MAT257 Analysis II on January 11, 2021: Proof of the COV formula (3).

Read Along: Spivak 66-74.

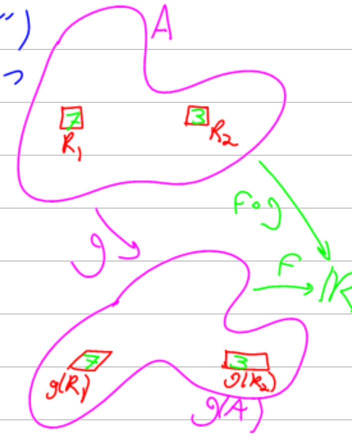
TT2 next week, discussion Wednesday.

COV Strategy: Show that every g is a composition of layer-preserving maps, and use dimensional reduction on those.

Where we were:

<u>Thm</u> (Change of Variables, "COV") Let $A\subset\mathbb{R}^n$ be open, $g:A\to\mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x\in A$ $g'(x)$ is invertible. If $F:g(A)\to\mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F\circ g)\,|\det g'|$$



Debts.

1. ~~COV(n−1) ⇒ COV(n) for l.p. maps~~
2. ~~Every g is a composition of l.p. maps & coord swaps.~~
3. ~~COV(g), COV(h) ⇒ COV(g∘h)~~
4. COV holds for coordinate swaps
5. local COV ⇒ global COV.
6. ~~Prove COV(1)!~~
7. COV(cont) ⇒ COV(integ)

Layer preserving maps:

$\xrightarrow{l.p.}$ layer preserving

$$g:\mathbb{R}^n_x\to\mathbb{R}^n_y \qquad g\begin{pmatrix}x_1\\ \vdots\\ x_{n-1}\\ x_n\end{pmatrix}=\begin{pmatrix}g_1(x)\\ \vdots\\ g_{n-1}(x)\\ x_n\end{pmatrix}$$

$$g_n(x)=x_n$$

<u>Lemma 3</u> $COV(g), COV(h)\Rightarrow COV(g\circ h)$

<u>Pf</u>
$$\int_{g(h(A))} F \overset{COV(g)}{=} \int_{h(A)} \underbrace{(F\circ g)\,|\det g'|}_{\tilde F} \overset{COV(h)}{=} \int_A (\tilde F\circ h)\,|\det(h')|$$
$$=\int_A (F\circ g\circ h)\,|\det(g'\circ h)|\,|\det(h')|$$
$$=\int_A F\circ(g\circ h)\cdot|\det(g'\circ h)\cdot h'| = \int_A F\circ(g\circ h)\cdot|\det(g\circ h)'| \quad \square$$

COV(n−1) ⇒ COV(n)

<u>Lemma 1</u> Assume COV(n−1). Let $U\subset\mathbb{R}^n$ be open & bndd. Let $g:U\to\mathbb{R}^n$ be a l.p. map cont. diff'able, 1-1, $g'(x)$ invertible & $g(U)$ is bndd. then a restricted COV holds. Namely, if $F:g(U)\to\mathbb{R}$ is bndd & cont. & supp $F\subset g(U)$, then

$$\int F = \int (F\circ g)\,|\det g'|$$

<u>Proof</u> Use Fubini on both sides.

<u>Lemma 2</u> In the conditions of the main thm, for every $a\in A$ there is some open $U\ni a$, s.t. on U g is a composition of coord. swaps & l.p. maps.

<u>Def</u>
$T_{ij}:\mathbb{R}^n\to\mathbb{R}^n$
$$T_{ij}\begin{pmatrix}x_1\\ \vdots\\ x_i\\ \vdots\\ x_j\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}x_1\\ \vdots\\ x_j\\ \vdots\\ x_i\\ \vdots\\ x_n\end{pmatrix}$$

<u>Pf</u>
$g:\mathbb{R}^n_x\to\mathbb{R}^n_y \quad y_i=g_i(x_1,\dots,x_n)$

$$\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\xmapsto{\alpha_k}\begin{pmatrix}x_1\\ \vdots\\ x_{n-1}\\ y_k\end{pmatrix}\xmapsto{\beta_k}\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}$$

<u>Lemma 6</u> COV(1).

<u>Pf</u>. Follows from the Fundamental thm of Calculus

Lemma 5 Local CoV $\Rightarrow$ global CoV:

Find a cover $\mathcal{V}=\{V\}$ of $g(A)$ by bndd open sets s.t. $\forall V\in\mathcal{V}$ $g^{-1}(V)$ is bndd & on it g is a composition of 1.p. maps & coord. swaps.

here we use g is a bijection!

Let $\{\psi_i\}$ be a PO1 for $g(A)$ sub to $\mathcal{V}$.

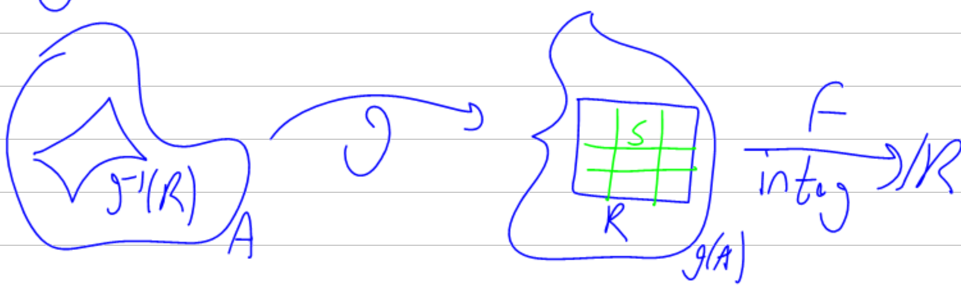
Then $\{\varphi_i=\psi_i\circ g\}$ is a PO1 for A sub to $\mathcal{U}=\{g^{-1}(V)\}$

So
$$\int_{g(A)} f=\sum_i\int_{\mathbb{R}^n}\psi_i f=\sum_i\int_{\mathbb{R}^n}(\psi_i\circ g)(f\circ g)|\det g'|=\ldots$$

Lemma 7 Suppose CoV holds for ~~cont~~ const functions f.
Then it also holds for arbitrary integrable f.

Pf We'll prove a local version which can be globalized as before

NTS:
$$\int_R f=\int_{g^{-1}(R)}(f\circ g)|\det g'|$$



Pf $L(f,P)=\sum_{S\in P}V(S)m_S(f)=\sum_S\int_S m_S(f)=$

$$=\sum_S\int_{g^{-1}(S)} m_S(f)|\det g'|\leq\sum_S\int_S (f\circ g)|\det g'|$$

$$= \sum_s \oint_{g^{-1}(R)} \chi_s (F\circ g)\,|\det g^{-1}|$$

$$\le \oint_{g^{-1}(R)} \sum_s \chi_s (F\circ g)\,|\det(g^{-1})|$$

$$= \oint_{g^{-1}(R)} (F\circ g)\,|\det(g')| \le \oint \_\_$$

$$\dots \le U(f,P) \quad \square$$

Aside: $\oint h_1 + \oint h_2 \le \oint (h_1+h_2)$

PF

$$L(h_1+h_2,P) = \sum_s V(s)\, m_s(h_1+h_2)$$

$$\ge \sum_s V(s)\big(m_s(h_1)+m_s(h_2)\big)$$

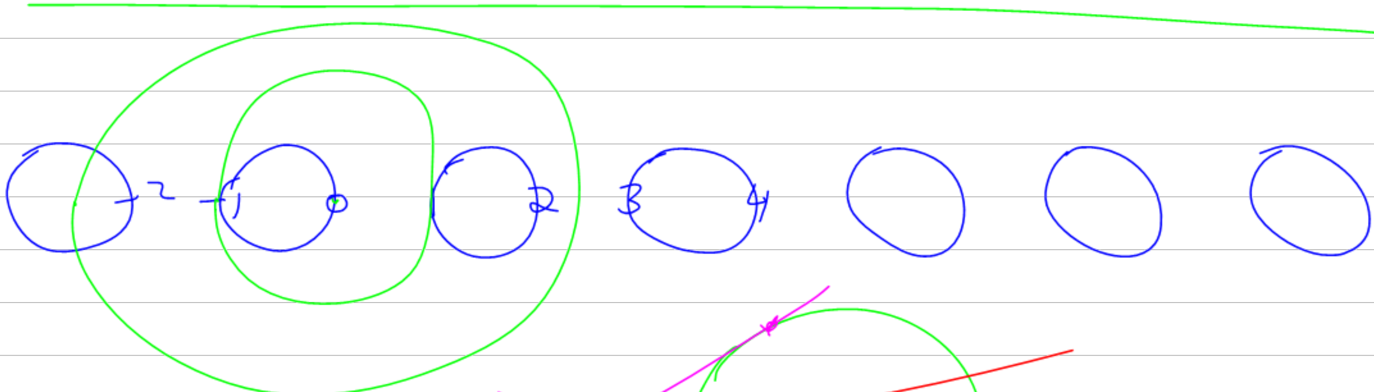
$$= L(h_1,P) + L(h_2,P)$$

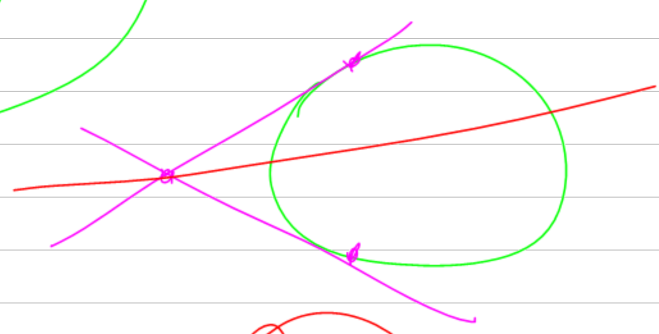
Lemma 4 COV holds for good swaps T_{ij}.

