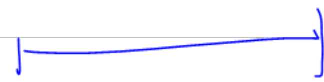
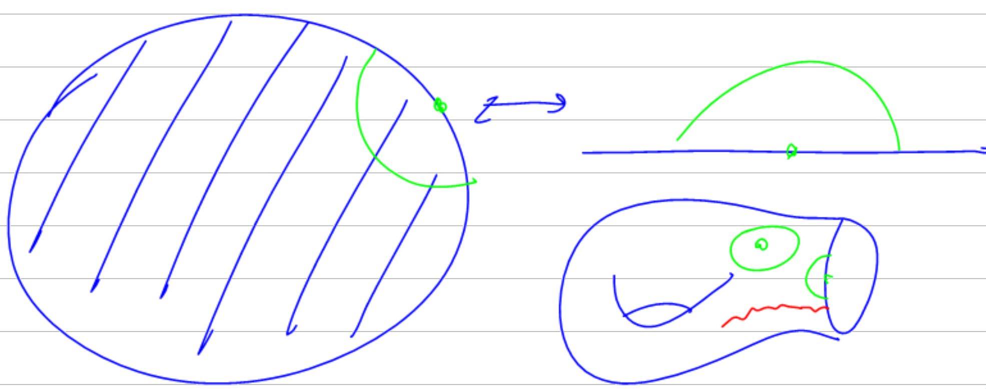
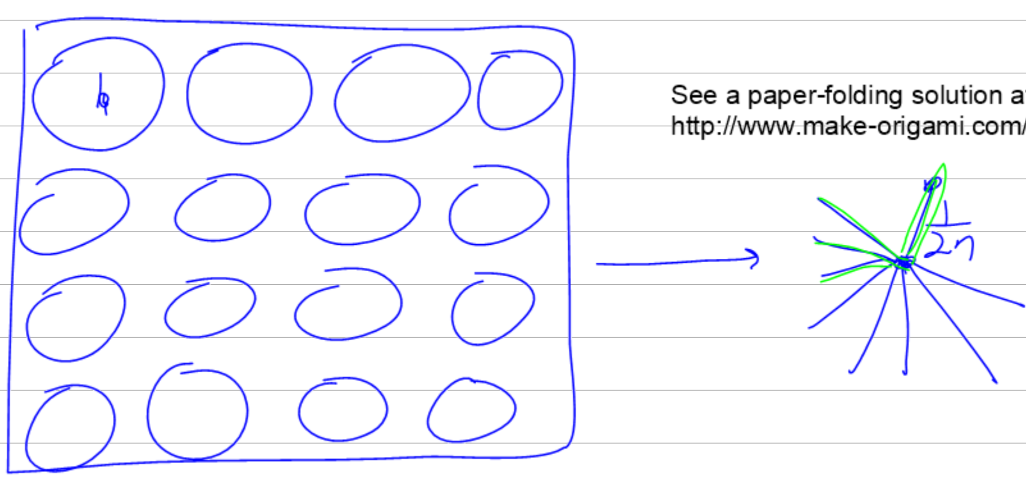
$S^1 =$  $I =$ 



$\subset \mathbb{R}^3$

See a paper-folding solution at
http://www.make-origami.com/HelenaVerrill/perimeter.php



$\frac{1}{2n}$

$\} n^2$

$\ell(P_n) = n^2 \cdot \frac{1}{n} = n$

MAT257 Analysis II on April 5, 2021: Volume forms, the 3D theorems.
Read Along: Spivak 126 to \infty.
HW19 is due tonight, HW20 (last!) will be online by midnight.
Old riddles (sol'n at end). Can you fold a rectangular piece of paper (perhaps many times) so that the result will have a longer perimeter than the original? If f:A->R^2 is a distance-non-increasing map from a rectangle to the plane, is it always the case that the length of the boundary of f(A) is less than the length of the boundary of A?

Blame COVID!

