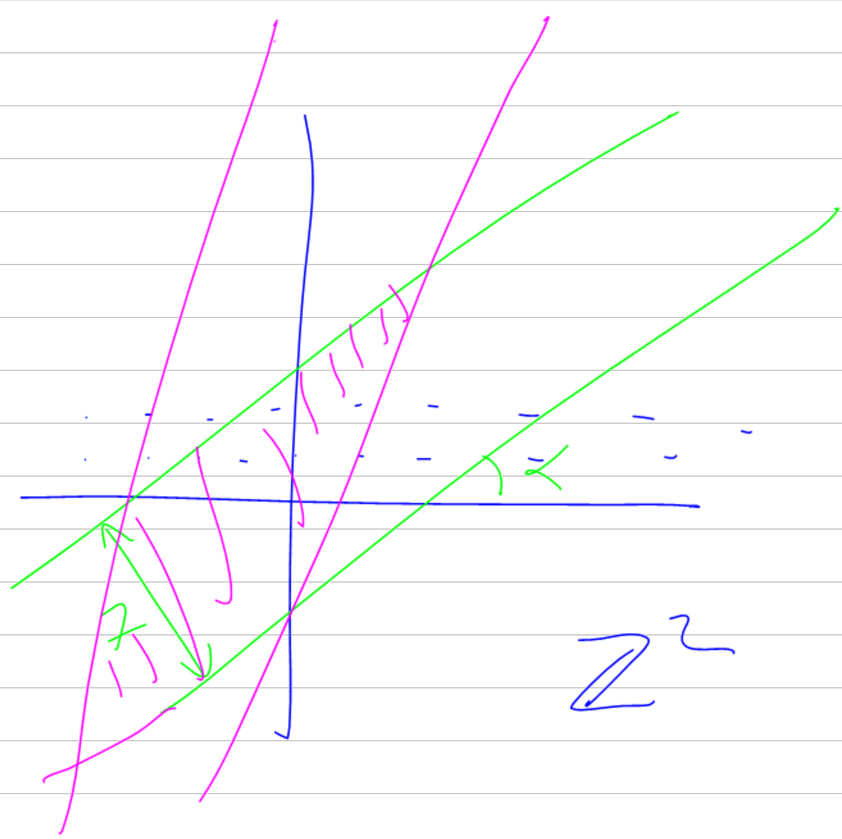


z^2

$\mathbb{Q} \hookrightarrow \mathbb{R}$

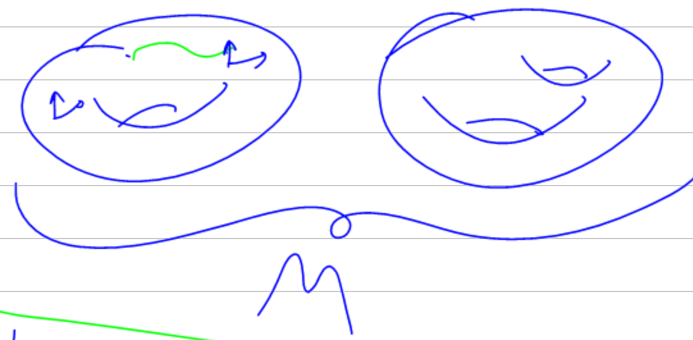
$\{r_n\}$ $\quad r_n \xrightarrow[n\to\infty]{} x$

MAT257 Analysis II on March 22, 2021: Orientation, integration. signtology
Read Along: Spivak 117-125.
HW17 due on Wednesday, HW18 on web by midnight.
Old riddle (sol'n at end). Can you find uncountably many sets of natural numbers, such that the intersection of any two of them is finite?