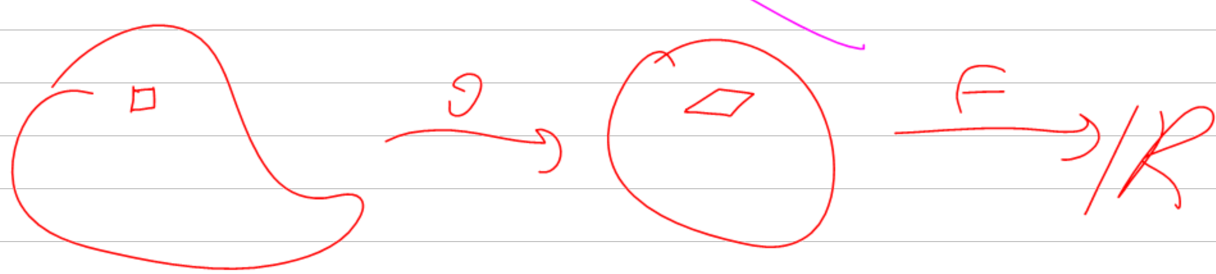
PF NTS $\displaystyle\int_{T_{ij}(A)} F = \int_A F\circ T_{ij}$ E.g., $\displaystyle\int_{TA} F(x,y) = \int_A F(y,x)$

; , , ,

$\boxed{\text{COV}}$

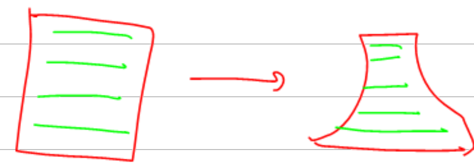






Layer-preserving Fncn:

$$g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \cancel{x} \\ \cancel{x} \\ \cancel{y} \\ x_n \end{pmatrix}$$



$$g : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad \begin{aligned} y_1 &= g_1(x_1 \ldots x_n) \\ &\vdots \\ y_n &= g_n(x_1 \ldots x_n) \end{aligned}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_K \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Office Hours and Term Test 2

Dror Bar-Natan

All Sections

Dear All,

Due to clashes of my current office hours schedule with MAT267, as of next week my office hours will take place on Tuesdays at 9-10 (as before) and at 12-1 (replacing 1-2).

Here's what I'll say in class tomorrow about Term Test 2.

- The test will take place on Tuesday January 19, 5-7PM (Toronto time), on Crowdmark (you will get a link by email about one minute before the official starting time). Other than documented accessibility matters, no exceptions!
- Our TAs Sebastian and Shuyang will hold extra pre-test office hours, in their usual zoom rooms. Sebastian on Monday 11-2 at **Sebastian's Zoom** (https://us02web.zoom.us/j/88310099689?pwd=S1dTNmJDZkRBNW5QWENWcWtyWm5QQT09) (password vchat), and Shuyang on Friday and on Tuesday at 10:30-11:30 at **Shuyang's Zoom** (https://utoronto.zoom.us/j/83428997680) (password vchat). These office hours replace some of their regular office hours; so Sebastian will not hold his regular office hours on January 25 and on February 1, and Shuyang will not hold her regular office hours on January 20 and 27.
- I will hold my regular office hours on Tuesday at 9-10 and 12-1, at **http://drorbn.net/vchat** (http://drorbn.net/vchat).
- I will be available to answer questions throughout the exam, at my usual office (**http://drorbn.net/vchat** (http://drorbn.net/vchat), but I'll add a waiting room). I will also be monitoring my regular email address (**drorbn@math.toronto.edu** (mailto:drorbn@math.toronto.edu)) throughout the exam.
- There will be mishaps! I just hope that not too many. If you encounter one, document everything with specific details, times, and screen shots, and send me a message by Wednesday January 20 at 7PM. I will deal with these situations on a case by case basis.
- Don't let unanswered questions and/or mishaps paralyze you! If you need an answer but for whatever reason you cannot reach me, think hard, come up with what you think is the most reasonable answer/resolution, document as best as you can (for example, by adding a note on your submission), and act following your conclusions.
- Material: Everything up to and including the Change of Variables Theorem, with greater emphasis on the material that was not included in Term Test 1 (meaning, starting with integration).
- Open book(s) and open notes but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor/TAs to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "5 of 5".
- You will be required to copy in your handwriting and sign an academic integrity statement and submit it on Crowdmark along with the rest of your exam. If you wish, you may save time by preparing the academic integrity statement in advance as in **this sample** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT1-LastTask.png).
- You will be given an extra 20 minutes at the end of the exam to upload it and to copy/sign the academic integrity statement.
- The vast majority of students will do honest work, and I appreciate that. Out of respect for the honest students I will do my best to pursue and punish any cheating that may occur. I'm more experienced than you! If you plan to be dishonest, think again.
- To prepare: Do the TT2 "rejects" available **here** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT2-Rejects.pdf), but more important: make sure that you understand every single bit of class material so far!
- It is not the test I want! Class material and HW are important, but there won't be questions straight from class/HW. Many things in 2020/21 are not as we want them.

Best,

Dror.

MAT257 Analysis II on January 13, 2021:
Proof of the COV formula (4) and Baby Sard.

Read Along: Spivak 66-74.

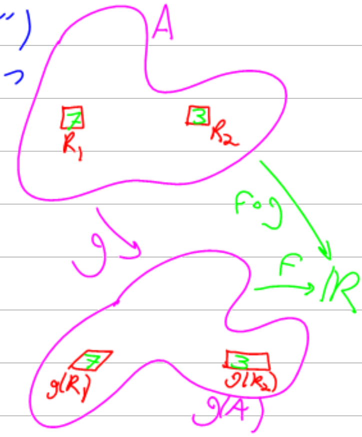
TT2 next week on Tuesday, 5-7PM.

COV Strategy: Show that every g is a composition of layer-preserving maps, and use dimensional reduction on those.

Where we were:

Thm (Change of Variables, "COV") Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diffable, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \to \mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F \circ g)|\det g'|$$



Debts.

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7. COV(cont) ⇒ COV(integ)

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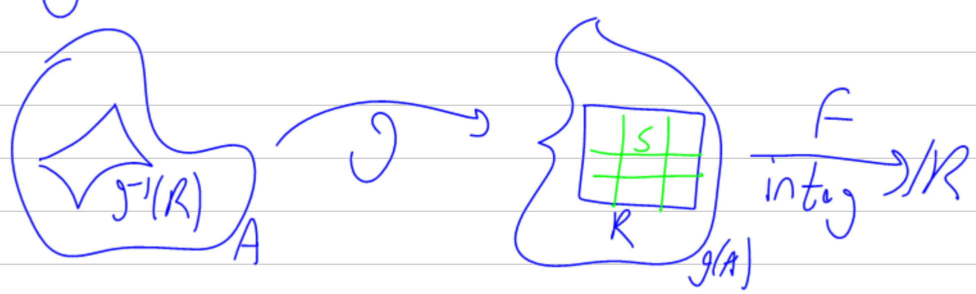
Pf We'll prove a local version which can be globalized as before.



NTS:

$$\int_R F = \int_{g^{-1}(R)} (F \circ g)|\det g'|$$

Pf

$$L(F,P) = \sum_{S \in P} v(S) m_S(F) = \sum_S \int_S m_S(F) =$$

$$= \sum_S \int_{g^{-1}(S)} m_S(F)|\det g'| \leq \sum_S \int_S (F \circ g)|\det g'|$$

$$= \sum_S \oint_{g^{-1}(R)} \chi_S (f\circ g) \cdot |\det g'|$$

$$\leq \oint_{g^{-1}(R)} \sum_S \chi_S (f\circ g) |\det(g')|$$

$$= \oint_{g^{-1}(R)} (f\circ g) |\det(g')| \leq \oint\_$$

$$\dots \leq U(f,P) \quad \square$$

Aside $\oint h_1 + \oint h_2 \leq \oint (h_1+h_2)$

PF

$L(h_1+h_2, P) = \sum_S V(S) m_S(h_1+h_2)$

$\geq \sum_S V(S)(m_S(h_1)+m_S(h_2))$

$= L(h_1,P)+L(h_2,P)$

Lemma 4 COV holds for cool swaps T_{ij}.

