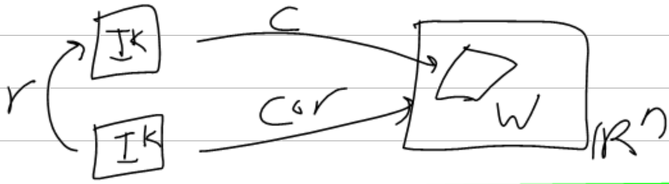
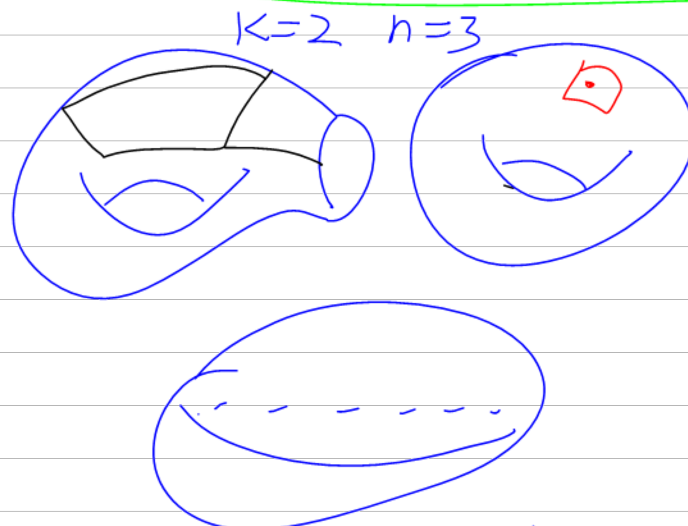
MAT257 Analysis II on March 8, 2021: Manifolds.
Read Along: Spivak 109-115.
TT3 is tomorrow! Info and prep questions at http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT3-Info.pdf

Exercise Given $c: I^k \to A \subset \mathbb{R}^n$, $\omega \in \Omega^k(A)$ and smooth $r: I^k \to I^k$ 1-1, onto, w/ $\det r' > 0$, $\int_c \omega = \int_{c\circ r} \omega$



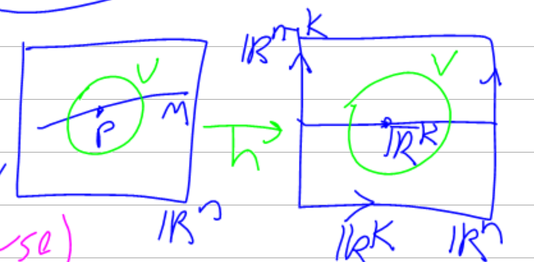
These are "k-dim manifolds in $\mathbb{R}^n$" →



Thm Given $k \leq n$ $M \subset \mathbb{R}^n$ $p \in M$, TFAE:

(M) $\exists U \ni p$ open $\exists V$ open in $\mathbb{R}^n_{\text{other}}$ & $h: U \longrightarrow V$ "a diffeomorphism" (smooth, w/ smooth inverse) s.t. $h(U \cap M) = V \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\})$

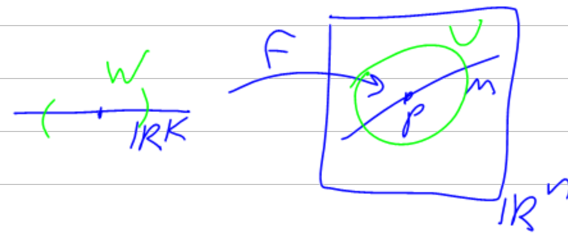
M:



(Z) $\exists$ open $U \ni p$ & smooth $g: U \to \mathbb{R}^{n-k}$ s.t. $U \cap M = U \cap g^{-1}(0)$ & $\operatorname{rank}(g') = n-k$

(C) $\exists$ open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth $\underbrace{F}_{1-1}: W \to \mathbb{R}^n$ s.t.

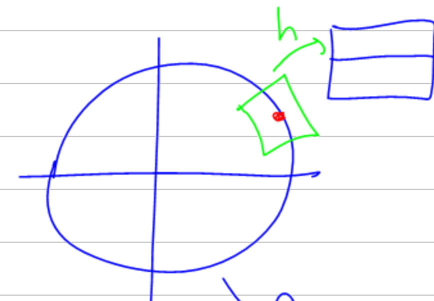
① $F(W) = M \cap U$ ② $F^{-1}: M \cap U \to W$ is cont.