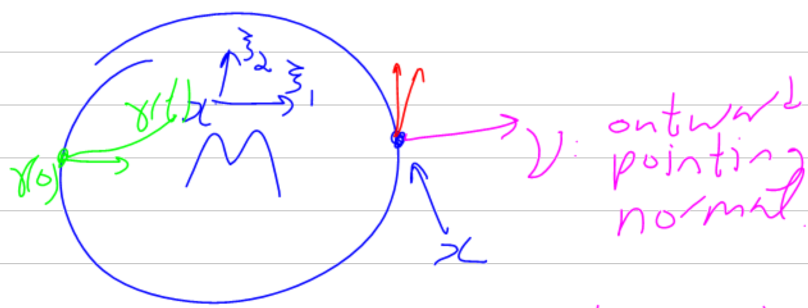
Orientation of M^k: A choice of an orientation for each T_xM, which can be presented in a locally ~~continuous~~ constant manner.
"orientable": has an orientation. "oriented": comes with one.

Comment A connected mfd has either 0 or 2 orientations.

Suppose M^k is an oriented mfld w/ bdry, orient the bdry of M, ∂M, as follows



Suppose $x \in \partial M$ orient $T_x(\partial M)$ by choosing t.v. $\eta_1 \dots \eta_{k-1}$ in $T_x(\partial M)$



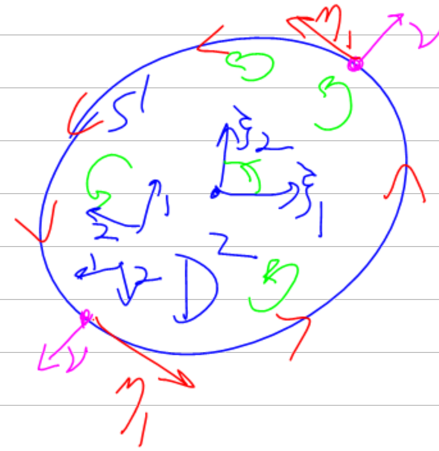
$T_x(M) \ni \nu \perp T_x(\partial M)$ in $\mathbb{R}^n$

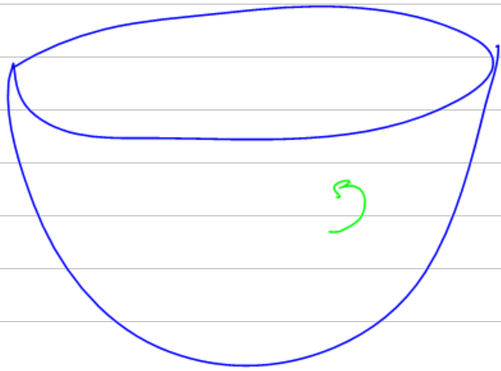
s.t. $\nu, \eta_1 \dots \eta_{k-1}$ is the given orientation of $T_x(M)$

Claim (LinAlg) This is well-defined!

Examples ② $M = D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$

$\partial D^2 = S^1$:

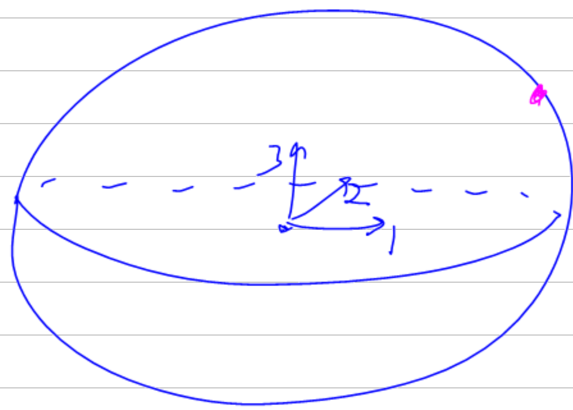




$\subset \mathbb{R}^3$

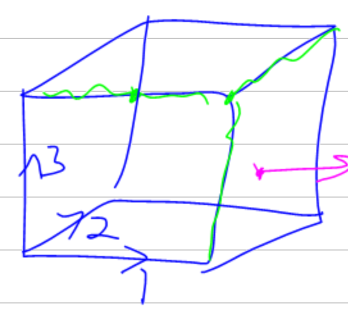
③ $D^3 = \{x \in \mathbb{R}^3 : |x| \le 1\}$ $\quad \partial D^3 = S^2$

oriented using the "std orientation of $\mathbb{R}^3$" (e_1, e_2, e_3)



$\Longrightarrow$ $\partial D^3 = S^2$ is oriented as last time.

