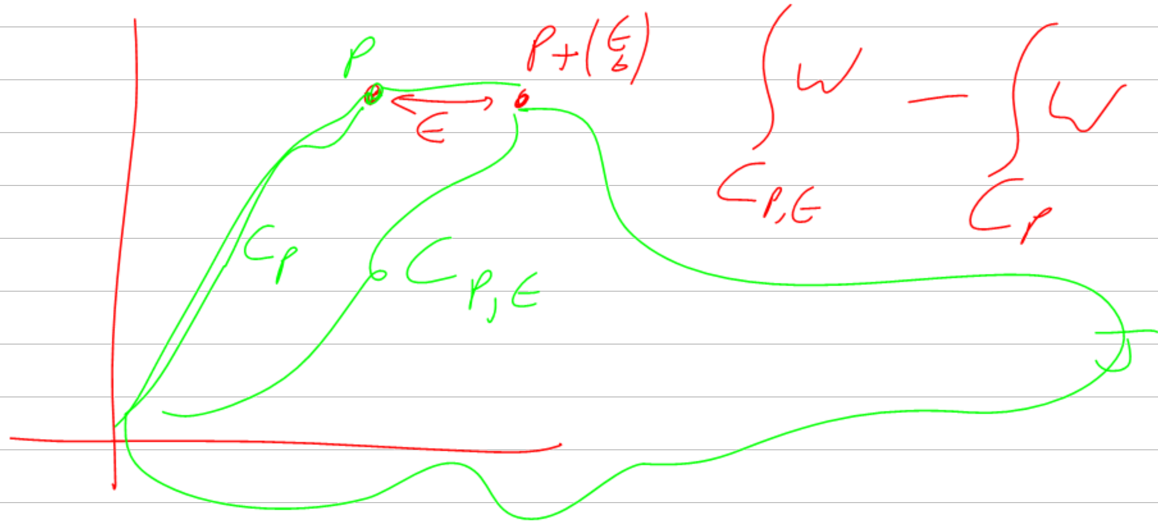
Precise $\Rightarrow$ Exact: wlog $\lambda(0) = 0$

$C_p(0) = 0$

$C_p(1) = p$



$\lambda(p) = \int_{C_p} w$



$$\int_{C_{p,\epsilon}} \omega - \int_{C_p} \omega = \int_p^{p+(\epsilon,0)} \omega$$

$$\sum_{i=1}^{\infty} \varphi_i A_{W_i}$$