Next tasks: If $M^3 \subset \mathbb{R}^3$ and F is a vector field,

$$\int_M \operatorname{div} F\, dV = \int_{\partial M} F \cdot n_1\, dA$$

n_1: unit normal to ∂M

If $M^2 \subset \mathbb{R}^3$ is compact orient,

$$\int_M (\operatorname{curl} F)\cdot n_2\, dA = \int_{\partial M} F \cdot T\, ds$$

n_2: unit normal to M; T: unit tangent to ∂M

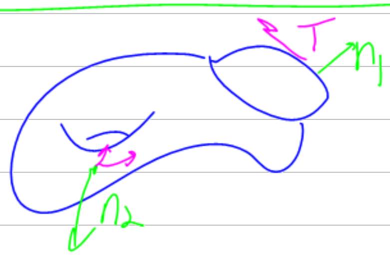
Integration on ◯? on ⬭?

From Munkres' Analysis on Manifolds:

***Theorem 35.2.** *Let M be a compact oriented k-manifold in $\mathbf{R}^n$. Let ω be a k-form defined in an open set of $\mathbf{R}^n$ containing M. Suppose that $\alpha_i : A_i \to M_i$, for $i = 1, \dots, N$, is a coordinate patch on M belonging to the orientation of M, such that A_i is open in $\mathbf{R}^k$ and M is the disjoint union of the open sets $M_1, \dots, M_N$ of M and a set K of measure zero in M. Then*

$$\int_M \omega = \sum_{i=1}^N \Big[\int_{A_i} \alpha_i^* \omega\Big].$$

But first, what are dV, dA, ds?



$M^k \subset \mathbb{R}^n$ oriented $\rightsquigarrow$ the volume form $\underline{dV}$ on M^k ($dV \in \Omega^k(M)$)

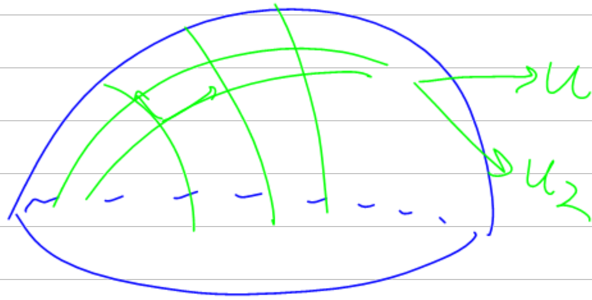
by some $\eta \in \Omega^k(M)$ defined up to mult by $\mathbb{R}_{>0}$.

Warning ∇ isn't a function, dV isn't a 1-form,

by declaring that dV is that multiple of η (by a function) for which

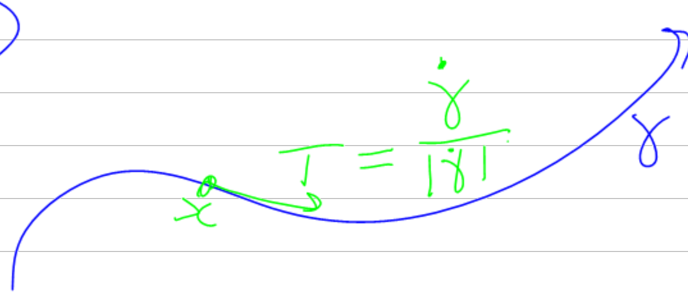
$$dV(u_1 \dots u_k) = 1$$

if $u_1 \dots u_k$ make an orthonormal basis of T_xM which agrees w/ the orientation of M.