PF NTS $\int_{T_{ij}(A)} f = \int_A f\circ T_{ij}$ E.g., $\int_{TA} f(x,y) = \int_A f(y,x)$

...

$\boxed{\text{COV}}$ $\boxed{\text{TT2}}$

The Baby Sard Theorem $A\subset\mathbb{R}^n$ open, $g: A\to\mathbb{R}^n$ cont. diffable, $C=\{x\in A : \det(g'(x))=0\}$. Then $g(C)$ is of meas-0.

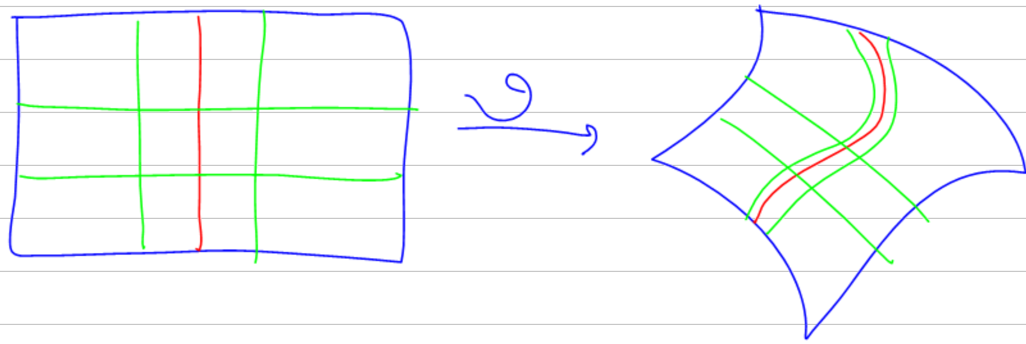
Corollary COV holds even w/o the condition "$g'(x)$ is invertible".

The Adult Sard Thm (Way harder!) $g: A \xrightarrow{} \mathbb{R}^m$, A open $\subset\mathbb{R}^n$, k-times cont. diffable where $k=\max(n-m+1,1)$, $C:=\{x : \operatorname{rank} g'(x) < m\}$. Then $g(C)$ is meas-0.

<u>Proof of Baby Sard.</u> WLOG A is a closed rectangle.

Pick $\epsilon>0$, find a partition P of A s.t. if

$S\in P$ & $S\cap C\neq\emptyset$, then all $|D_i g_j|<\epsilon$ on S.



MAT257 Analysis II on January 15, 2021: Coordinate swaps and Baby Sard.
Read Along: Spivak 66-74.
TT2 next week on Tuesday, 5-7PM.
Riddle Along: Four points form a perfect square in the plane. Can you turn that square into a larger one by a sequence of moves of the form (x,y)->(x,2x-y) performed on pairs from within our four points? (Namely, by reflections of y about x?)

Leftover:

4. cov holds for coordinate swaps

Lemma 4 cov holds for coord swaps T_{ij}.

PF NTS $\int_{T_{ij}(A)} f = \int_A f\circ T_{ij}$ E.g., $\int_{TA} f\circ T = \int_A f$

Q. How do you write the proof of something so disturbingly obvious?

A. You go back to the defs.

PF Given $A=[a_1,b_1]\times[a_2,b_2]$; $TA=[a_2,b_2]\times[a_1,b_1]$

$P=((a_1=t_{10},t_{11},\dots,t_{1n}=b_1),(a_2=t_{20},\dots,t_{2n}=b_2))$ $TP=\dots$

$L(f\circ T,\underset{\text{of } TA}{P})=\sum_{S\in P} v(S)m_S(f\circ T)=\sum_{S\in P} v(TS)m_{TS}(f)=$

$\sum_{TS\in TP}\_\_\_ = \sum_{S\in TP}\_\_\_ = L(f,TP)\dots$

$\boxed{4}\ \boxed{\text{COV}}$

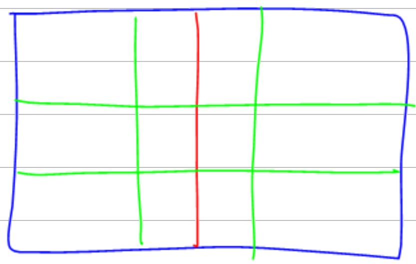
The Baby Sard Theorem $A\subset\mathbb{R}^n$ open, $g:A\to\mathbb{R}^n$ cont. diffable, $C=\{x\in A: \det(g'(x))=0\}$. Then $g(C)$ is of meas-0.

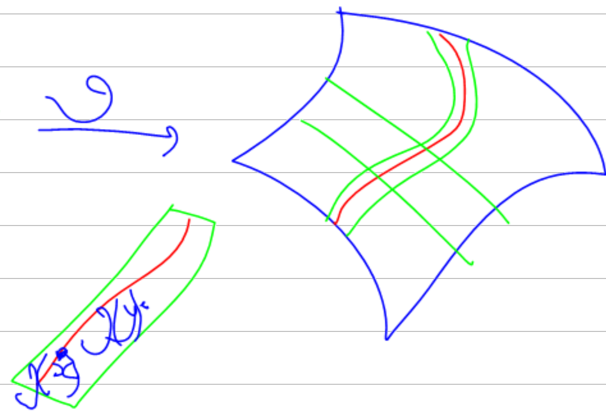
Corollary COV holds even w/o the condition "$g'(x)$ is invertible".

The Adult Sard Thm (Way harder!) $g: \underset{\text{open}\subset\mathbb{R}^n}{A}\to\mathbb{R}^m$ k-times cont. diffable where $k=\max(n-m+1,1)$, $C:=\{x:\operatorname{rank} g'(x)<m\}$. Then $g(C)$ is meas-0.

<u>Proof of Baby Sard.</u> WLOG A is a closed rectangle.

Pick $\epsilon > 0$, find a partition P of A s.t. if $S \in P$ & $S \cap C \neq \emptyset$, then $|g'(x) - g'(y)| < \epsilon$ on S.





Now pick $x \in C$ & $y \in S$,

$$\Big|\underbrace{g(y) - \big(g(x) + g'(x)(y-x)\big)}_{\text{Ided, w/ } h(y) = \Gamma,\ h(x)=0\ \&\ |h'| < \epsilon, \dots}\Big| < C_1 \epsilon |y - x| < C_1 \epsilon\, d(S)$$

diameter of S

$\underbrace{g'(x)(y-x)}$ lies in an $(n-1)$-dim subspace of $\mathbb{R}^n$

$\underbrace{g(x) + g'(x)(y-x)}$ lies in an $(n-1)$-dim hyperplane

$\Rightarrow g(S)$ lies in a cylinder of height $\epsilon C_1 d(S)$ and base area $< C_2 d(S)^{n-1}$

$\Rightarrow g(S)$ can be covered by rectangles of total volume $< C_1 C_2 \epsilon\, d(S)^n < C_3 \operatorname{Vol}(S) \epsilon$

So $g(C)$ can be covered by rects of tot vol

$$\sum_{S : S \cap C \neq \emptyset} C_3 \operatorname{Vol}(S) \cdot \epsilon < \sum_{S \in P} C_3 \operatorname{Vol}(S) \epsilon < \operatorname{Vol}(A) \cdot C_3 \cdot \epsilon \qquad \square$$

MAT257 Analysis II on January 18, 2021.
Read Along: Spivak 75-85.
TT2 tomorrow, 5-7PM.
Office hours today: Sebastian 11-2. Tomorrow Shuyang 10:30-11:30, and Dror 9-10 and 12-1.
Please be gentle with zoom chat!
* Class-related only.
* Try not to reach the point where the chat is a distraction.
* Be gentle! Be kind! You don't see everyone - your words may offend somebody you don't even see.
Riddle Along: Can you fold a rectangular piece of paper (perhaps many times) so that the result will have a longer perimeter than the original?

Remember, $\int_c d\omega = \int_{\partial c} \omega$

Def $T: V^k \to \mathbb{R}$ multi-linear; "k-tensor"

$\mathcal{T}^k(V)$, a v.s.

$\otimes: \mathcal{T}^k(V) \times \mathcal{T}^l(V) \to \mathcal{T}^{k+l}(V)$

multilinear and associative, non-commutative.

$\mathcal{T}^1(V) = V^*$ $\left[\text{E.g. } (\mathbb{R}^2)^* \text{ so } \left[\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right]^* = \begin{pmatrix}-2 & \frac{3}{2}\\ 1 & -\frac{1}{2}\end{pmatrix}\right]$

Thm V w/ basis $v_1 \ldots v_n$; $\varphi_1 \ldots \varphi_n$ the dual basis

Then $\{\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_k} : 1 \leq i_j \leq n\}$ is a basis of $\mathcal{T}^k(V)$. Hence $\dim \mathcal{T}^k(V) = n^k$.

Pf ...

Def Given $f: V \to W$, the pullback $f^*: \mathcal{T}^k(W) \to \mathcal{T}^k(V)$

(linear, respects $\otimes$)

Example T is an "inner-product"

Gram-Schmidt

$f^* T = \langle \quad \rangle$

MAT257 Analysis II on January 20, 2021.
Read Along: Spivak 75-85.
Please be gentle with zoom chat!
* Class-related only.
* Try not to reach the point where the chat is a distraction.
* Be gentle! Be kind! You don't see everyone - your words may offend somebody you don't even see.

