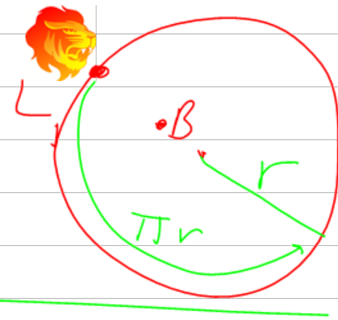
MAT257 Analysis II on November 30, 2020: Integration NT.
Read Along: Spivak 63-66 and 66-74. HW9 deadline extended to Friday!
Riddle Along: Save Ben!

$V_L = 4 V_B$



$\boxed{\text{PO1: Given } A, \mathcal{U} \ \exists\, \varphi_i \text{ loc fin, sum}=1\text{, subordinate.}}$

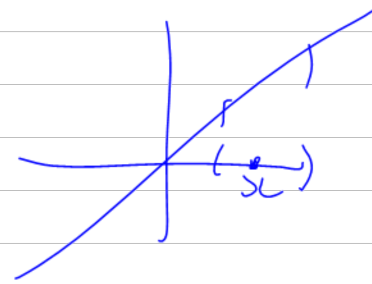
Suppose $F: A \longrightarrow \mathbb{R}$ not nec. bndd.
$A \subset \mathbb{R}^n$, open, not-necessarily bndd

Yet, locally bndd: $\forall x \in A \ \exists \epsilon > 0$ s.t. $B_\epsilon(x) \subset A$ & F is bndd on $B_\epsilon(x)$.

Also assume disc(F) is meas-0.

bndd $\Rightarrow$ loc. bndd.

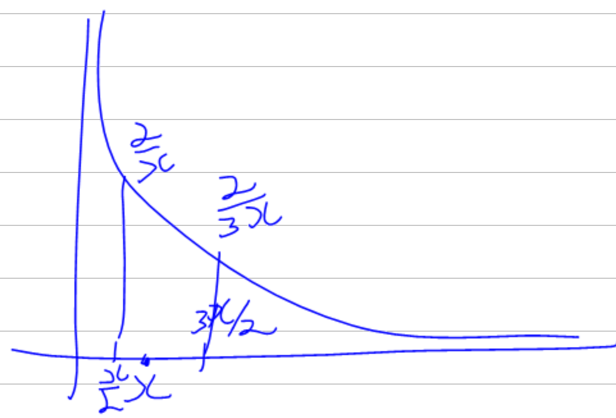
$y = x$ isn't bndd on $A = \mathbb{R}$ yet it is loc. bndd



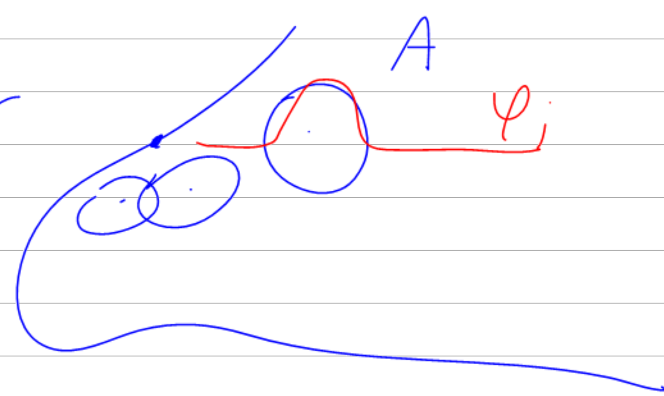
$y = \frac{1}{x}$ on $A = [x > 0]$

Given x

$B_{\frac{x}{2}}(x)$



Let $\mathcal{U}$ be some cover of A by bndd open sets on which F is bndd. Let $\Phi = \{\varphi_i\}$ be a PO1 for A subordinate to $\mathcal{U}$.



$$\int_A F = \underbrace{\left[\int_A 1\cdot F = \int_A \left(\sum \varphi_i\right) F\right]}_{\text{in Dror's imagination}} = \sum_{i=1}^{\infty} \overbrace{\int_U \varphi_i F}^{\text{makes sense!}}$$

for some $U \in \mathcal{U}$ $\operatorname{supp} \varphi_i \subset U$

Alas, summation is a problem.