This is indep of the choice of O.N. basis b/c change of basis matrices between O.N. bases have det $= \pm 1$,

Examples ①

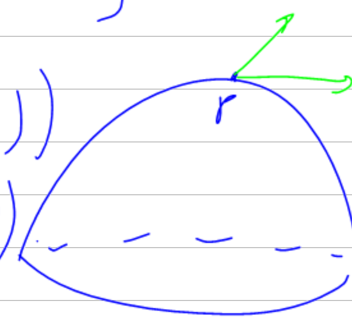


$$dV(T) = ds(T) = 1$$

② $S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$

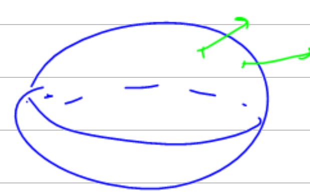
$p = (0,0,1)$ $\quad u = (p,(1,0,0))$

$\quad\quad\quad\quad\quad\quad\;\; v = (p,(0,1,0))$



$$(\alpha\, dx \wedge dy)(u,v) = \alpha \implies \alpha = 1$$

$$dA_p = dx \wedge dy$$



In general, if $M^2 \subset \mathbb{R}^3$ is an oriented 2-manifold in $\mathbb{R}^3$, & $n : M \to \bigcup_{x \in M} T_x\mathbb{R}^3$

s.t.,
1. $n(x) \in T_x\mathbb{R}^3$
2. $n(x) \perp T_x M$, $|n(x)| = 1$

"a unit normal to M"

3. If u, v define the orientation of M at x, then (u, v, n) define the orientation of $\mathbb{R}^3$ at x.

In this case,

$$dV^{(u,v)} = dA(\underbrace{u, v}_{\text{any two } u,v \in T_xM}) = \underbrace{\begin{vmatrix} - & u & - \\ - & v & - \\ - & n & - \end{vmatrix}}_{\substack{\text{bilinear \& alternating as function of } u \,\&\, v. \\ \text{\& has the right value (1) if } (u,v) \text{ make an} \\ \text{oriented O.N. basis.}}} = (u \times v) \cdot n$$

$$\begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

(Reminder: $u \times v$ is normal to both u, v, $|u \times v| =$ area of the parallelogram spanned by u, v,

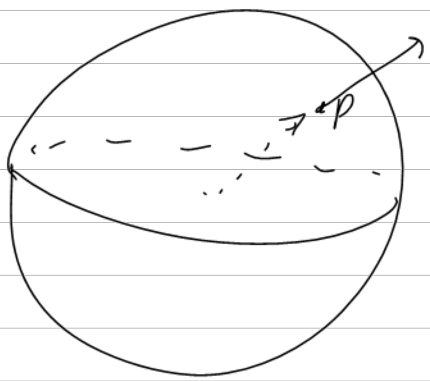
$(u, v, u \times v)$ defined the orientation of $\mathbb{R}^3$)

$$dA(u, v) = \begin{vmatrix} - & u & - \\ - & v & - \\ - & n & - \end{vmatrix} = (u \times v) \cdot n$$

$$= (n_1\, dy \wedge dz + n_2\, dz \wedge dx + n_3\, dx \wedge dy)(u, v)$$

$$(dy \wedge dz)(u, v) = u_2 v_3 - v_2 u_3 \quad \ldots\ldots$$

<u>Example</u> What's dA on S^2 at $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S^2$



$p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad n = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$dA = n_1\, dy \wedge dz + \ldots$

$= x\, dy \wedge dz + y\, dz \wedge dx + z\, dx\, dy$

(as seen earlier)

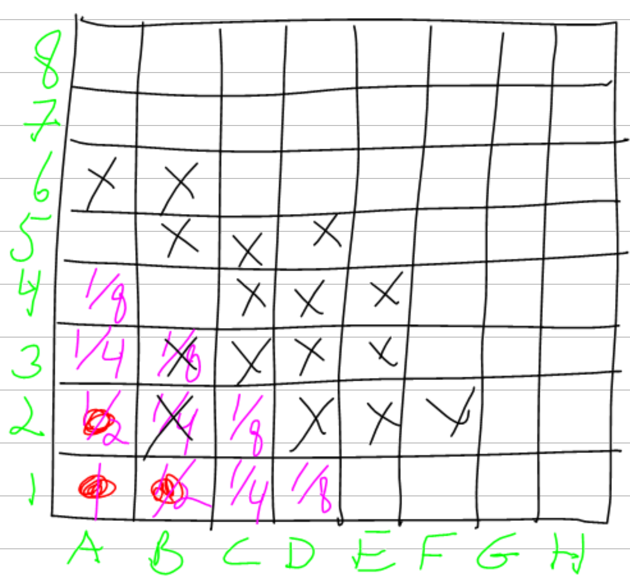
$$\left(\frac{1}{1-x}\right)'\Big|_{x=\frac12} = (1+x+x^2+\dots)'\Big|_{x=\frac12} = (0+1\cdot x^0+2x^1+3x^2+\dots)\Big|_{\frac12}$$