Riddle Along: Let f be a distance-non-inreasing function from the plane to the plane (d(x,y) >= d(f(x),f(y)), for all x,y), and let R be a rectangle in the plane. Is it always true that the length of the boundary of R is greater or equal to the length of the boundary of f(R)?

Remember, $\int_C d\omega = \int_{\partial C} \omega$

Warning: Elsewhere $\mathcal{T}^k(V^*)$

Recall $\mathcal{T}^k(V)$ = "k-tensors on V" = $\left\{ \begin{array}{c} k\text{-linear maps} \\ V^k \to \mathbb{R} \end{array} \right\}$

a v.s.!

Also, $\mathcal{T}^k \times \mathcal{T}^l \to \mathcal{T}^{k+l}$ via $(T_1, T_2) \mapsto T_1 \otimes T_2$

$\mathcal{T}^1(V) = V^*$ $\left[\text{E.g. } (\mathbb{R}^2)^* = \text{Sec } \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right]^* = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \right]$

<u>Thm</u> V w/ basis $v_1 \dots v_n$; $\varphi_1 \dots \varphi_n$ the dual basis

Then $\{\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k} : 1 \leq i_j \leq n\}$ is a basis of $\mathcal{T}^k(V)$. Hence $\dim \mathcal{T}^k(V) = n^k$.

<u>Pf</u> ...

<u>Def</u> Given $f: V \to W$, the pullback $f^*: \mathcal{T}^k(W) \to \mathcal{T}^k(V)$

(linear, respects $\otimes$)

<u>Example</u> T is an "inner-product"

Gram-Schmidt, $f^* T = \langle \ \rangle$

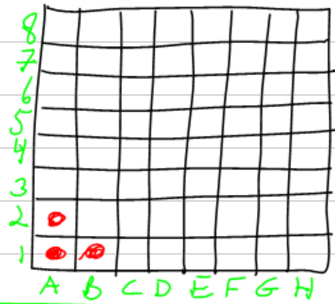
<u>Claim</u> Alternating $\Leftrightarrow$ kills repetitions

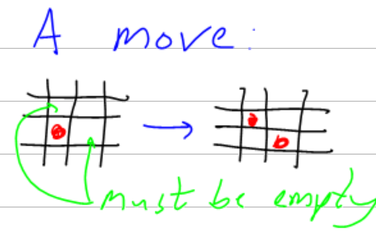
<u>Def</u> $\Lambda^k(V)$ a sub-v.s. of $\mathcal{T}^k(V)$ (Warning: $\Lambda^k(V^*)$)

Examples: Determinants, minors

Permutations & signs.

MAT257 Analysis II on January 22, 2021: Alternating Tensors.
HW11, due next Wednesday, is on the web!
Read Along: Spivak 75-85.
Riddle Along: On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied.





Notation $\underline{n}=\{1,\dots,n\}$ $\quad J=I\in\underline{n}^k$ means $I=J=(j_1,\dots,j_k)$

If $v_j\in V$, $\quad v_I=(v_{j_1},\dots,v_{j_k})\in V^k$

If $\varphi_j\in V^*$, $\quad \varphi_I=\varphi_{j_1}\otimes\dots\otimes\varphi_{j_k}\in\mathcal{T}^k(V)$

Thm V w/ basis $v_1,\dots,v_n$; $\varphi_1\dots\varphi_n$ the dual basis

Then $\{\varphi_I : I\in\underline{n}^k\}$ is a basis of $\mathcal{T}^k(V)$.

Hence $\dim\mathcal{T}^k(V)=n^k$.

Done: 1. $T_1=T_2$ in $\mathcal{T}^k(V)$ iff $\forall I\ T_1(I)=T_2(I)$.

2. $\varphi_I(v_J)=\delta_{IJ}$.

Span Given $T\in\mathcal{T}^k(V)$, we want $T=\sum_{I\in\underline{n}^k} a_I\varphi_I$

Then L.I. ...

Def Given $f:V\to W$, the pullback

(linear, respects $\otimes$) $\quad f^*:\mathcal{T}^k(W)\to\mathcal{T}^k(V)$ .....

Example T is an "inner-product"

Gram-Schmidt, $f^*T=\langle\ \ \rangle$

Claim Alternating $\Longleftrightarrow$ kills repetitions

Def $\Lambda^k(V)$ Warning: $\Lambda^k(V^*)$ a sub-v.s. of $\mathcal{T}^k(V)$

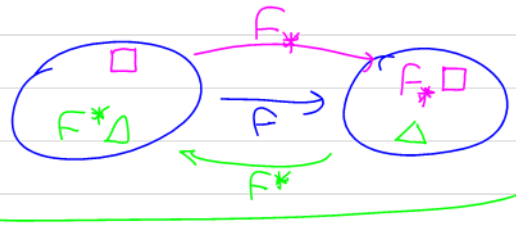
Examples: Determinants, minors.

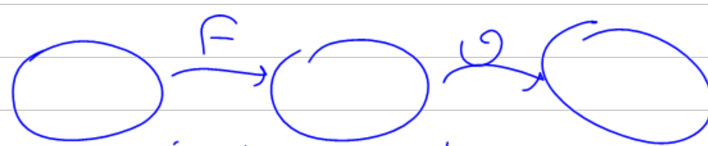
Permutations & signs.

MAT257 Analysis II on January 25, 2021: Alternating Tensors.
TT2: Not yet.
Read Along: Spivak 75-85.
Riddle Along: The Cantor set C is of measure 0. Is the same true for C+C={x+y: x,y in C}?

Recall: $I, J \in \underline{n}^k$, $\varphi_I = \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ $\quad V_I = (V_{i_1}, \dots, V_{i_k})$ $\quad \varphi_I(V_J) = \delta_{IJ}$

$\Rightarrow$ φ_I is a basis of $\mathcal{T}^k(V)$; $\dim \mathcal{T}^k(V) = n^k$





covariant & contravariant.

Claim Alternating $\Leftrightarrow$ kills repetitions

Def $\Lambda^k(V)$ a sub-v.s. of $\mathcal{T}^k(V)$ $\quad$ Warning: $\Lambda^k(V^*)$

Examples: Determinants, minors.

Permutations [make a group S_k] signs [Axiomatics, strand diagrams, $\prod \operatorname{sign}(\sigma_j - \sigma_i)$, det]

$T \in \Lambda^k \Leftrightarrow T \circ \sigma = (-1)^\sigma T$

$$(\operatorname{Alt} T)(V_1 \dots V_k) := \frac{1}{k} \sum_{\sigma \in S_k} \frac{1}{k!} T(V_\sigma)$$

Prop 1. $\operatorname{Im} \operatorname{Alt} \subset \Lambda^k$, so $\operatorname{Alt}: \mathcal{T}^k \to \Lambda^k$

2. If $w \in \Lambda^k$, $\operatorname{Alt}(w) = w$

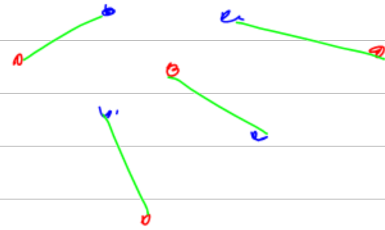
3. $\operatorname{Alt} \circ \operatorname{Alt} = \operatorname{Alt}$

$$w \wedge \eta := \frac{(k+l)!}{k!\, l!} \operatorname{Alt}(w \otimes \eta)$$

Bilinear, associative, super-commutative

"commutes" with pullbacks.

MAT257 Analysis II on January 27, 2021: More on alternating Tensors.
Read Along: Spivak 75-85.
TT2 results sent! Appeals deadline Friday Feb 5 at 5PM.
HW12 is on Crowdmark!
Riddle Along: n red points and n blue points are placed in the plane with no 3 on the same line. Prove that it is possible to pair them up using n straight line segments so that no two of the segments will



<u>Def</u> $\Lambda^k(V) = \{T \in \mathcal{T}^k(V) : T$ is alternating / anti-symmetric$\}$

A subspace!

Examples: Determinants & minors!

$S_k = \{\text{bijections } \underline{k} \to \underline{k}\}$

"The k-th symmetric/permutation group"

<u>Thm</u> $\exists!$ $\text{sign}: S_k \to \{\pm 1\}$, $\sigma \mapsto (-1)^\sigma$ s.t.

$(-1)^{\sigma\circ\lambda} = (-1)^\sigma (-1)^\lambda$, $(-1)^{\tau_{ij}} = -1$

board line

<u>Notes</u> strand diagrams, $\prod_{i<j} \ldots$, Det

$T \in \Lambda^k \iff T\circ\sigma = (-1)^\sigma T$

$\underline{n}_a^k \sim \binom{\underline{n}}{k} \subset \underline{n}^k$

ω_I

basis for $\Lambda^k(V)$

$\dim \Lambda^k(V)$

by imitating $\mathcal{T}^k(V)$!