Eg. $\sum (-1)^n \frac{1}{n}$ converges to $\frac{\pi}{6}$

<u>Thm</u> (retro, I beg for forgiveness)

If a_i is a seq converging to 0,

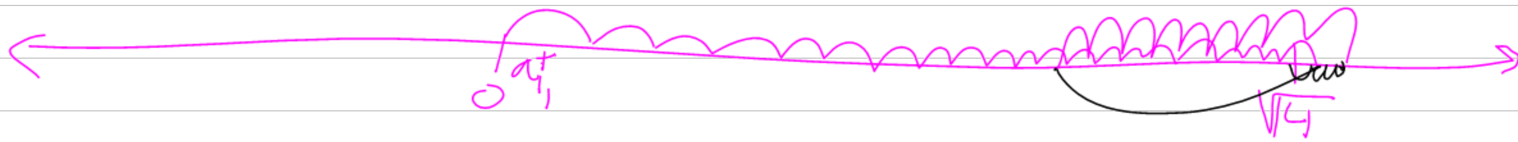
and $a_i^+ = \begin{cases} a_i & a_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$ } Good

$a_i^- = \begin{cases} a_i & a_i \leq 0 \\ 0 & \text{otherwise} \end{cases}$ } Evil

if $\underbrace{\sum a_i^+}_{\text{infinite good}}$ & $\underbrace{\sum a_i^-}_{\text{infinite evil}}$ both diverge,

then there is some permutation (b_i) of the (a_i)'s s.t. $\sum b_i = \sqrt{14}$.

$\xrightarrow[\text{good}]{\hspace{6em}}$ $\xleftarrow[\text{evil}]{\hspace{6em}}$



However, if $\sum |a_i| < \infty$ (Finite evil / Finite good)

$\sum a_n$ converges, & if (b_n) is a perm of (a_n)'s, $\sum b_n = \sum a_n$.

If $\sum |a_i| < \infty$ we say that (a_i) is absolutely convergent.

<u>Def</u> We say that f is $(\mathcal{U},\Phi)$-integrable if $\int \varphi_i |f|$ is absolutely convergent,

$$\vee\!/$$

hence $\left|\int \varphi_i f\right|$ is also absolutely conv.

In this case set $$\int_A^{(\mathcal{U},\Phi)} f = \sum_{i=1}^{\infty} \int_{\substack{R \text{ containing} \\ \text{supp } \varphi_i}} \varphi_i f$$

<u>Thm</u> If $(\mathcal{U}',\Phi')$ are another cover &

another POI, satisfying same cond., &

$\Phi' = \{\varphi_j'\}$. Then

$$\int_A^{(U,\Phi)} F = \int_A^{(U',\Phi')} F \qquad \left[\text{call this quantity } \int_A^{NT} F\right]$$

Proof

$$\int_A^{(U,\Phi)} g \overset{(1)}{=} \sum_{i=1}^{\infty} \int \varphi_i g \overset{(2)}{=} \sum_{i=1}^{\infty} \int \left(\sum_{j=1}^{\infty} \varphi_j'\right) \varphi_i g$$

$$\overset{(3)}{=} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int \varphi_j' \varphi_i g \overset{(4)}{=} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \int \varphi_i \varphi_j' g$$

$$= \sum_{j=1}^{\infty} \int \left(\sum_{i=1}^{\infty} \varphi_i\right) \varphi_j' g = \sum_{j=1}^{\infty} \int \varphi_j' g = \int_A^{(U',\Phi')} g$$

MAT257 Analysis II on December 2, 2020: Integration NT, @9:50: changes of variable.
Read Along: Spivak 63-66 and 66-74. Comment on Solution Sets!
Riddle Along: Save Vidhu!