③ $\forall a \in W$ $\operatorname{rank} F'(a) = k$

Def Given $k \le n$, $M \subset \mathbb{R}^n$ is a k-dim manifold
k-manifold
M^k
if $\forall p \in M$ any one of the conditions above hold.

Examples 1. $S^1 \subset \mathbb{R}^2$, indeed



(2): $S^1 = \{p \in \mathbb{R}^2 : |p| = 1\}$

$= \{p \in \mathbb{R}^2 : |p|^2 - 1 = 0\}$

$= g^{-1}(0)$

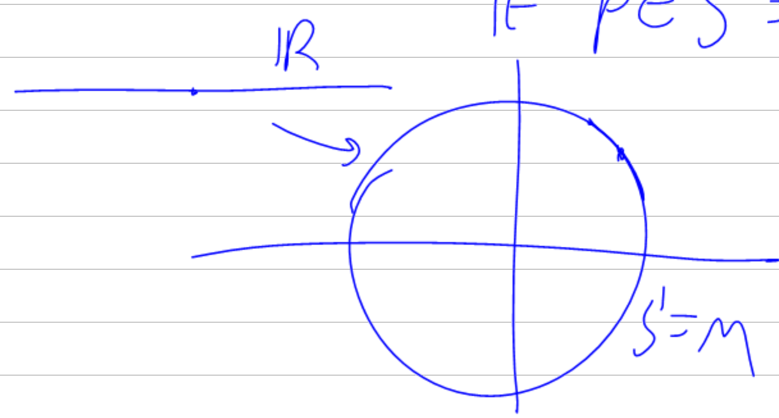
$p \mapsto |p|^2 - 1$, g into $\mathbb{R}$

$g(x,y) = x^2 + y^2 - 1$

$$g' = \left(\frac{\partial g}{\partial x} \ \ \frac{\partial g}{\partial y}\right) = \left(2x \ \ 2y\right) \ne 0$$

if $p \in S^1 = M$

(c) Use



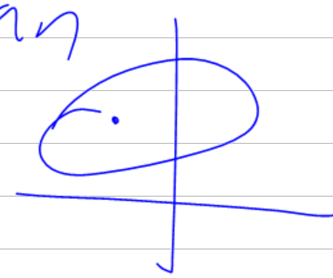
$F : \mathbb{R}_t \to \mathbb{R}^2_{x,y}$

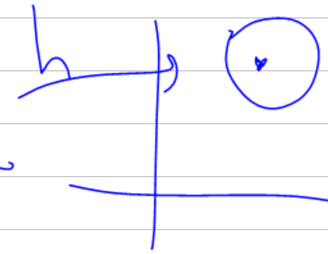
$$F(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

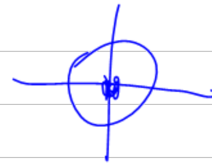
$$F' = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \ne 0 \qquad \operatorname{rank} F' = 1$$