If time, count non-dec seqs.

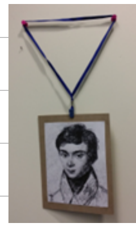
$(\text{Alt}\, T)(v_1 \ldots v_k) := \frac{1}{k}\sum_{\sigma\in S_k} \frac{1}{k!} T(v_\sigma)$

<u>Prop</u> 1. $\text{Im Alt} \subset \Lambda^k$, so $\text{Alt}: \mathcal{T}^k \to \Lambda^k$.
2. If $\omega \in \Lambda^k$, $\text{Alt}(\omega) = \omega$
3. $\text{Alt}\circ\text{Alt} = \text{Alt}$

$\omega\wedge\eta := \frac{(k+l)!}{k!\,l!}\text{Alt}(\omega\otimes\eta)$

Bilinear, associative, super-commutative, "commutes" with pullbacks.

MAT257 Analysis II on January 29, 2021:
More on alternating Tensors.
Read Along: Spivak 75-85.
Zoom recording today!
Riddle Along: Can you hang a medal on 2 nails so that if you remove any one of them, the medal will fall? Same with 3 nails? And 4?



Thm $\exists!\ \text{sign}: S_k \to \{\pm 1\}$, $\sigma \mapsto (-1)^\sigma$ s.t. $(-1)^{\sigma\circ\lambda} = (-1)^\sigma(-1)^\lambda$, $(-1)^{\tau_{ij}} = -1$

Def $\Lambda^k(V) = \{T \in \mathcal{T}^k(V) : \forall \sigma \in S_k,\ T\circ\sigma = (-1)^\sigma T\}$

$\{1 \le i_1 < i_2 < \dots < i_k \le n\} =: \underline{n}^k_a \sim \binom{n}{k} \subset \underline{n}^k$

$\lambda \in \Lambda^k(V)$ is determined by $\lambda(V_I)$, $I \in \underline{n}^k_a$!

$$W_I := \sum_{\sigma\in S_k} (-1)^\sigma \varphi_I \circ \sigma$$

Claim $W_I \in \Lambda^k(V)$, $W_I(V_J) = \delta_{IJ}$

hence $\{W_I\}_{I\in\underline{n}^k_a}$ is a basis of $\Lambda^k(V)$, hence $\dim \Lambda^k(V) = \binom{n}{k}$.

Aside: $\sigma^*: V^k \to V^k$ by $(\sigma^* u)_i = u_{\sigma(i)}$ then $(\sigma\circ\tau)^* = \tau^*\circ\sigma^*$

PF of claim.

Aside $|\underline{n}^k_{nd}| = \binom{n+k-1}{k}$ **PF** $n=5$, $k=7$: $1223555 \Leftrightarrow \boxed{1*2**3*45***}$

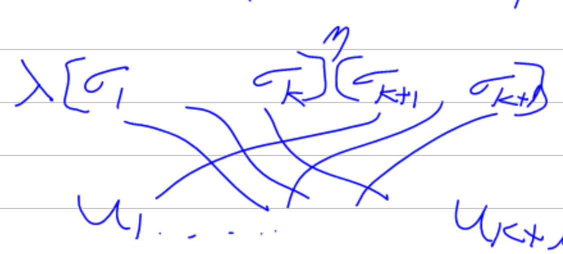
Thm $\exists!$ Bilinear $\wedge: \Lambda^k(V)\times\Lambda^l(V) \to \Lambda^{k+l}(V)$, $(\lambda, \eta) \mapsto \lambda\wedge\eta$ s.t.

1. Associative

2. Supercommutative 3. $W_I = W_{i_1}\wedge\dots\wedge W_{i_k}$.

PF Set $(\lambda\wedge\eta)(u_1 \dots u_{k+l}) = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} (-1)^\sigma (\lambda\otimes\eta)(\sigma^*(u_1 \dots u_{k+l}))$

$$= \sum_{\substack{\sigma\in S_{k+l} \\ \sigma 1 < \dots < \sigma k \\ \sigma(k+1) < \dots < \sigma(k+l)}} (-1)^\sigma \lambda(u_{\sigma 1} \dots u_{\sigma k})\ \eta(\quad)$$



MAT257 Analysis II on February 1, 2021: More on alternating Tensors: wedge products, minors, pullbacks, degrees 0 and 1, degree n and orientations, degree n-1 and cross products.
Mon -> Mon HW schedule: maybe later. Zoom recording on Friday failed :(.
Read Along: Spivak 75-85.
Riddle Along: Can you colour the points of the plane with 3 colours, so that no two points of distance exactly 1 have the same colour? How about 4, 5, 6, or 7?
Riddled before: Is there a continuous surjection f:[0,1]->[0,1] which is constant on a set of intervals whose lengths sum to 1?

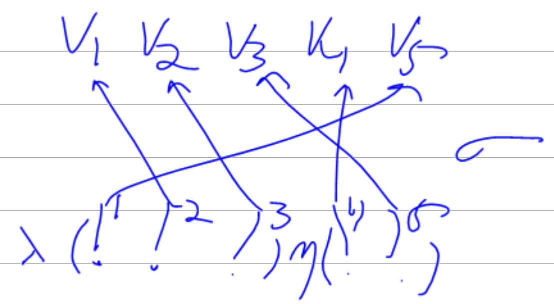
$$\int_C d\omega = \int_{\partial C} \omega$$

Thm $\exists!$ Bilinear $\wedge : \underset{\lambda}{\Lambda^k(V)} \times \underset{\eta}{\Lambda^l(V)} \to \underset{\lambda\wedge\eta}{\Lambda^{k+l}(V)}$ s.t.

1. Associative.
2. Supercommutative.
3. $\omega_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$.

Pf Set $(\lambda\wedge\eta)(u_1 \dots u_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma (\lambda \otimes \eta)(\sigma^*(u_1 \dots u_{k+l}))$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma 1 < \dots < \sigma k \\ \sigma(k+1) < \dots < \sigma(k+l)}} (-1)^\sigma \lambda(u_{\sigma 1} \dots u_{\sigma k})\, \eta(\quad)$$



Comments 1. On $\mathbb{R}^n$, $\omega_I = \dots = \lambda_I$ of Jan 27.

2. Pullbacks a. Work.

b. compatible w/ $+, \cdot, \wedge$

c. $L : V \to V$ $\quad L^* \omega_{top} = \dots$

may skip { d. $L : V_n \to V'_{n'}$, $\quad L^* \omega_{I \in (\underline{n}')^k} = \sum_{J \in \underline{n}^k} c_{IJ} \omega_J$ $\quad c_{IJ} = \det(A_{IJ})$

3. Λ^0, Λ^1

4. Λ^n and orientations.

5. Λ^{n-1} and cross products.

TA meeting notes, Feb 1 2021

$$W_I = \sum_\sigma (-1)^\sigma (\varphi_{i_1} \, \varphi_{i_2} \cdots \varphi_{i_k}) \circ \sigma^*$$

$$I = (1 \leq i_1 < i_2 < \cdots < i_k \leq n)$$

$$\Lambda^k \times \Lambda^\ell \longrightarrow \Lambda^{k+\ell}$$

1. Assoc
2. Super-commutative
3. $W_I = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$

$$(\overset{k}{\lambda} \wedge \overset{\ell}{\eta})(v_1 \, \ldots \, v_{k+\ell})$$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \lambda(v_{\sigma 1} \ldots v_{\sigma k}) \, \eta(v_{\sigma(k+1)} \cdots v_{\sigma \ell})$$

HW 13:
1. $\times$ in 3D
2. orientation pres / rev?
3. $(\Lambda^k)^* = \Lambda^{n-k}$
4.

Wed class:
pf of $\wedge$ thm
pull backs
vol elements
orientations

MAT257 Analysis II on February 3, 2021: Wedge products, minors, pullbacks, volume elements and orientations.
HW13 (a tough one!) is on the web.   Mon -> Mon HW schedule: maybe later.
Read Along: Spivak 75-85.
Riddled before: Is there a continuous surjection f:[0,1]->[0,1] which is constant on a set of intervals whose lengths sum to 1?
Riddled before: A unit cube in R^3, the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

<u>Thm</u> $\underset{\text{nts}}{\exists}\ \underset{\text{done}}{!}$ $(\underset{\Lambda^k}{\lambda}, \underset{\Lambda^l}{\eta}) \longmapsto \underset{\Lambda^{k+l}}{\lambda\wedge\eta}$ s.t.