(k) I^k oriented in the std manner $(e_1 \dots e_k)$



$I_{(j,\alpha)}$ has two orientation that come to mind:

ⓐ As a part of ∂I^k

ⓑ By identifying $I_{(j,\alpha)}$ w/ I^{k-1} via

$I^{k-1} \ni y \mapsto (y_1, \ldots, \overset{j}{\alpha}, \ldots y_{k-1}) \in I_{(j,\alpha)}$

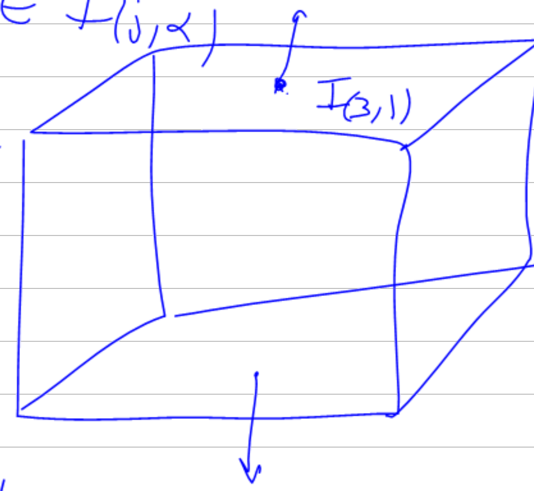


Figure out the two orientations on $I_{(j,\alpha)}$:

$\Theta_b : (e_1, \ldots, e_{j-1}, e_{j+1}, \ldots e_k)$

this is consistent w/ Θ_a if when we prepend ν to it, we get $(e_1, \ldots e_k)$ of I^k

$\nu = -(-1)^{\alpha} e_j = \begin{cases} +e_j & \alpha = 1 \\ -e_j & \alpha = 0 \end{cases}$

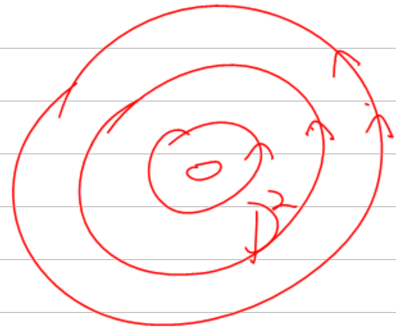
$(-(-1)^{\alpha} e_j, e_1, \ldots, e_{j-1}, e_{j+1}, \ldots e_k) \overset{?}{\sim} (e_1, \ldots, e_k)$

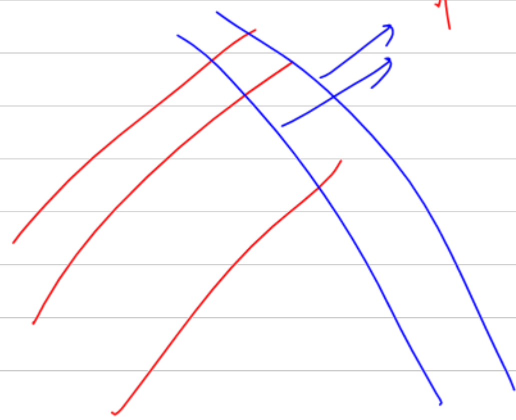
differ by a sign s,

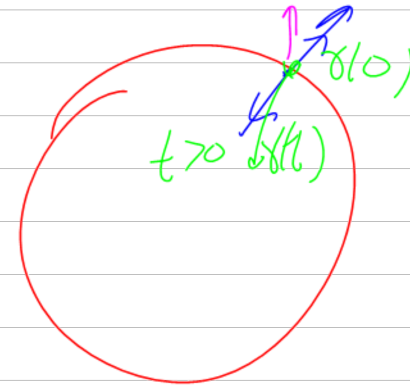
$s = -(-1)^{\alpha} \cdot (-1)^{j-1} = (-1)^{j+\alpha}$