0. Any open set in $\mathbb{R}^n$ is an n-manifold in $\mathbb{R}^n$

Any finite set in $\mathbb{R}^n$ is a 0-manifold in $\mathbb{R}^n$

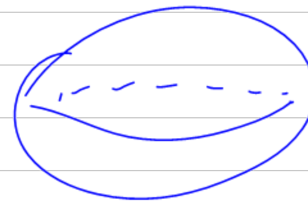






2. $S^n = \{ p \in \mathbb{R}^{n+1} : |p| = 1 \}$

$S^2 \subset \mathbb{R}^3$

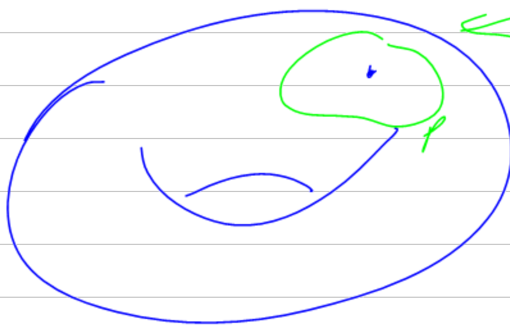


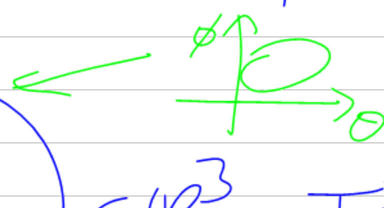
a manifold as

$S^n = g^{-1}(0)$ where

$g : \mathbb{R}^{n+1} \to \mathbb{R} \quad p \mapsto |p|^2 - 1$

3. $T^2 =$

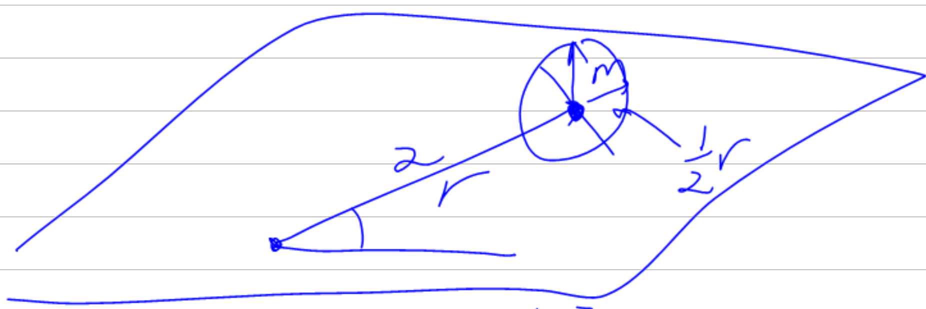




$\subset \mathbb{R}^3$

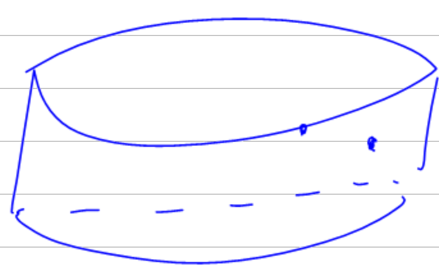
$$T^2 = \left\{ \begin{pmatrix} (2+\cos\phi)\cos\theta \\ (2+\cos\phi)\sin\theta \\ \sin\phi \end{pmatrix} \right\}$$

$$\begin{pmatrix} r \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} r \\ 0 \end{pmatrix} \cos\phi + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin\phi$$



$\mathbb{R}^2$

$$r = 2 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$\int_a^b f(x)\,\underline{\underline{dx}} \qquad \int_{[a,b]} f\,\underline{\underline{dx}}$$

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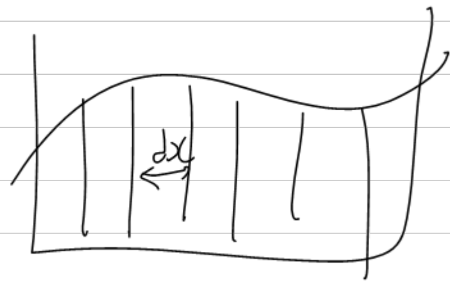


$dx \wedge dy$

$dy \wedge dx = -dx \wedge dy$

$$dx_I = dx_{i_1} \wedge \overbrace{(dx_{i_2} \wedge \cdots \wedge dx_{i_k})}^{k-1}$$

$$\underset{1}{\omega} \wedge \underset{k-1}{\eta}$$

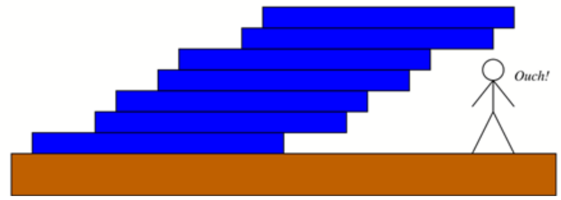
$$\underbrace{d\omega}_{k} = \sum \underbrace{dx_i}_{1} \wedge \underbrace{\frac{\partial \omega}{\partial x_i}}_{k}$$

MAT257 Analysis II on March 10, 2021: Manifolds.
Read Along: Spivak 109-115.
HW16 is on the web!
Our final assessment: Tuesday April 20 9-12 (morning TO time).

Within 10 minutes of the start of the test yesterday, the questions were posted on a question-sharing web site. Somebody thinks they are smart! We'll work to prove them wrong.

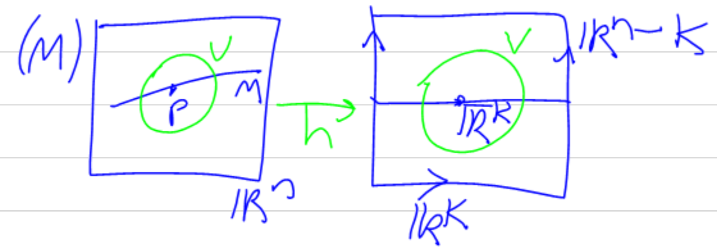
There was an issue with Q4; it will be managed after I know how many students were affected.