$V_L = V_V$

$f : A^{\text{open}} \longrightarrow \mathbb{R}$ not nec. bndd. locally bndd but not necessarily bndd. disc(f) of meas-0

$\mathcal{U}$: Cover A by bndd open sets contained in A. $\Phi = \{\varphi_i\}$: PoI for A subordinate to $\mathcal{U}$

f "$(\mathcal{U},\Phi)$-integrable" means $\sum \int_U^{\text{old}} \varphi_i |f| < \infty$; $\int_A^{(\mathcal{U},\Phi)} f := \sum \int \varphi_i f$

Thm $\cancel{(\mathcal{U},\Phi)}$ NT-integrable $\Leftrightarrow$ $\cancel{(\mathcal{U}',\Phi')}$-integrable & $\int_A^{\cancel{(\mathcal{U},\Phi)}} f = \int_A^{\cancel{(\mathcal{U}',\Phi')}\, NT} f$.

PF 1.

$$\int_A^{(\mathcal{U},\Phi)} g \underset{(1)}{=} \sum_i \int_U \varphi_i g \underset{(2)}{=} \sum_i \int_U \Big(\sum_j \varphi_j'\Big)\varphi_i g \underset{(3)}{=} \sum_i \sum_j \int_{U \cap U'} \varphi_j' \varphi_i g$$

$\operatorname{supp} \varphi_i \subset U$
$\operatorname{supp} \varphi_j' \subset U'$

$$\underset{(4)}{=} \sum_j \sum_i \int_{U\cap U'} \varphi_i \varphi_j' g \underset{(3)}{=} \sum_j \int_{U'} \Big(\sum_i \varphi_i\Big)\varphi_j' g \underset{(2)}{=} \sum_j \int_{U'} \varphi_j' g \underset{(1)}{=} \int_A^{(\mathcal{U}',\Phi')} g$$

Justifications for $g = |f|$

(1) – ignore

(2) – Sum $= 1$ $\Rightarrow$ f is $(\mathcal{U},\Phi)$-integ iff it is $(\mathcal{U}',\Phi')$-integrable.

(3) – A finite sum as supp of φ_i is compact & φ_j' is loc. finite.

(4) – because all terms are ≥ 0

Justifications for $g = f$ assuming f is NT-integrable

(1) – def.

(2) – Sum $= 1$

(3) – A finite sum

(4) – by absolute convergence.

Thm 1. If $\overbrace{A}^{\text{also}}$ & f are bndd, ~~and supp $f \subset A$~~, then f is integ (NT).

2. If in addition A is Jordan-measurable, then then $\int_A^{NT} f = \int_A^{\text{old}} f$

Pf 1. Assume $|f| \leq M$, and A is contained in some rectangle R. Take some $(\mathcal{U}, \Phi)$,

$$\sum_{i=1}^{N} \int_R \varphi_i |f| = \int_R \underbrace{\left(\sum_{i=1}^{N} \varphi_i\right)}_{\leq 1} \underbrace{|f|}_{\leq M} \leq 1 \cdot M \cdot \mathrm{Vol}(R)$$

bndd increasing seq. as a fnctn of N, so it is convergent, so $\sum_{i=1}^{\infty} \int \varphi_i |f|$ converges.

(2) It is now also given that A is Jordan measurable (bd A has meas 0). So we can find a cmpct $C \subset A$ s.t. $\mathrm{Vol}(A - C) < \epsilon$ (assuming ϵ was given in advance)

For only finitely many i's $\mathrm{supp}\, \varphi_i \cap C \neq \emptyset$

Let N be bigger than all of these i's.

$$\left| \int_A f - \sum_{i=1}^{N} \int \varphi_i f \right| = \left| \int \left(1 - \sum_{i=1}^{N} \varphi_i\right) f \right|$$

$$\leq \int_A \left(1-\sum_{i=1}^N \varphi_i\right)|F| = \int_{\underbrace{A-C}_{\text{Vol} \leq \varepsilon}} \underbrace{\left(1-\sum \varphi_i\right)}_{\leq 1} \underbrace{|F|}_{\leq M}$$