That's why we've defined

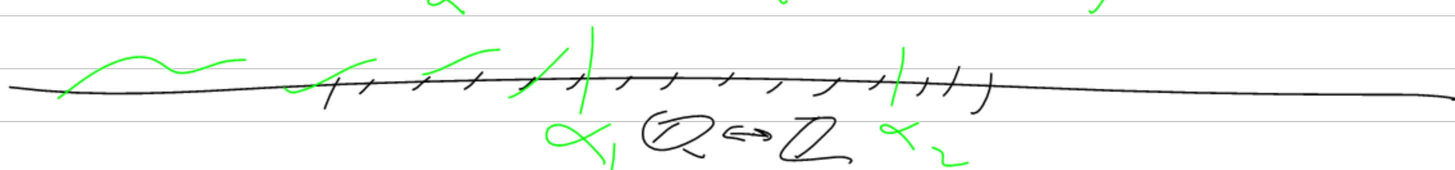
$$\partial c = \sum_{j,\alpha} (-1)^{j+\alpha} c_{(j,\alpha)}$$





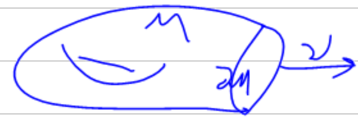


$$A_\alpha := \mathbb{Q} \cap (-\infty, \alpha)$$



MAT257 Analysis II on March 24, 2021: Orientation, integration.
Read Along: Spivak 117-125.
HW17 is due tonight, HW18 is on web.
Old riddle (sol'n at end). Can you find uncountably many sets of natural numbers, such that for any two of them, one contains the other?

Suppose $x \in \partial M \subset \underset{\text{oriented}}{\overset{\text{given}}{M}}$ and $\nu \in T_x M$ is the outward-pointing normal to $T_x \partial M$

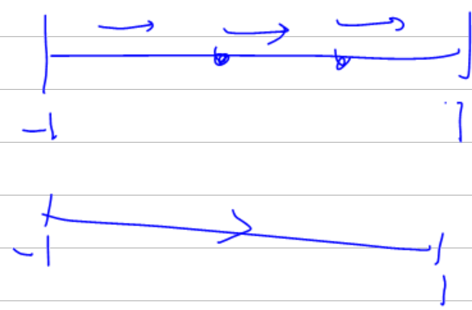


then $\mathcal{O}_{T_x M} = [\nu, u_1, \ldots, u_{k-1}]$ $\mathcal{O}_{T_x \partial M} = [u_1, \ldots, u_{k-1}]$

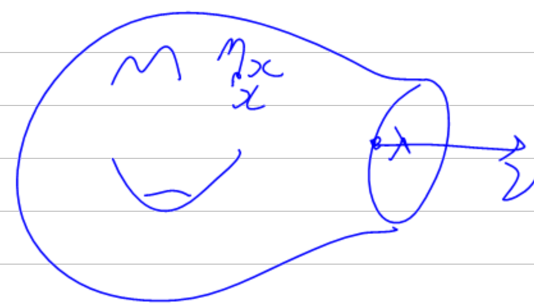
(2) D^2 (3) D^3 (k) I^k (1) $D^1 \sim I^1$

$D^1 = [-1, 1] \subset \mathbb{R}$, $I^1 = [0,1]$

$\partial D^1 = \{-1, +1\}$ 0-dim mfld.