How far sideways can you reach by stacking up n identical blue domino pieces, before your tower will lean over and fall? What if n goes to infinity?
(no glue is allowed, and as shown, the stacking isn't necessarily "even")

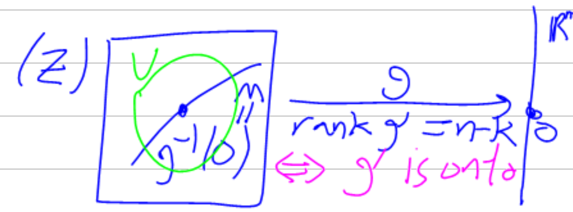


Thm Given $k \le n$, $M \subset \mathbb{R}^n$, $p \in M$, TFAE:

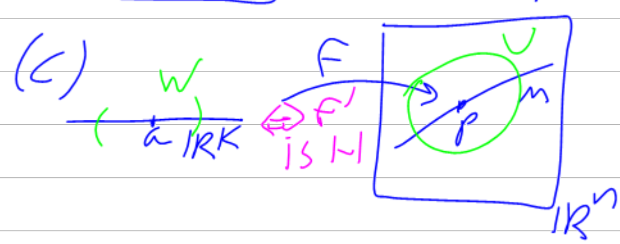
(M) $\exists$ open $U \ni p$, open $V \subset \mathbb{R}^n_{\text{other}}$ & a diffeomorphism $h: U \to V$ s.t. $h(U \cap M) = V \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\})$.



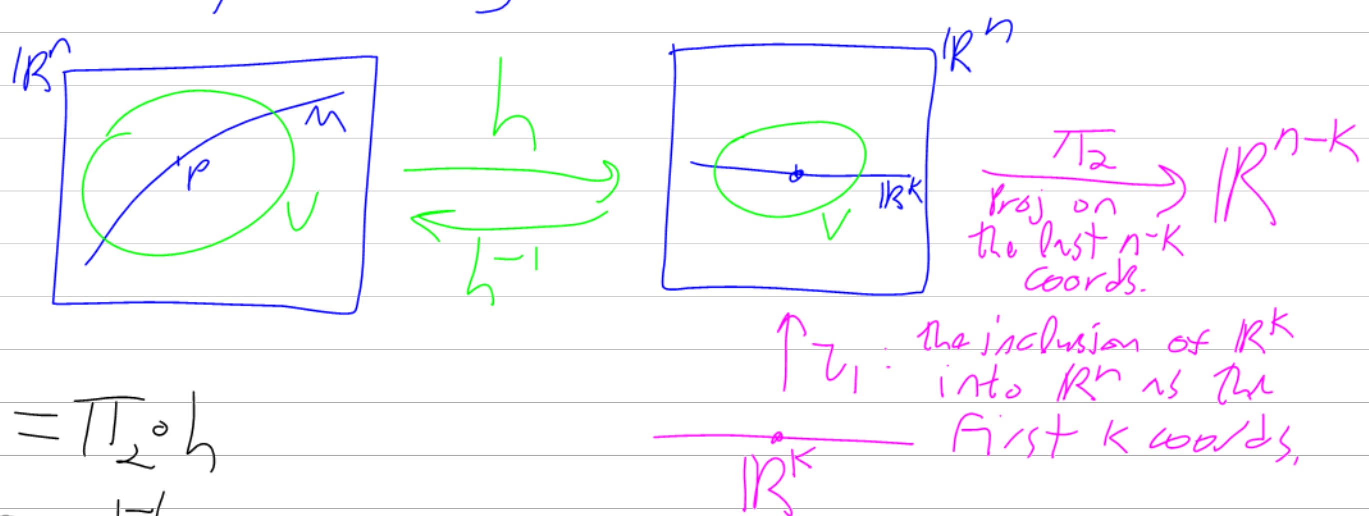
(Z) $\exists$ open $U \ni p$ & smooth $g: U \to \mathbb{R}^{n-k}$ s.t. $U \cap M = U \cap g^{-1}(0)$ & $\operatorname{rank}(g') = n-k$



(C) $\exists$ open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth 1-1 $f: W \to \mathbb{R}^n$ s.t. ① $f(W) = M \cap U$ ② $f^{-1}: M \cap V \to W$ is cont. ③ $\forall a \in W$, $\operatorname{rank} f'(a) = k$.