$$= \underbrace{1\left(\tfrac12\right)^0+2\left(\tfrac12\right)^1}_{2} + \underbrace{3\left(\tfrac12\right)^2+4\left(\tfrac12\right)^3+\dots}_{\text{what we want}}$$

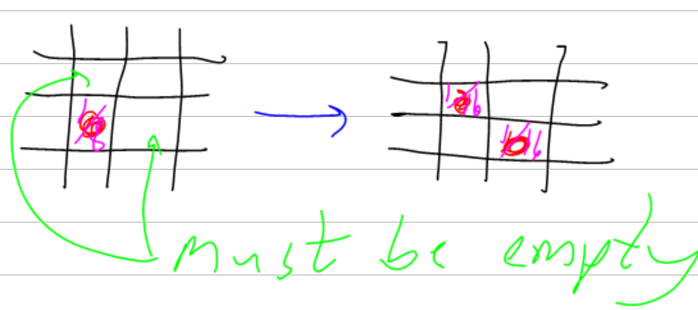
$$W_i = 1+1/2+1/2 = 2$$

$$W_e < \boxed{\sum_{k=2}^{\infty} 2^{-k}(k+1)} = \left(\frac{1}{1-x}\right)'\Big|_{\frac12} - 2$$

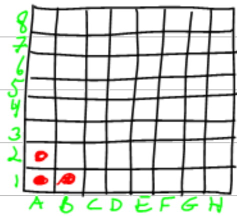
$$= \frac{1}{(1-x)^2}\Big|_{x=\frac12} - 2 = 2$$



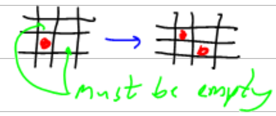
A move:



MAT257 Analysis II on April 7, 2021: The 3D theorems.
Read Along: Spivak 126 to \infty.
Old riddle (sol'n at end). On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you clear the original 3 pawns?



A move:



Next tasks: If $M^3 \subset \mathbb{R}^3$ and F is a vector field, $\displaystyle\int_M \operatorname{div} F\,dV = \int_{\partial M} F\cdot n\,dA$ (n: unit normal to ∂M)

If $M^2 \subset \mathbb{R}^3$ is compact orient., $\displaystyle\int_M (\operatorname{curl} F)\cdot n\,dA = \int_{\partial M} F\cdot T\,ds$ (n: unit normal to M; T: unit tangent to ∂M)

But first, what are dV, dA, ds?

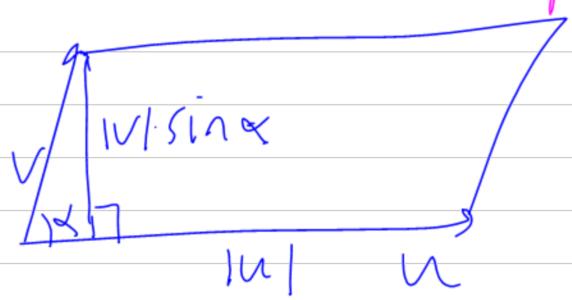
If $M^2 \subset \mathbb{R}^3$ & n is the positive unit normal,

$$dA(u,v) = \begin{vmatrix} - & u & - \\ - & v & - \\ - & n & - \end{vmatrix} = (u\times v)\cdot n = (n_1\,dy\,dz + n_2\,dz\,dx + n_3\,dx\,dy)(u,v)$$

$$= \pm|u\times v| = \pm|u|\cdot|v|\cdot\sin\alpha = \pm\sqrt{|u|^2\cdot|v|^2 - u\cdot v}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} \qquad \cos\alpha = \frac{u\cdot v}{|u|\cdot|v|}$$

$+$ if (u,v) make a positive basis (basis that agrees w/ orientation)



<u>Example</u> Suppose M is the image of a single cube: $M = c(I^2)$ where c is smooth, 1-1, w/ injective c'.