An orientation at $x \in M^k$ is also $\eta_x \in \Lambda^k(T_x M)$ ~~mult by pos scalar~~



Tools: Can use ν

2. Can use i^*: Forms on M $\to$ Forms on ∂M $\quad i: \partial M \hookrightarrow M$

Aside Given a vector field X, define an op:

$$\iota_X : \Omega^*(M) \longrightarrow \Omega^{*-1}(M)$$

"interior multiplication by X"

by $(\iota_X \underset{p\text{-form}}{w})(u_1, \ldots u_{p-1}) = w(X, u_1 \ldots u_{p-1})$

* well-defined
* linear in w
* linear in X: $\iota_X + \iota_Y = \iota_{X+Y}$

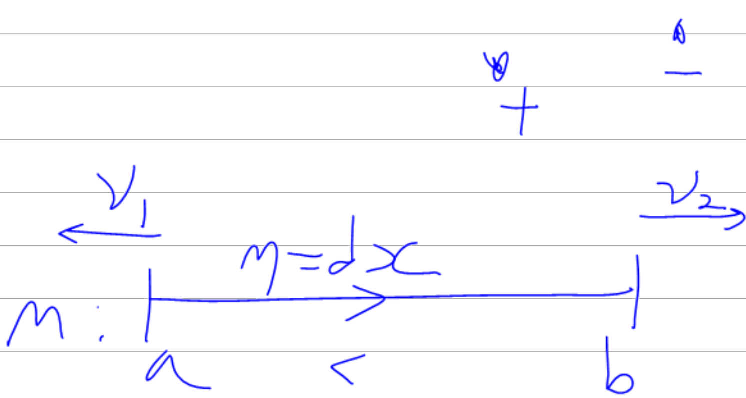
Exercise: Suppose M^k is oriented, ∂M has the induced orientation, ν is the outward pointing normal, $\eta_x \in \Lambda^k(T_x M)$ represents the orientation of M & $\lambda_x \in \Lambda^{k-1}(T_x \partial M)$ represents the orientation of ∂M, then for $x \in \partial M$,

$$\lambda_x \text{ "=" } i^*(\iota_\nu \eta_x)$$ up to multiplication by a pos scalar.

$$\eta \sim [v_1 \dots v_k] \iff \eta(v_1, \dots, v_k) > 0$$

What is an orientation of a single pt?

Ans. $\eta \in \Lambda^0(T_{pt}\, pt)$ ~~pos scalars~~ $=$ a sign $+$ or $-$



$$\lambda_a = i^* \iota_{\nu_1}(dx)$$
$$= i^* dx(\nu_1) = -1$$

$\partial M = \{a, b\}$

$$\lambda_b = \dots\ dx(\nu_2) = +1$$

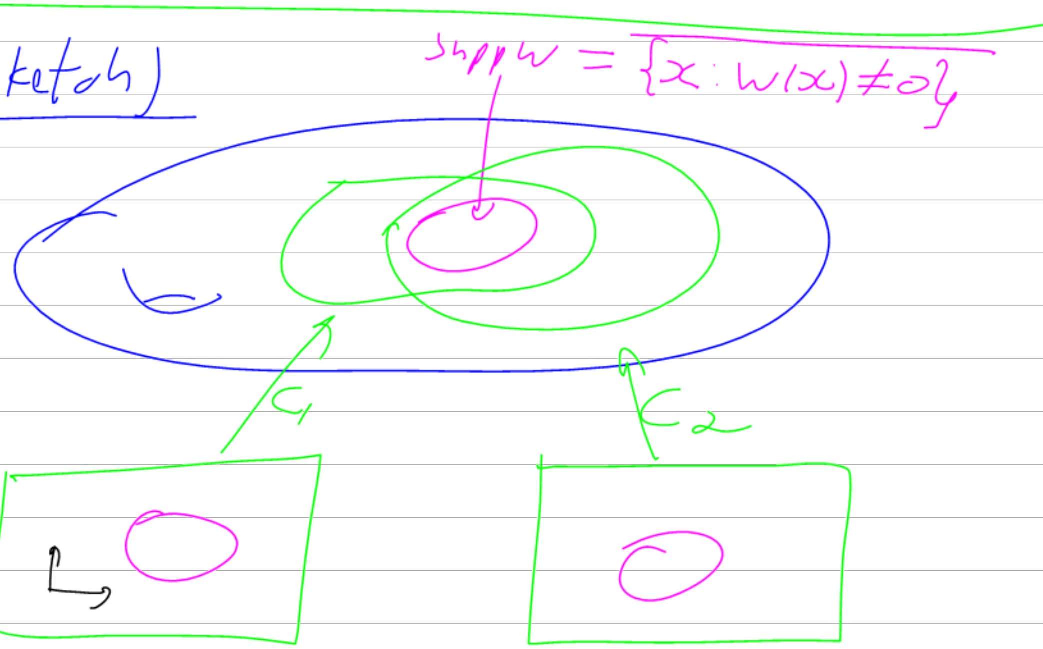
$$\partial\left(\underset{a}{|}\longrightarrow\underset{b}{|}\right) = \underset{a}{\overset{-}{\bullet}} \quad \underset{b}{\overset{+}{\bullet}} = [b]-[a]$$