Pf M $\Rightarrow$ C, Z easy. Indeed

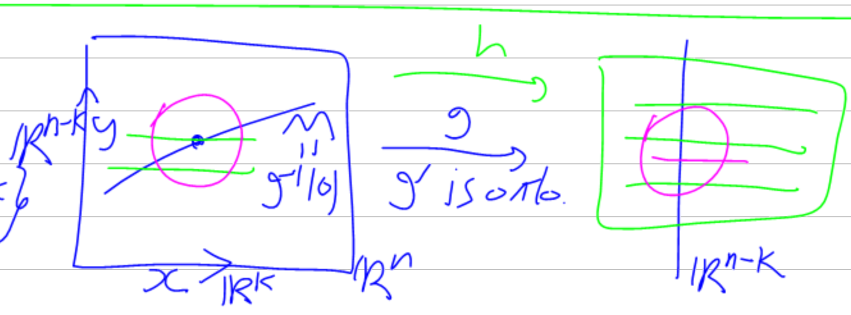


(Z) Set $g = \pi_2 \circ h$

(C) set $f = h^{-1} \circ \iota_1$

(Z) $\longrightarrow$ (M)

$\mathbb{R}^n = \{(x,y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\}$



$$g' = \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \underbrace{\left(\frac{\partial g}{\partial x} \,\middle|\, \frac{\partial g}{\partial y} \right)}_{n} \end{array} \Big\} n-k$$

wlog $\frac{\partial g}{\partial y}$ these cols are lin-indep meaning that $\frac{\partial g}{\partial y}$ is invertible

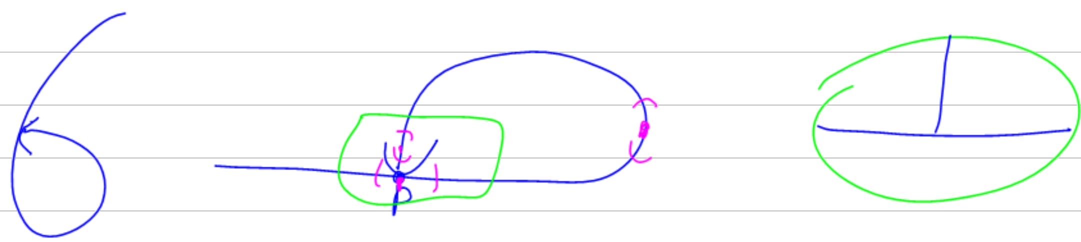
set $h(x,y) = (x, g(x,y))$

$$h' = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} I & 0 \\ * & \frac{\partial g}{\partial y} \end{pmatrix} \text{ is invertible}$$

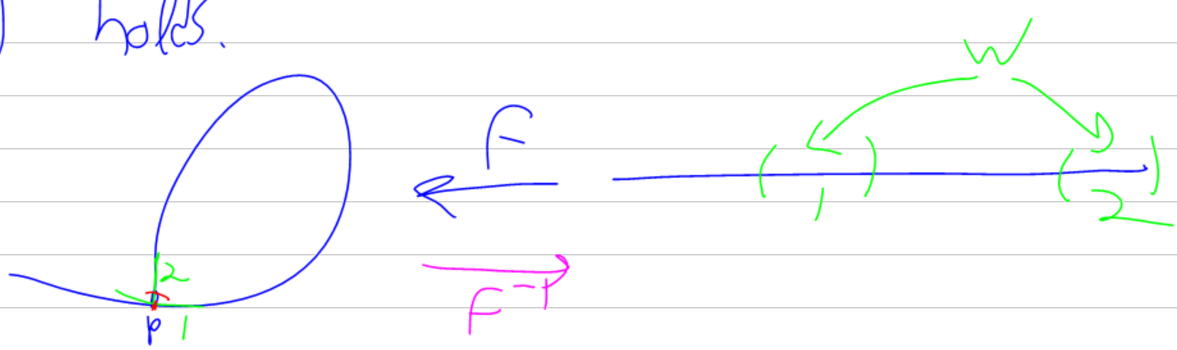
invertible

hence by the inverse function thm, h is invertible near p.