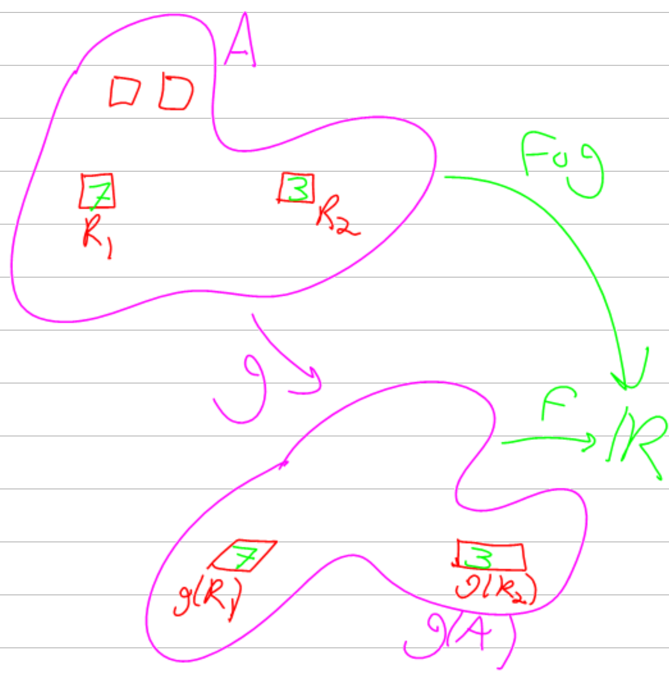
$\leq \varepsilon \cdot M$ & this can be made small.

$\square$.

<u>Thm</u> (Change of Variables, "COV")

Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \to \mathbb{R}$ is integrable, then

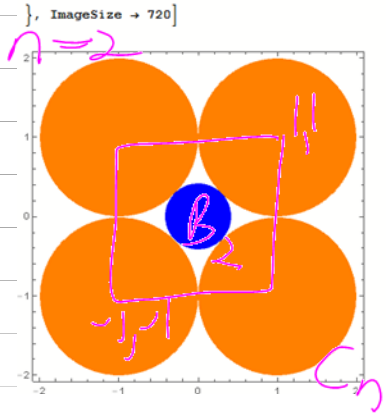
$$\int_{g(A)} F = \int_A (F \circ g)\underbrace{|\det g'|}_{\substack{\text{Jacobian} \\ \text{of } g}}$$

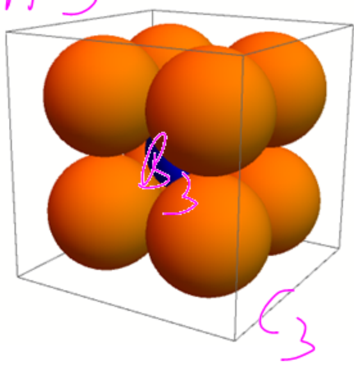


MAT257 Analysis II on December 4, 2020: Changes of variable, Gaussian integration, volumes of spheres and balls.
Read Along: Spivak 66-74.
Riddle Along: Compute lim_{n\to\infty}Vol(B_n)/Vol(C_n), where B_n is the largest ball bounded by 2^n balls of radius ones with centers at {-1,1}^n and C_n is the smallest cubes bounding same balls. Promise: You will learn something very surprising if you solve this riddle.