1. Assoc. $(\lambda\wedge\eta)\wedge\phi = \lambda\wedge(\eta\wedge\phi)$

2. Super-commutative: $(\lambda\wedge\eta) = (-1)^{kl}\eta\wedge\lambda$

3. $\omega_I = \varphi_{i_1}\wedge\dots\wedge\varphi_{i_k}$

<u>Existence</u> Set $(\lambda\wedge\eta)(u_1\dots u_{k+l}) = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}}(-1)^\sigma(\lambda\otimes\eta)(\sigma^*(u_1\dots u_{k+l}))$

$$= \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}}(-1)^\sigma\lambda(u_{\sigma 1}\dots u_{\sigma k})\,\eta(u_{\sigma(k+1)}\dots u_{\sigma(k+l)})$$

$$= \sum_{\substack{\sigma\in S_{k+l}\\ \sigma 1<\dots<\sigma k\\ \sigma(k+1)<\dots<\sigma(k+l)}}(-1)^\sigma\lambda(u_{\sigma 1}\dots u_{\sigma k})\,\eta(u_{\sigma(k+1)}\dots u_{\sigma(k+l)})$$

$k=3$ $l=2$:

u_1 u_2 u_3 u_4 u_5

$$\sum_{\text{"good"}}(-1)^\sigma\lambda(\_\ \_\ \_)\,\eta(\_\ \_)$$

"a splitting"

$\sigma = [2\,4\,5\,1\,3]$

on to the proofs of $\in\Lambda^{k+l}$ & 1, 2, 3.

Pullbacks [compatible w/ all ops]

$\dim V = n$ $\quad \Lambda^n(V)$, $\Lambda^{top}(V) = \{\text{Volume elements}\} \ni \omega$

$$L: V\to V \Rightarrow L^*\omega = (\det A)\,\omega$$

Orientation 1. A basis, up to a positive-det C.O.B.

2. A volume form, up to a pos. scalar

Both pushes & pulls!

MAT257 Analysis II on February 5, 2021: Orientations, tangent vectors.
Read Along: Spivak 82-92.
Riddled before: Is there a continuous surjection f:[0,1]->[0,1] which is constant on a set of intervals whose lengths sum to 1?
Riddled before: A unit cube in R^3, the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

Orientation 1. A volume form, up to a pos. scalar

2. A basis, up to a positive-det C.O.B.

Both pushes & pulls!

$\mathbb{R}^n_p \sim T_p\mathbb{R}^n = \{(p, v)\} = \{v_p\}$

a v.s., inner product

Vector fields & component fncts.
(cont., diff'ble)

$$F(p) = \sum F^i(p)\,(p, e_i)$$

Sums, inner products.

Diff. forms:

$$\lambda(p) = \sum_{I \in \Omega^k_n} \lambda_I(p)\, w_I(p)$$

Pushing & pulling.

MAT257 Analysis II on February 8, 2021: Vector fields and differential forms.
Read Along: Spivak 86-92.
HW14 will be on the web by midnight. It will be due on the next teaching Monday.
Riddled before: Is there a continuous surjection f:[0,1]->[0,1] which is constant on a set of intervals whose lengths sum to 1?
Riddled before: A unit cube in R^3, the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.
Just learned from Tanya Khovanova, https://blog.tanyakhovanova.com/2021/02/the-anniversary-coin/: Eight out of sixteen coins are heavier than the rest and weigh 11 grams each. The other eight coins weigh 10 grams each. We do not know which coin is which, but one coin is conspicuously marked as an “Anniversary” coin. Can you figure out whether the Anniversary coin is heavier or lighter using a balance scale at most three times?

Reminder. Given $p \in \mathbb{R}^n$,

$\mathbb{R}^n_p \sim T_p\mathbb{R}^n := \{(p, v) : v \in \mathbb{R}^n\} = \{v_p\}$ Pushes!

$F : \mathbb{R}^n \to \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n$ s.t. $F(p) \in T_pV$ is a "vector field"

Can add, scale, inner-multiply, but not push or pull

$F(p) = \sum F^i(p)\,(p, e_i)$

Direction derivatives $D_{(p,v)}$: "bi"linear, Leibnitz.
compatibility w/ push, pull D_F

Diff. Forms, $\lambda(p) = \sum_{I \in \Omega^k_n} \lambda_I(p)\, \omega_I(p)$

$+, \cdot, \wedge$, Pushing & pulling, compatibilities

$\Omega^k(\mathbb{R}^n)$: Cont. diff'able k-forms.

$d : \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$

$dx^i \quad dF$

Compatibility with pull-backs.

$$\mathbb{R}^n_p \sim T_p\mathbb{R}^n$$

$$= \{(p, v)\} \qquad \xi = (p, v)$$

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2)$$

$$D_\xi F = F'(p) \cdot v = (F \circ \gamma)'(0)$$

$$(\gamma(0), \gamma'(0)) = \dot{\gamma}(0) \}$$

$$F : \mathbb{R}^n \longrightarrow \bigcup_p T_p\mathbb{R}^n$$

$$F(p) \in T_p(\mathbb{R}^n) \qquad \sim (F^i)$$

$D_F F$ again a function

$$\Omega^k(\mathbb{R}^n) = \{\lambda : \mathbb{R}^n \longrightarrow \bigcup_p \Lambda^k(T_p\mathbb{R}^n)$$

$$\lambda = \sum_{I \in \underline{n}^k_a} \lambda_I \omega_I$$

$$d : \Omega^0 \longrightarrow \Omega^1 \qquad dx_i$$

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

$\exists! \; d : \Lambda^k \to \Lambda^{k+1}$

$d(\omega \wedge \eta) = \_ \quad \_ \quad \_ \; .$

$d\lambda = \sum dx^i \wedge \frac{\partial \lambda}{\partial x_i}$

MAT257 Analysis II on February 10, 2021: Differential forms and d.
Read Along: Spivak 86-92.
Reminder - TA Office Hours: Sebastian Monday 11-12, Shuyang Wednesday 3-4; a great resource!
Learned from Tanya Khovanova, https://blog.tanyakhovanova.com/2021/02/the-anniversary-coin/: Eight out of sixteen coins are heavier than the rest and weigh 11 grams each. The other eight coins weigh 10 grams each. We do not know which coin is which, but one coin is conspicuously marked as an “Anniversary” coin. Can you figure out whether the Anniversary coin is heavier or lighter using a balance scale at most three times?

Diff. forms, $\lambda(p) = \sum_{I \in n^k_\wedge} \lambda_I(p)\, \omega_I(p)$

$+, \cdot, \wedge$, Pushing & pulling, compatibilities

$\Omega^k(\mathbb{R}^n)$: (smooth) k-forms.

$d : \Omega^0(\mathbb{R}^n) \longrightarrow \Omega^1(\mathbb{R}^n)$ $\quad dx^i \quad df$

Compatibility with pull-backs

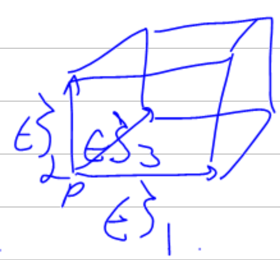
Example: $\phi : \begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$
$\phi^*(dx \wedge dy) = \ldots$

Stories about $\int_C d\omega = \int_{\partial C} \omega$, Thm

Moral definition

$$(d\omega)(\xi_1, \ldots, \xi_{k+1}) := \lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \int_{\partial P(\epsilon\xi_1, \epsilon\xi_2, \ldots \epsilon\xi_{k+1})} \omega$$

where $P(\epsilon\xi_1 \ldots \xi_{k+1}) = $



Should be 1. Linear in ω 2. Linear in the ξ_i's

3. Vanish if repetitions $\Rightarrow d\omega \in \Omega^{k+1}(\mathbb{R}^n)$

Compute $d\omega$ up to signs; get

$$d\omega = \sum_{i=1}^n dx_i \wedge \frac{\partial \omega}{\partial x_i} \longleftarrow \text{The practical def!}$$

MAT257 Analysis II on February 12, 2021: d.
Read Along: Spivak 86-92.
Reminder - TA Office Hours: Sebastian Monday 11-12, Shuyang Wednesday 3-4; a great resource!
Reminder - next class in 10 days!
Riddle Along. On Z x Z, a visible roach R starts at (0, 0) and once a minute jumps to the northeast, up to a distance of 10. Meanwhile, an exterminator E can poison one grid point per minute, away from R. Can E trap R?

Diff. k-forms on $\mathbb{R}^n$: $p \mapsto \Lambda^k(T_p\mathbb{R}^n)$ $\quad \Omega^k(\mathbb{R}^n) := \left\{\begin{matrix}\text{smooth}\\ k\text{-forms}\end{matrix}\right\}$

$d: \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ by $df(\xi) = D_\xi f$

dx_i are φ_i! so $\lambda \in \Omega^k(\mathbb{R}^n) \Rightarrow \lambda = \sum_{I \in \Omega_n^k} \lambda_I(p)\, dx_I$

Can $+, \alpha\cdot, \wedge$, pull, all nicely compatible. Compute df!

Example. $\phi: \mathbb{R}^2_{r,\theta} \to \mathbb{R}^2_{x,y}$ by $\begin{pmatrix} r\\ \theta\end{pmatrix} \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta\end{pmatrix}$. compute $\phi^*(dx \wedge dy)$.