<u>Aside</u> In (C) cond ② is necessary $6 \in \mathbb{R}^2$:



(C)\(2) holds.

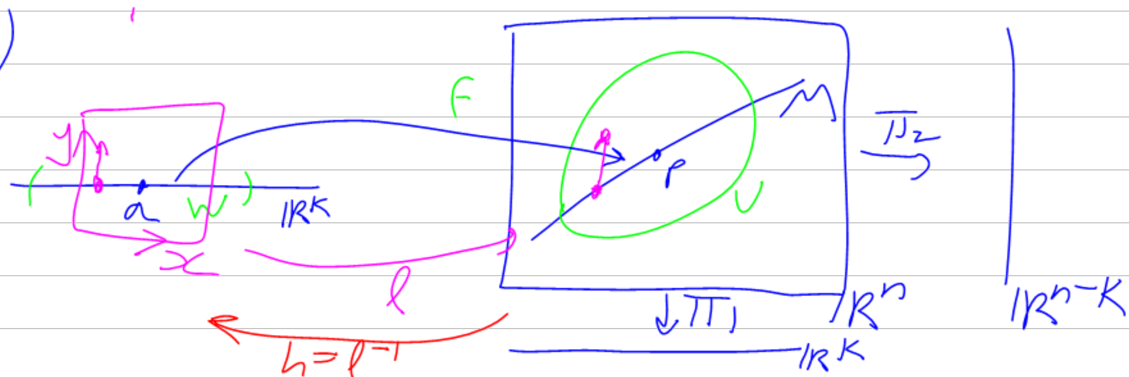


<u>Pf of (C) ⇒ (M)</u>

1 $F(W) = M \cap U$

2 $F^{-1}: M \cap U \to W$ is cont

③ rank $F'(a) = k$ on all of W

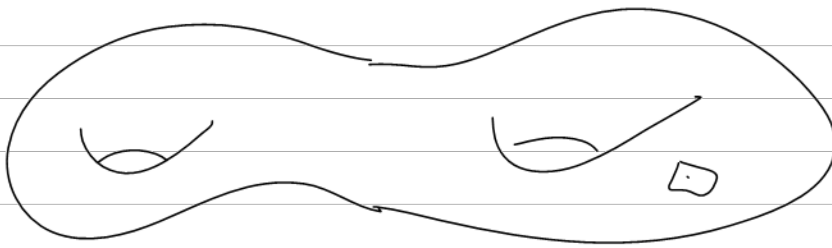


WLOG, $\operatorname{rank}(\pi_1 \circ f) = k$

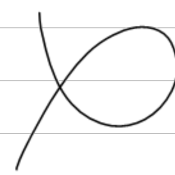
Define $\ell(x,y) = f(x) + \begin{pmatrix} 0 \\ y \end{pmatrix}$

$$\ell : \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^n \qquad \ell' = \begin{pmatrix} \frac{\partial f}{\partial x} & 0 \\ \frac{\partial f}{\partial y} & I \end{pmatrix}$$

$$\mathbb{R}^n \longrightarrow$$

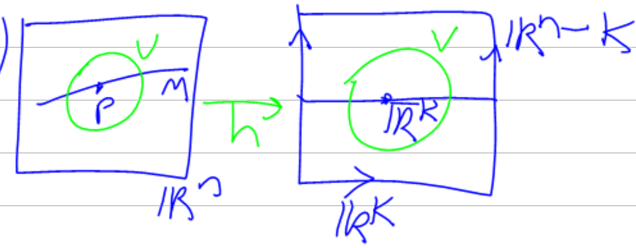
 $\subset \mathbb{R}^3$ 

$6 : (0,1) \xrightarrow[1-1]{} \mathbb{R}^2$



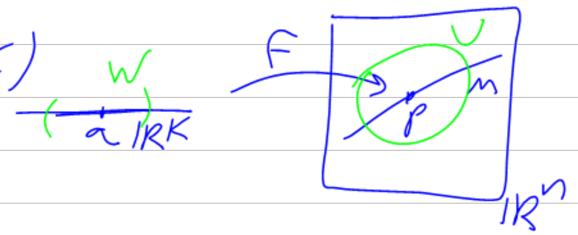
MAT257 Analysis II on March 12, 2021: Manifolds, manifolds with boundary.
Read Along: Spivak 109-115.
No news yet on TT3.
Our final assessment: Tuesday April 20 9-12 (morning TO time).
Old riddle (sol'n at end): Can you fit 125 4x2x1 boxes in one 10x10x10 cube?