Integration (sketch)

M^k is a k-dim oriented mfd,

$w \in \Omega^k(M)$

$$\int_M w = ?$$



Prop M^k oriented k-manifold, C_1, C_2 are smooth injective orientation preserving cubes in M^k, $w \in \Omega^k(M)$ s.t.

missing in Spivak 5.4.

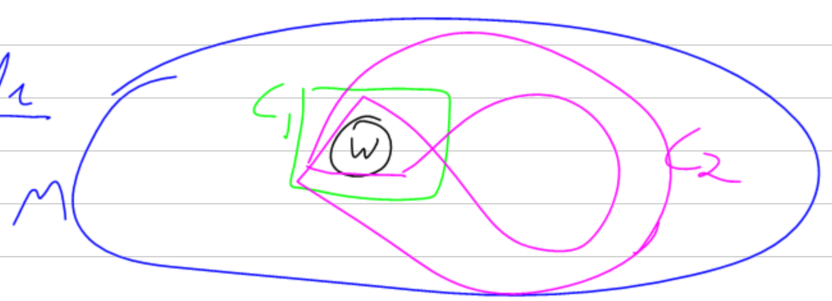
$$\operatorname{supp} w \subset C_1(I^k) \cap C_2(I^k)$$

C_i is 1-1 & pushes the std orientation of I^k to the orientation of M.

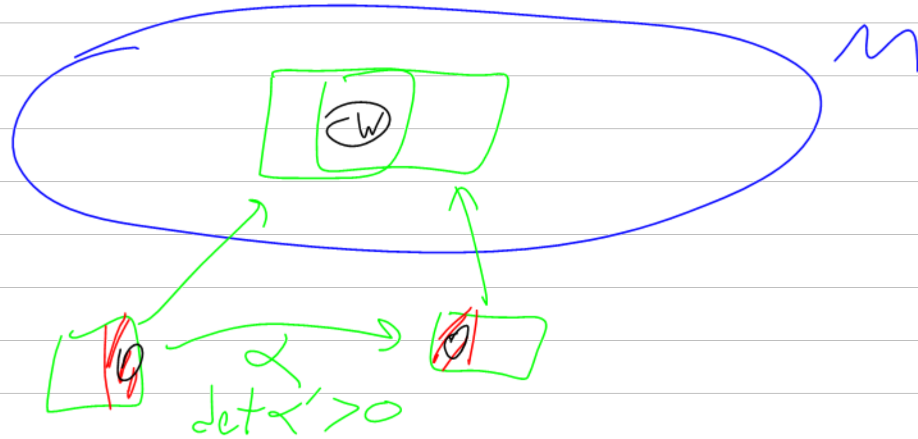
Then $$\int_{C_1} w = \int_{C_2} w$$

[Def call this common value $\int_M w$ (this is well defined)]

Example



Proof



P_1 0 1 0 1 0 0 1 1 0 1

P_2 0 0 0 0 1 0 1

P_3 0 1 1 1 0 0 1

P_4 1 1 1 0 1 1 1

1		4	
1	1	1	
1	0	1	
1	1	0	
1	1	1	

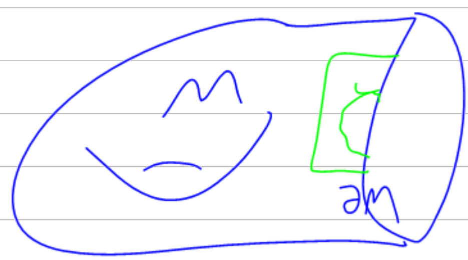
MAT257 Analysis II on March 26, 2021: Integration, Stokes' Theorem.
Read Along: Spivak 122-126.
Old riddle (sol'n at end). n prisoners. Each wears a tower of infinitely-many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right? Can they do better than 1/2^n?