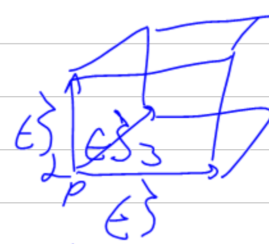
Define $d: \Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}$

Properties 1. linear 2. Leibnitz 3. pullbacks.
4. $d^2 = 0$

Stories about $\int_C d\omega = \int_{\partial C} \omega$, then Theorem: $d = d_g$ where

$$(d_g\omega)(\xi_1, \dots, \xi_{k+1}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \int_{\partial P(\epsilon\xi_1, \epsilon\xi_2, \dots \epsilon\xi_{k+1})} \omega$$

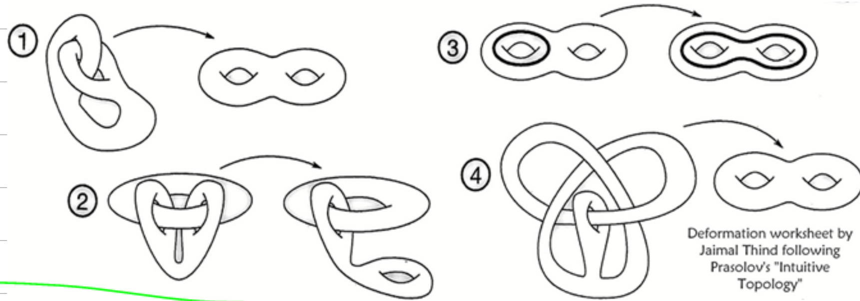
$P(\epsilon\xi_1 \dots \xi_{k+1}) =$



Should be 1. Linear in ω 2. Linear in the ξ_i's
3 Vanish if repetitions $\Rightarrow d_g\omega \in \Omega^{k+1}(\mathbb{R}^n)$

Compute $d_g\omega$ up to signs!

MAT257 Analysis II on February 22, 2021: d.
Read Along: Spivak 86-92.
Reminder - TA Office Hours: Sebastian Monday 11-12, Shuyang Wednesday 3-4; a great resource!
HW14 due tonight, HW15 on web by midnight!
Riddle Along. On Z x Z, a visible roach R starts at (0, 0) and once a minute jumps to the northeast, up to a distance of 10. Meanwhile, an exterminator E can poison one grid point per minute, away from R. Can E trap R?



$d : \Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}$

Properties 0. At $k=0$ $d_{new} f = d_{old} f$ $\square$

1. Linear $d(\omega_1 + \omega_2) = \dots$ $d(\alpha\omega) = \dots$ $\square$

2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$ Leibnitz law $\square$

3. $d(g^* \omega) = g^*(d\omega)$

4. $\Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \Omega^{k+2}$ $d \circ d = 0$, $d^2 = 0$

Prove 3 & 4, then:

Stories about $\int_C d\omega = \int_{\partial C} \omega$, then Theorem: $d = d_g$ where

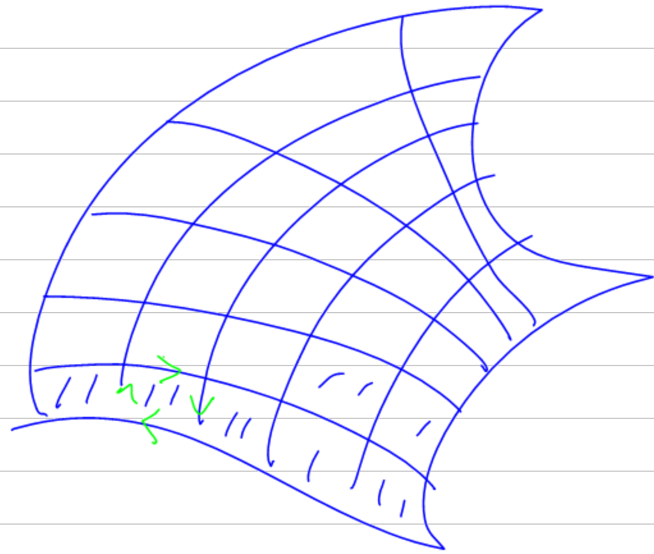
$(d_g \omega)(\xi_1, \dots, \xi_{k+1}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \int_{\partial P(\epsilon\xi_1, \epsilon\xi_2, \dots \epsilon\xi_{k+1})} \omega$

$P(\epsilon\xi_1 \dots \xi_{k+1}) =$

Should be 1. Linear in ω 2. Linear in the ξ_i's

3 Vanish if repetitions $\Rightarrow d_g\omega \in \Omega^{k+1}(\mathbb{R}^n)$

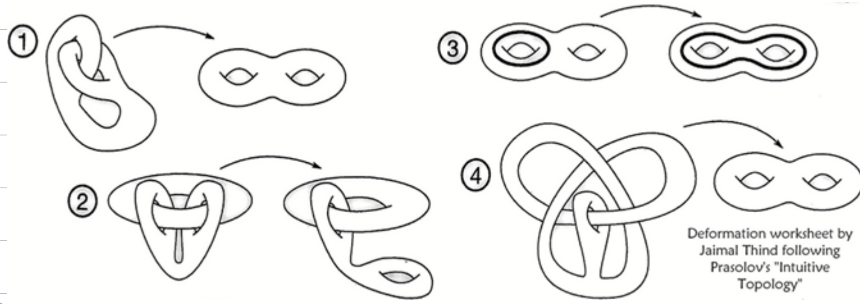
Compute $d_g\omega$ up to signs!



$$w = \sum_{i=1}^{n} (-1)^i \frac{x_i}{|x|^p} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

For which p is $dw = 0$?

MAT257 Analysis II on February 24, 2021: The geometric meaning of d.
Read Along: Spivak 86-92.
Riddle Along. Can you cover a diameter 100 disk with 99 (possibly overlapping) 100 x 1 bands?



Deformation worksheet by Jaimal Thind following Prasolov's "Intuitive Topology"

$d:\Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}$

$df(\xi) = D_\xi f$, linear, (super)-Leibnitz, compatible w/ pullbacks, $d^2 = d \circ d = 0$

Thm $(d\omega)(\xi_1, \dots, \xi_{k+1}) =$ where $\xi_i = (p, v_i)$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big[\omega(p+\epsilon v_i)(\epsilon v_1, \dots \widehat{\epsilon v_i}, \dots v_{k+1}) - \omega(p)(\epsilon v_1, \dots \widehat{\epsilon v_i}, \dots, v_{k+1}) \Big)$$

(on board ~sin!)

@ $k=0$: ...

geom meaning, relation w/ $\int_C d\omega = \int_{\partial C} \omega$.

Proof Enough to take $\omega = f \cdot \lambda$, where λ has constant coeffs. Then....

Aside closed & exact, Poincaré's Lemma,
Forms on an open set, $\lambda = \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$

Singular k-cube in $A \subset \mathbb{R}^n$: Cont. $c:[0,1]^k \to A$.
(0-cubes, 1-cubes)

$\begin{pmatrix}\text{The space of } k\text{-chains in } A\end{pmatrix} :=$ The free Abelian group generated by all k-cubes $= \left\{ \sum_{i=1}^{m} a_i c_i \right\}$

"shopping lists"

1. order immaterial
2. $0 \cdot c_i$ can be omitted
3. repeating cubes can be merged

Post-Mortem: $(t,0)+(0,t) \neq (t,t)$

Can add! $(2c_1 + 3c_2) + (c_4 + c_1 - 3c_2) = \ldots$

Can multiply by integers!

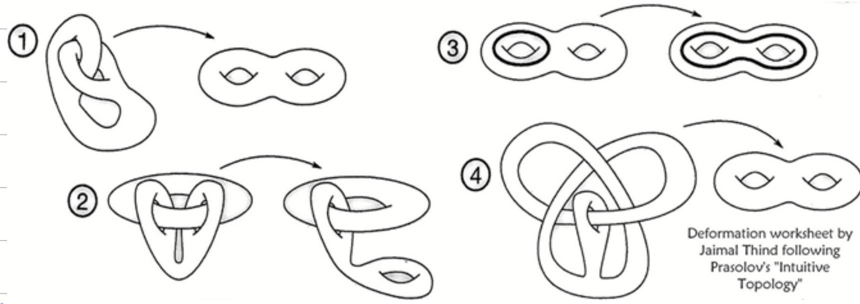
Has inverses! (Has "0")

$I^k_{(j,\alpha)} : I^{k-1} \longrightarrow I^k \qquad c_{(j,\alpha)} := c \circ I^k_{(j,\alpha)}$ "Face"

$\partial c := \sum_{j=1}^{k} (-1)^{j+\alpha} c_{(j,\alpha)}$ Extends to k-chains!

<u>Thm</u> $\partial^2 = 0$

MAT257 Analysis II on February 26, 2021: Chains.
Read Along: Spivak 92-93, (skip Poincare), 97-108.
Riddle Along: A function f:Z^3→R is superharmonic if for every x, f(x) is greater than or equal to the average of f on the 6 points adjacent to x. Is there a non-negative, non-constant, superharmonic function on Z^3?