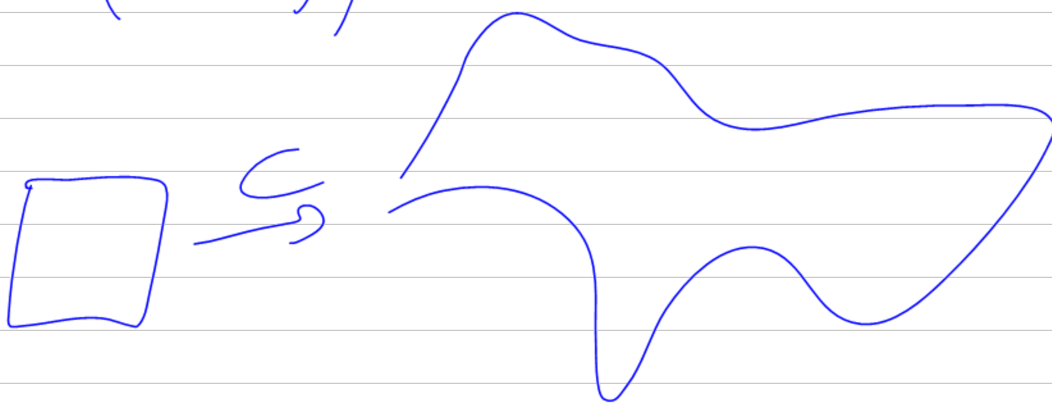
Area(M) =  $= \displaystyle\int_M dA$

$$\int_{I^2} c^*(dA) = \int_{I^2_{x,y}} \underbrace{c^*(dA)(e_1,e_2)}\, dx\wedge dy$$

$$= \int_{I^2} dA\big(\underset{\substack{\parallel \\ c'\cdot e_1}}{c_* e_1}, c_* e_2\big) = \int_{I^2} \overbrace{dA\Big(\underbrace{\frac{\partial c}{\partial x}}_{u}, \underbrace{\frac{\partial c}{\partial y}}_{v}\Big)}$$

$$\begin{pmatrix} \frac{\partial c_1}{\partial x} & \frac{\partial c_1}{\partial y} \\ \frac{\partial c_2}{\partial x} & \frac{\partial c_2}{\partial y} \\ \frac{\partial c_3}{\partial x} & \frac{\partial c_3}{\partial y} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \int_{I^2} \sqrt{\left|\frac{\partial c}{\partial x}\right|^2 \cdot \left|\frac{\partial c}{\partial y}\right|^2 - \frac{\partial c}{\partial x}\cdot\frac{\partial c}{\partial y}}$$



On $\mathbb{R}^3$

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] \arrow[d, shift right] & \Omega^1 \arrow[r, "d"] \arrow[d, shift right] & \Omega^2 \arrow[r, "d"] \arrow[d, shift right] & \Omega^3 \arrow[d, shift right] \\
\mathcal{F} \ni f \arrow[r, "grad"] \arrow[u, shift right, "f"'] & \mathcal{V} \ni F \arrow[r, "curl"] \arrow[u, shift right, "w^1_F = F_1 dx + F_2 dy + F_3 dz"'] & \mathcal{V} \ni G \arrow[r, "div"] \arrow[u, shift right, "w^2_G = G_1 dy\wedge dz + \dots"'] & g \in \mathcal{F} \arrow[u, shift right, "w^3_g = g\, dx\wedge dy\wedge dz"']
\end{tikzcd}$$