Coord patches.

(M) 

$\frac{1}{2}$Thm Given $k \le n$, $M \subset \mathbb{R}^n$, $p \in M$, (C) $\Rightarrow$ (M)

(M) $\exists$ open $U \ni p$, open $V \subset \mathbb{R}_{\text{other}}$ & a diffeomorphism $h: U \to V$ s.t. $h(U \cap M) = V \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\})$.

(C) 

(C) $\exists$ open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth 1-1 $f: W \to \mathbb{R}^n$ s.t.
① $f(W) = M \cap U$ ② $f^{-1}: M \cap U \to W$ is cont. ③ $\forall a \in W$, rank $f'(a) = k$.

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wlog $f: W \longrightarrow \mathbb{R}^k_x \times \mathbb{R}^{n-k}_y = \mathbb{R}^n$, $W \subset \mathbb{R}^k$, $\pi_1: \to \mathbb{R}^k$, $\pi_2: \to \mathbb{R}^{n-k}$; $f' = \Big(\underbrace{\equiv}_{k}\Big)\, n$

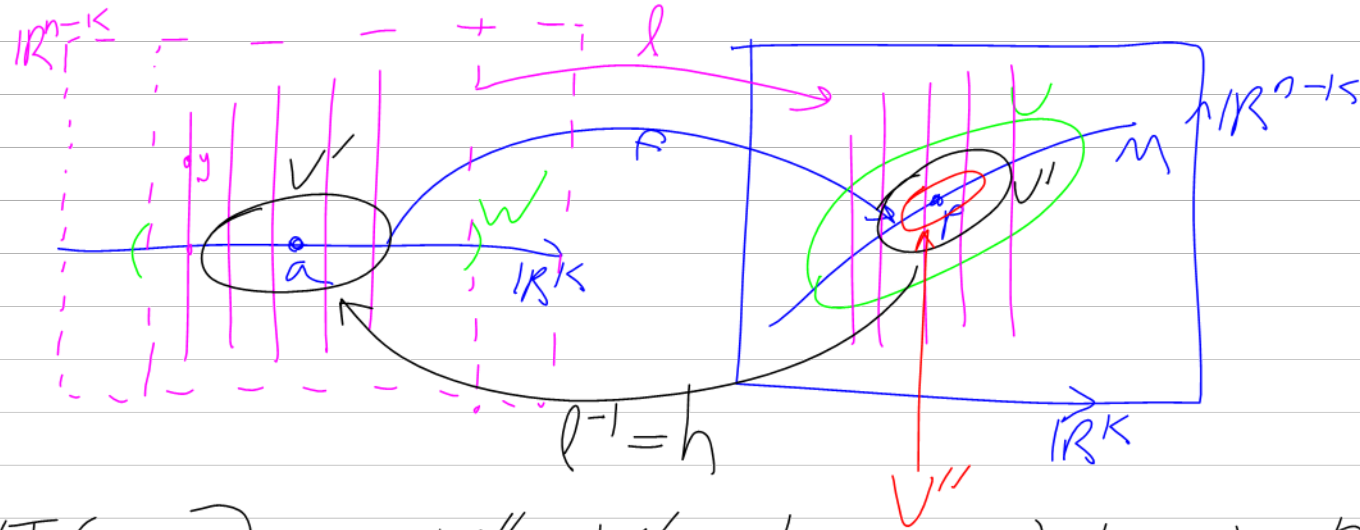
$(\pi_1 \circ f)'$ is invertible.

$\ell: W_x \times \mathbb{R}^{n-k}_y \longrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$

by $\ell: (x, y) \longmapsto f(x) + \begin{pmatrix} 0 \\ y \end{pmatrix} \in \mathbb{R}^n$

$$\ell' = \begin{array}{c|cc} & k & n-k \\ \hline k & \frac{\partial(\pi_1 \circ f)}{\partial x} & 0 \\ n-k & \frac{\partial(\pi_2 \circ f)}{\partial x} & I \end{array}$$ invertible at a.