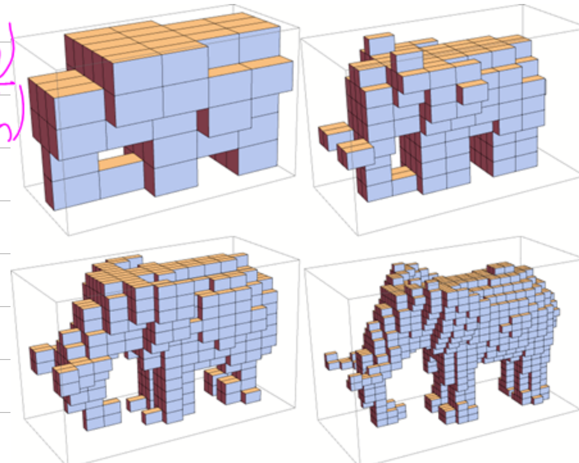
```
GraphicsRow[{
  Graphics[{Orange, Disk /@ Tuples[{1, -1}, 2], Blue, Disk[{0, 0}, √2 - 1]}, Frame -> True],
  Graphics3D[{Orange, Ball /@ Tuples[{1, -1}, 3], Blue, Ball[{0, 0, 0}, √3 - 1]}]
 }, ImageSize → 720]
```

$n=2$ $n=3$





$$\lim_{n\to\infty}\frac{Vol(B_n)}{Vol(C_n)} = ?$$

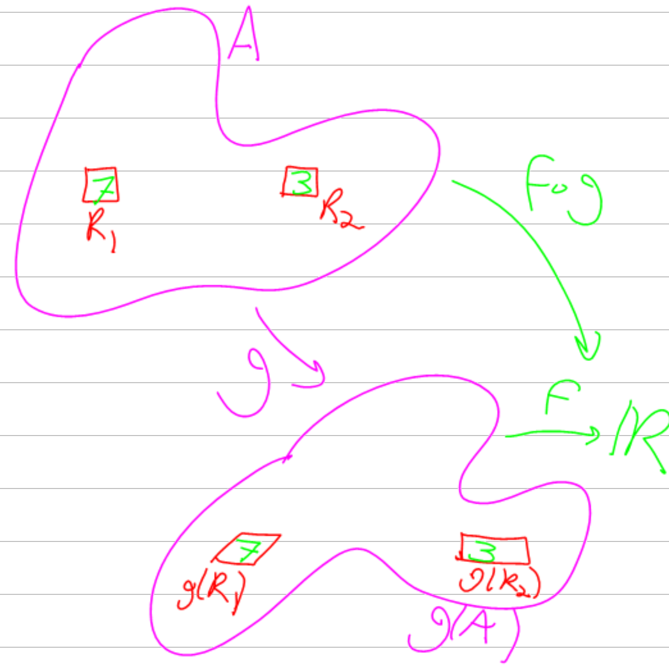


<u>Thm</u> (Change of Variables, "COV")

Let $A\subset\mathbb{R}^n$ be open, $g:A\to\mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x\in A$ $g'(x)$ is invertible. If $F:g(A)\to\mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F\circ g)\,|\underbrace{\det g'}|$$

Jacobian of g at x



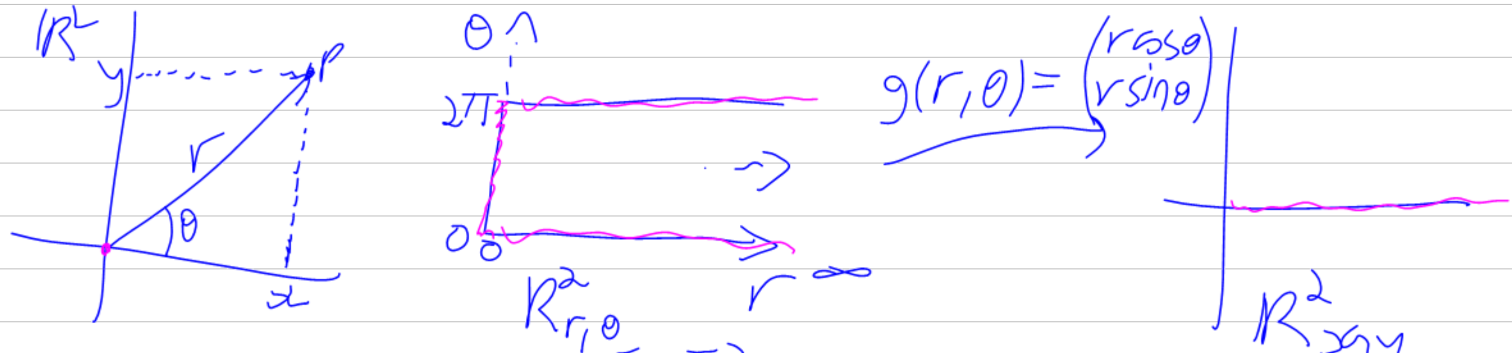
Compute $I_1=\int_{\mathbb{R}} e^{-x^2/2}dx$ "most important integral in mathematics".