$\mathcal{F} = \{\text{Functions}\}$

$\mathcal{V} = \{\text{Vect. Fields}\}$

$dF = W^1_{\mathrm{grad} F}$ $\quad dW^1_F = W^2_{\mathrm{curl} F}$ $\quad dW^2_G = W^3_{\mathrm{div} G}$

<u>Claim</u> 1 If $N^1 \subset \mathbb{R}^3$, $W^1_F = (T \cdot F)\, ds$ on N^1
oriented 1-mfd.

positive unit tangent to N $\qquad$ the length form of N^1

2. If $M^2 \subset \mathbb{R}^3$ $\qquad W^2_G = (G \cdot n)\, dA$ on M^2.
oriented 2-manifold

Positive unit normal $\qquad$ Area form

<u>Pf of 1</u> Eval both sides on T.

$$(T \cdot F)\underbrace{ds(T)}_{1} = T \cdot F \qquad W^1_F(T)$$

$$= (F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3)(T_1 e_1 + T_2 e_2 + T_3 e_3)$$

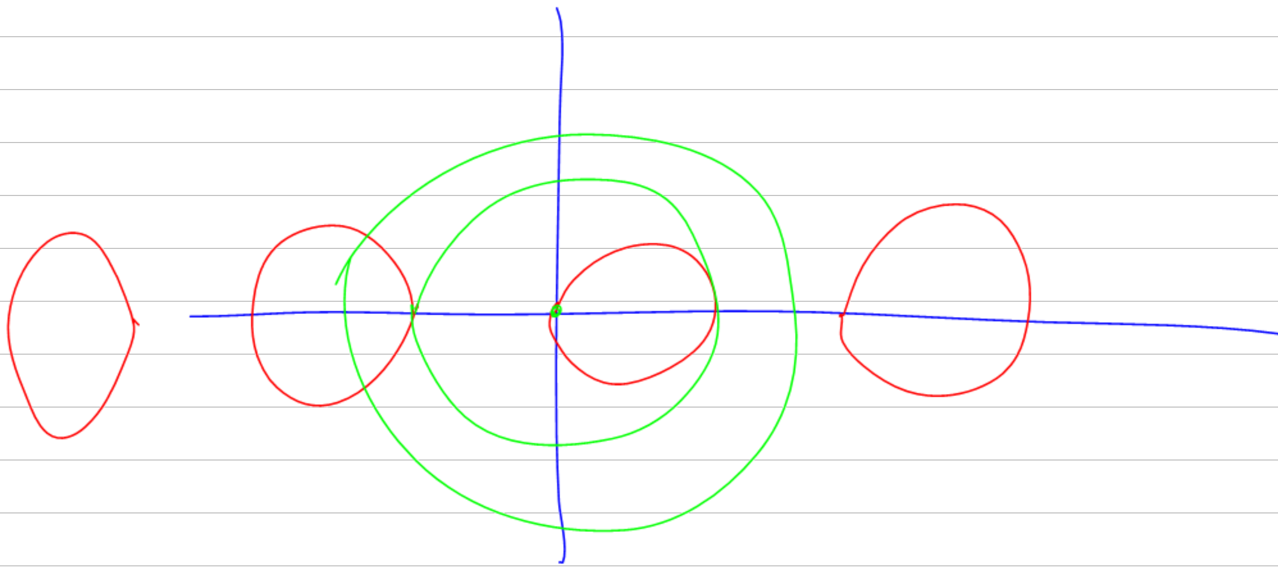
$$= F_1 \cdot T_1 + F_2 T_2 + F_3 T_3 = F \cdot T \quad \square$$

<u>Pf 2</u> NTS $W^2_G = (G \cdot n)\, dA$ $\quad$ Enough to verify on (u, v), a positive basis of $T_p M$

$$W^2_G(u, v) = (G_1\, dx_2 \wedge dx_3 + \ldots)\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) =$$

$$= G \cdot (u \times v) = G \cdot (|u \times v|\, n)$$

$$= (G \cdot n)|u \times v| = G \cdot n\, dA(u, v) \quad \square$$



$$\mathrm{Area}_{xy}(u,v) = (u \times v)\cdot n \qquad n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

u, v, w make an O.N. basis.