$(a+b)(a-b) \neq a^2-b^2$? No.

Proposition Let C_i $(i=1,2)$ be smooth injective orientation preserving $\underbrace{k\text{-cubes}}_{\text{"good cubes"}}$ in an oriented M^k, and let $\omega \in \Omega^k(M)$ be s.t.
$\operatorname{supp}\omega \subset C_1(I^k) \cap C_2(I^k)$

$\boxed{\text{works near } \partial M \text{ too!}}$

then $\int_{C_1}\omega = \int_{C_2}\omega$; **Def** Call this $\int_M \omega$

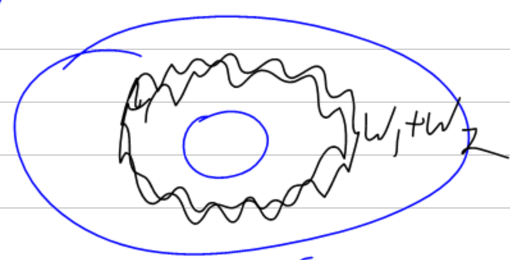


..... everything works.

Note 1 $\int_M \omega$ is linear in ω whenever this makes sense. (when all forms are supported in the same cube)

example

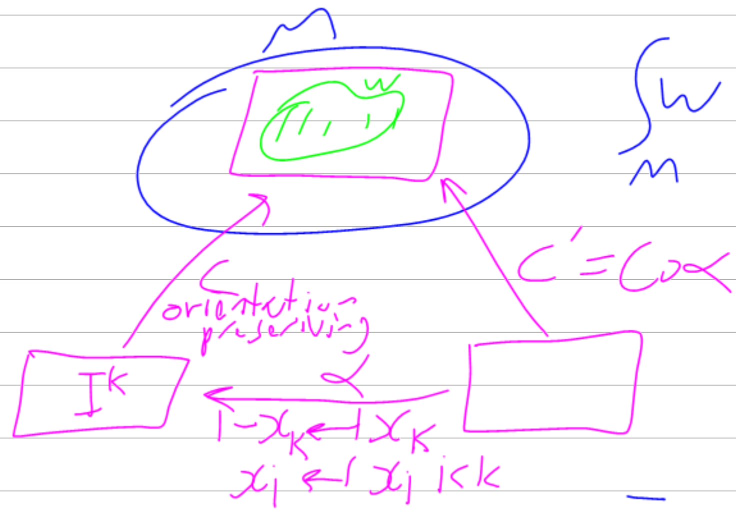
$1<|x|<2$



Note 2 $\int_{-M} \omega = -\int_M \omega$

$-M$: Same as M, but with opposite orientation!