$\Rightarrow \exists$ V' & U' s.t. $\ell|_{V'}: V' \longrightarrow U'$ is invertible where $V' \subset W \times \mathbb{R}^{n-k}$ & $U' \subset U$



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N.T.S. $\exists$ open $U'' \subset U'$ s.t. $V'' = h(U'')$, then

$$h(U'' \cap M) = V'' \cap \mathbb{R}^k$$
$$\Updownarrow$$
$$U'' \cap M = \ell(V'' \cap \mathbb{R}^k)$$

NTS. $F(V'' \cap \mathbb{R}^k) = U'' \cap M$

Recall $F^{-1} : M \cap U \longrightarrow W$ is cont.

meaning $F : W \longrightarrow M \cap U$ is "open"

carries open sets to sets of the form $\tilde{U} \cap (M \cap U)$ where $\tilde{U} \subset \mathbb{R}^n$ is open

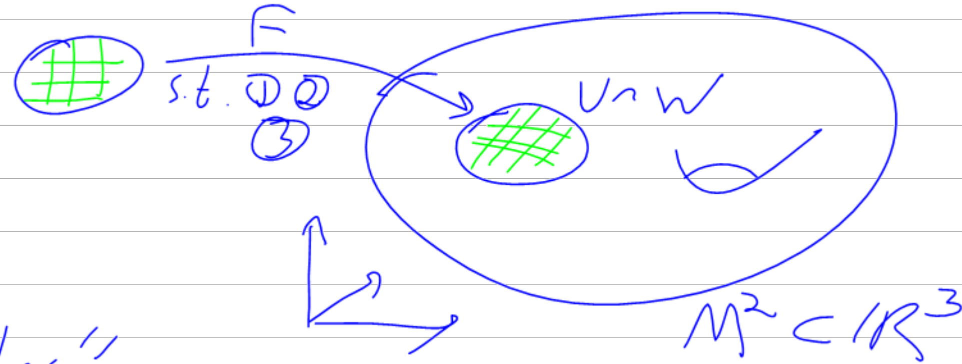
$\Rightarrow F(V' \cap \mathbb{R}^k) = \tilde{U} \cap (M \cap U)$

where $\tilde{U} \subset \mathbb{R}^n$ is open

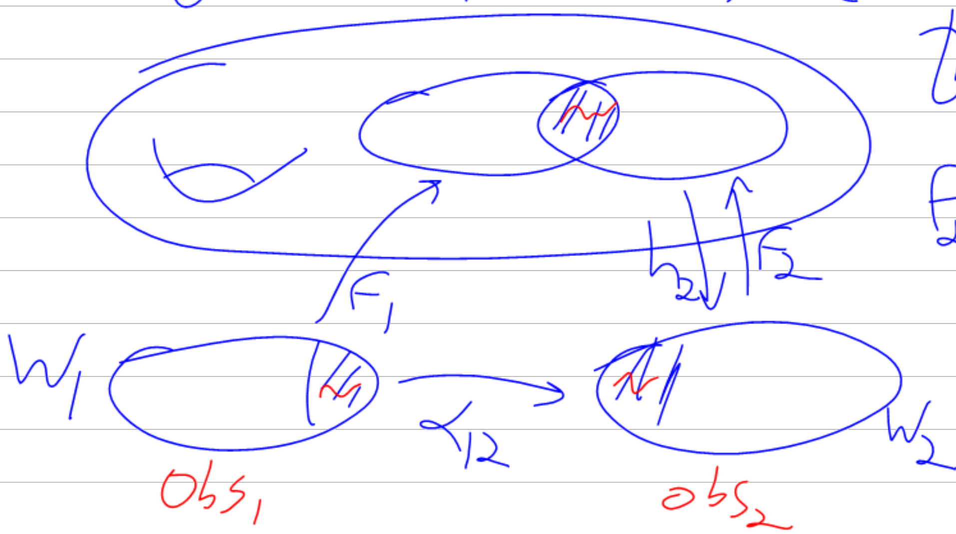
$$= M \cap \underbrace{(U \cap \tilde{U})}_{U''} = M \cap U''$$

what we wanted, w/ $V'' = V'$.

Def If $M^k \subset \mathbb{R}^n$, the f's of (C) are called "coordinate patches".



Corollary Suppose f_1, f_2 are coord. patches then $\alpha_{12} = f_2^{-1} \circ f_1 : f_1^{-1}(f_2(W_2)) \to f_2^{-1}(f_1(W_1))$ is smooth w/ a smooth inverse.



$f_2^{-1} \circ f_1 = \pi_1 \circ h_2 \circ f_1$ so it is smooth.