$$A(\mathrm{Proj}_{xy} C) = (u \times v)\cdot n + (v \times w)\cdot n + (w \times u)\cdot n$$

$$= (u + v + w)\cdot n$$

$$= \text{length.}$$

MAT257 Analysis II on April 9, 2021: The 3D theorems, towards the Maxwell equations. See http://drorbn.net/2021-257/ap/Maxwell.pdf
Read Along: Spivak 126 to \infty and beyond.
Note that we have a class on Monday as usual! But no Petr tutorial.
Old riddle (sol'n at end). A unit cube in R^3, the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

1. On $N^1 \subset \mathbb{R}^3$, $W^1_F = (T\cdot F)\,ds$

2. On $M^2 \subset \mathbb{R}^3$, $W^2_G = (G\cdot n)\,dA$

$$\int_M dw = \int_{\partial M} w$$

In $\mathbb{R}^3$:

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] \arrow[d, shift left, "f"] & \Omega^1 \arrow[r, "d"] \arrow[d, shift left, "F"] & \Omega^2 \arrow[r, "d"] \arrow[d, shift left, "G"] & \Omega^3 \arrow[d, shift left, "g"] \\
\mathcal{F} \arrow[r, "\mathrm{grad}"] \arrow[u, shift left, "f"] & \mathcal{V} \arrow[r, "\mathrm{curl}"] \arrow[u, shift left, "W^1_F"] & \mathcal{V} \arrow[r, "\mathrm{div}"] \arrow[u, shift left, "W^2_G"] & \mathcal{F} \arrow[u, shift left, "W^3_g"]
\end{tikzcd}$$

Fnctns V.F. V.F Fnctns.

<u>Case 3</u> $M = M^3$ $w = W^2_G$ $\qquad dW^1_F = W^2_{\mathrm{curl}F}$

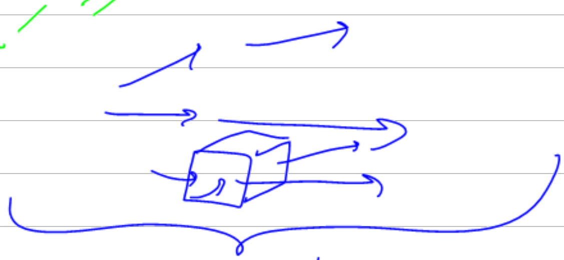
$$dw = W^3_{\mathrm{div}\,G} = (\mathrm{div}\,G)(dx\,dy\,dz)$$

$$\int_M dw = \int_{\partial M} w$$

$$\int_M (\mathrm{div}\,G)\,dV = \int_{\partial M} (G\cdot n)\,dA \quad \text{"Gauss' thm"}$$

The normal part of the flow

$$\int_M \underbrace{\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\right)}_{=\,\mathrm{tr}\,G'} dV = \int_{\partial M} (G\cdot n)\,dA$$



How much flow is "created" within M

= total outflow.

$M = M^2 \quad W = W^1_F \quad dW = W^2_{\operatorname{curl} F}$

$$\int_M dW = \int_{\partial M} W$$

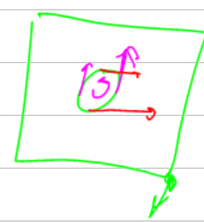
$$\int_M W^2_{\operatorname{curl} F} = \int_{\partial M} W^1_F$$

$$\int_M (\operatorname{curl} F)\cdot n \, dA = \int_{\partial M} (F \cdot T)\, ds$$

"Stokes' thm"

$$\operatorname{curl} F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$

The rotation of a ball carried by the flow (whirling of the flow)



$$\int_M (\operatorname{curl} F)\cdot n \, dA = \int_{\partial M} (F \cdot T)\, ds$$