$$I_2=\int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}dxdy \overset{(1)}{=} \int_{\mathbb{R}} dx\int_{\mathbb{R}} dy\, e^{-x^2/2}e^{-y^2/2}$$

$$\boxed{(2)\quad \int^{NT} F7 = \sum\int\varphi_i f\, 7 = 7\sum\int\varphi_i F \overset{?}{=} 7\int^{NF}}$$

$$\overset{(2)}{=}\int_{\mathbb{R}} dx\, e^{-x^2/2}\int_{\mathbb{R}} dy\, e^{-y^2/2} = I_1\int_{\mathbb{R}} dx\, e^{-x^2/2} = I_1^2$$

Compute I_2 by switching to "polar coordinates"



$A=[0,\infty)\times[0,2\pi)$

$$\operatorname{Jac}(g) = \det g' = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r$$

$$I_2 \overset{(3)}{=} \int_A dr\,d\theta\, e^{-r^2/2}\cdot r$$

$$\int_{g(A)} \overset{e^{-(x^2+y^2)/2}}{\overset{\shortparallel}{f}} = \int_A (f\circ g)\,|\underbrace{\det g'}_{}|$$

Jacobian of g at x

$$= \int_0^{2\pi} d\theta \int_0^{\cancel{\infty}\ R} e^{-r^2/2}\, r\,dr \qquad \Big(\lim_{R\to\infty}\Big)$$

$$(f\circ g)(r,\theta) = e^{-[(r\cos\theta)^2+(r\sin\theta)^2]/2}$$

$$= e^{-r^2/2}$$

$$\overset{(4)}{=} \int_0^{2\pi} d\theta \left[-e^{-r^2/2}\right]_0^{\infty}$$

$$\left(e^{-r^2/2}\right)' = e^{-r^2/2}(-r)$$

$$= \int_0^{2\pi} d\theta\cdot 1 = 2\pi \implies I_1 = \sqrt{2\pi}$$

Fubini for NT: Suppose $A\subset\mathbb{R}^n$ & $B\subset\mathbb{R}^m$ are open suppose $f: A\times B\to\mathbb{R}$ loc. bdd, $\operatorname{disc}(f)$ of meas-0.

$$\int_{A\times B} f \sim \int_A dx \int_B dy\, f$$

Suppose $(\mathcal{U}, \Phi)$ are a cover, POI for A

Suppose $(\mathcal{V}, \Psi)$ are a cover, POI for B

Then $\mathcal{W} = \left\{ U \times V : \begin{array}{l} U \in \mathcal{U} \\ V \in \mathcal{V} \end{array} \right\}$ is an approp. cover of $A \times B$

& $\Phi \times \Psi = \left\{ \varphi_i(x) \psi_j(y) \right\}$ is a POI for $A \times B$ sub. to $\mathcal{W}$

$$\int_{A \times B}^{NT} f = \sum_{i,j} \int^{old} \varphi_i(x) \psi_j(y) f(x,y)\, dx\, dy$$

$\left[\text{assume } f \text{ is cont.} \right]$

$$= \sum_i \int^{old} dx\, \varphi_i(x) \int^{old} dy \sum_j \psi_j(y) f(x,y)$$

$$= \sum_i \int^{old} dx\, \varphi_i(x) \int_B^{NT} dy\, f(x,y) = \int_A^{NT} dx \int_B^{NT} dy\, f$$