Picture

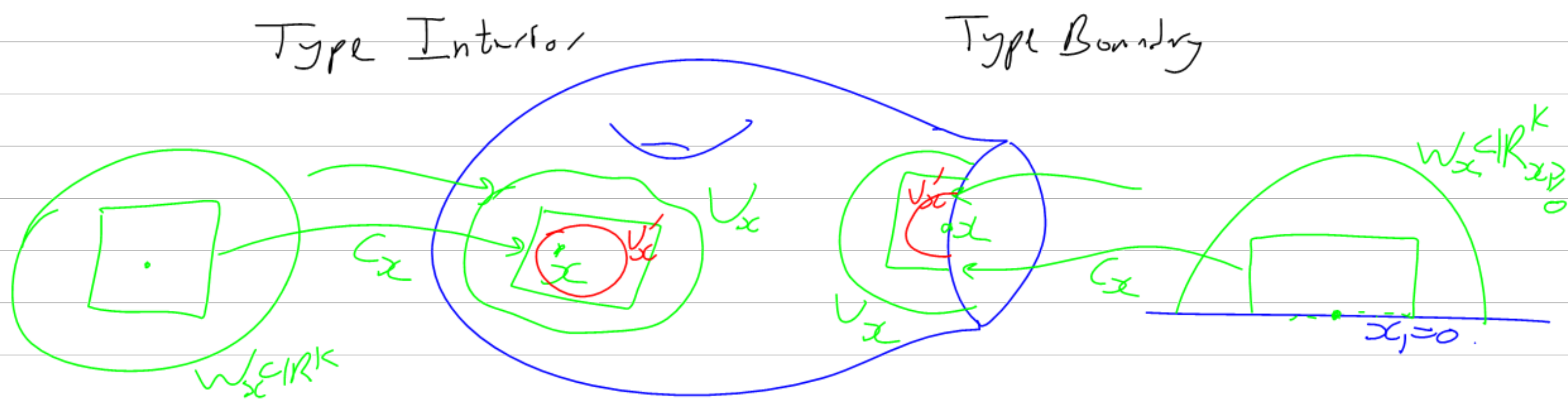


$$\int_M \omega = \int_{I^k} c^*\omega$$

$$\int_{-M} \omega = \int_{c'} \omega$$

$$= \int_{I^k} \alpha^* c^* \omega = -\int_{I^k} c^*\omega$$

To integrate a general M, use PO1.



We know how to integrate any form whose support $\subset U_x'$

Find a PO1 for M subordinate to V_x where the V_x's are open in $\mathbb{R}^n$ & such that $U_x = M \cap V_x$, call it (φ_i)

Define $\omega \in \Omega^k(M)$

$$\int_M \omega = \sum_i \int_M \varphi_i \omega$$

This makes sense!

Comments 1. If M is cmpct, (φ_i) can be chosen to be finite & no conv. issues. Otherwise, we say that w is "integrable"

if $$\sum_i \int_{C_i} \varphi_i |w| < \infty \qquad \begin{array}{l} \operatorname{supp} \varphi_i \subset \operatorname{im} C_i \\ C_i^* w = f\, dx_1 \ldots dx_k \end{array}$$

$$\int_{I^k} \varphi_i |f|$$

and only then we use the def. above.

2. Independence of the POI is proven as before.

$$\varphi_i \quad \psi_j \qquad \varphi_i \psi_j \ldots$$

3. The type B pts/nbds, restricted to ∂M give cubes & a POI for ∂M.

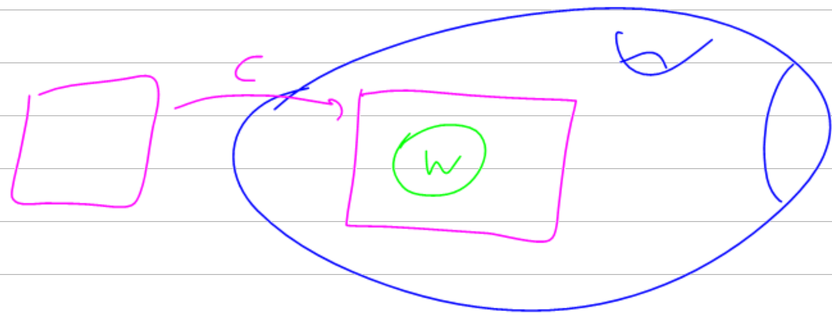
Stokes' thm If M is compact & oriented, if $w \in \Omega^{k-1}(M)$, then

$$\int_M dw = \int_{\partial M} w$$

Pf Type I, then type B, then combine using a POI.

Type I Suppose $\operatorname{supp} w \subset$

$$\operatorname{int}(\operatorname{im} c) \subset \operatorname{int} M = M - \partial M$$



$$\int_M d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega = \int 0 = 0$$

$$\int_{\partial M} \omega = 0.$$

[Not the interesting case!]