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$df(\xi) = D_\xi f$, linear, (super)-Leibnitz,

compatible w/ pullbacks, $d^2 = d \circ d = 0$, satisfies "infinitesimal

$\int_P d\omega = \int_{\partial P} \omega$

Aside closed & exact, Poincaré's Lemma,

Forms on an open set, $\lambda = \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$

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(0-cubes, 1-cubes)

$\left(\begin{array}{c}\text{The space} \\ \text{of k-chains} \\ \text{in } A\end{array}\right) :=$ The free Abelian group generated by all k-cubes $= \left\{ \sum_{i=1}^{m} a_i c_i \right\}$

"shopping lists"
"inventories"

~~1. order immaterial~~

3. repeating cubes can be "merged"

2. $0c_i$ can be omitted

Can add! $(2c_1 + 3c_2) + (c_4 + c_1 - 3c_2) = \dots$

Can multiply by integers!

Has inverses! (Has "0")

Post-Mortem:
$(0,t) + (t,0) \neq (t,t)$.

$I^k_{(j,\alpha)}: I^{k-1} \longrightarrow I^k$ $\qquad c_{(j,\alpha)} := c \circ I^k_{(j,\alpha)}$ "Face"

$\partial c := \sum_{j=1}^{k} (-1)^{j+\alpha} c_{(j,\alpha)}$ Extends to k-chains!

Thm $\partial^2 = 0$. Lemma $(c_{(i,\alpha)})_{(j,\beta)} = (c_{(j+1,\beta)})_{(i,\alpha)}$ if $i \le j$

MAT257 Analysis II on March 1, 2021: Chains and Integration.
Read Along: Spivak 97-108.
HW15 is due tonight!
TT3 is next week, discussion on Wednesday.
Riddle Along: Can you write the function 1+xy+(xy)^2 as a sum f(x)g(y)+h(x)k(y) of two products of functions that depend on just one variable?

$$C_k(A) = \begin{pmatrix} k\text{-chains} \\ \text{in } A \end{pmatrix} = \left\{ \sum_{i=1}^{m} a_i c_i : \begin{matrix} a_i \in \mathbb{Z} \\ c_i : I^k \to A \\ \text{cont.} / \boxed{\text{diff'ble}} / \text{smooth} \end{matrix} \right\} \Big/ \begin{matrix} \text{shopping} \\ \text{list} \\ \text{rules} \end{matrix}$$

An Abelian group / a $\mathbb{Z}$-module

$I^k_{(j,\alpha)} : I^{k-1} \to I^k$ by $(x_1 \ldots x_{k-1}) \mapsto (x_1, \ldots, \overset{j}{\alpha}, x_{j+1} \ldots x_k)$

$c_{(j,\alpha)} := c \circ I^k_{(j,\alpha)}$ "face" $\quad \partial c := \sum_{j=1}^{k} \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} c_{(j,\alpha)}$

Thm $\partial^2 = 0$. Extends to k-chains!

Lemma $(c_{(i,\alpha)})_{(j,\beta)} = (c_{(j+1,\beta)})_{(i,\alpha)}$ if $i \leq j$

Pf of thm

Push or pull? Compatibility w/ ∂.

$\int_{I^k} f\, dx_1 \wedge \ldots \wedge dx_k$ No examples!

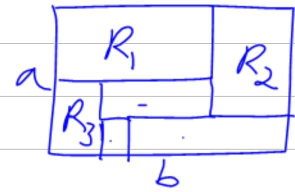
$A \subset \mathbb{R}^n$, $c \in C_k(A)$, $\omega \in \Omega^k(A)$, $\int_c \omega$

example $c = \begin{pmatrix} \cos 2\pi\theta \\ \sin 2\pi\theta \end{pmatrix}$ $\omega = \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$

example Suppose $c : I^2 \to \mathbb{R}^2_{x,y}$ is 1-1 diff'ble w/ $\det(c') > 0$ & $A = c(I^2)$. Then $\int_c dx \wedge dy = \ldots$,

Compatibility of push forwards & pull backs.

MAT257 Analysis II on March 3, 2021: Integration and Stokes' Theorem.
Read Along: Spivak 97-108.
TT3 is next Tuesday! Info and prep questions at
http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT3-Info.pdf
Riddle Along: A rectangle R has sides that are (both) not integers, and is tiled with rectangles R_i. Show that at least one of the R_i's has sides that are (both) not integer.
A Question for Deep Reflection: Mirrors flip left and right, but not up and down. How can mirrors tell horizontal from vertical?



$a, b \notin \mathbb{Z}$

$I^k_{(j,\alpha)} : I^{k-1} \longrightarrow I^k$ by $(x_1 \dots x_{k-1}) \mapsto (x_1, \dots, \overset{j}{\alpha}, x_{j+1} \dots x_k)$

$c_{(j,\alpha)} := c \circ I^k_{(j,\alpha)}$ "face" $\quad \partial c := \sum_{j=1}^{k} \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} c_{(j,\alpha)}$

$\int_{I^k} f\, dx_1 \wedge \dots \wedge dx_k := \int_{I^k} f \qquad \partial^2 = 0$

$A \subset \mathbb{R}^n, \quad c \in C_k(A), \quad \omega \in \Omega^k(A), \quad \int_c \omega := \int_{I^k} c^* \omega$

example $c = \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix} \quad \omega = \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$

[by Stokes' thm, ω is not exact]

example Suppose $c : I^2 \to \mathbb{R}^2_{xy}$ is 1-1 cube w/ $\det(c') > 0$ & $A = c(I^2)$. Then $\int_c dx \wedge dy = \dots$

Push or pull? Compatibility w/ ∂ & w/ $\int$

Reminder HW15Q1 $\quad \Omega^0(\mathbb{R}^3) \xrightarrow[\text{grad}]{d} \Omega^1(\mathbb{R}^3) \xrightarrow[\text{curl}]{d} \Omega^2(\mathbb{R}^3) \xrightarrow[\text{div}]{d} \Omega^3(\mathbb{R}^3)$

Thm $c \in C_k(A)$, $\omega \in \Omega^{k-1}(A)$, $A \subset \mathbb{R}^n$

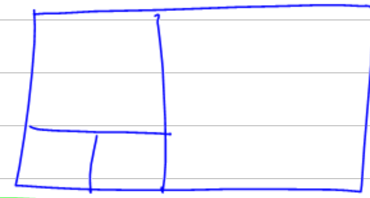
then $\int_c d\omega = \int_{\partial c} \omega$

PF WLOG, $c : I^k \to A$ is a single cube.

$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega) \overset{?}{=} \int_{\partial I^k} c^*\omega = \int_{c_* \partial I^k} \omega = \dots$

First do I^k, $\omega = f\, dx_1\, \widehat{dx_j} \dots$

MAT257 Analysis II on March 5, 2021: Stokes' Theorem.
Read Along: Spivak 97-108.
TT3 is next Tuesday! Info and prep questions at
http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT3-Info.pdf
Riddle Along: Can you partition a rectangle exactly one of whose sides is irrational into finitely many squares?
A Question for Deep Reflection: Mirrors flip left and right, but not up and down. How can mirrors tell horizontal from vertical?



$(x_1 \ldots x_{k-1}) \xmapsto{I^k_{(j,\alpha)}} (x_1, \ldots, \overset{j}{\alpha}, x_{j+1} \ldots x_k)$ $\quad \partial C := \sum_{j=1}^{k} \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} C \circ I^k_{(j,\alpha)}$

$C \in C_k(A \subset \mathbb{R}^n), \omega \in \Omega^k(A)$ $\quad \int_C \omega := \int_{I^k} \underset{\underset{f\,dx_1 \ldots dx_k}{\parallel}}{C^*\omega} := \int_{I^k} f$

Thm $C \in C_k(A \subset \mathbb{R}^n), \omega \in \Omega^{k-1}(A) \Rightarrow \int_C d\omega = \int_{\partial C} \omega$

PF WLOG, $C: I^k \to A$ is a single cube.

$\int_C d\omega = \int_{I^k} C^*(d\omega) = \int_{I^k} d(C^*\omega) \overset{?}{=} \int_{\partial I^k} C^*\omega = \int_{C_*\partial I^k} \omega = \ldots$

Then do I^k, $\omega = F\,dx_1 \ldots \widehat{dx_i} \ldots dx_k$ $: F\,dx_{\text{no } i}$

(use y_i for the variables inside faces)

Post Mortem: I should have prepared in advance the "middle term" of lhs = ... = rhs.

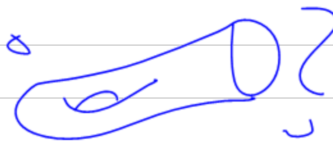
Exercise Given $C: I^k \to A \subset \mathbb{R}^n$, $\omega \in \Omega^k(A)$ and $r: I^k \to I^k$ 1-1, onto, w/ $\det r' > 0$, $\int_C \omega = \int_{C \circ r} \omega$

(show spivak p 104) done link.

MAT257 Analysis II on March 8, 2021: Manifolds.
Read Along: Spivak 109-115.
TT3 is tomorrow! Info and prep questions at http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT3-Info.pdf

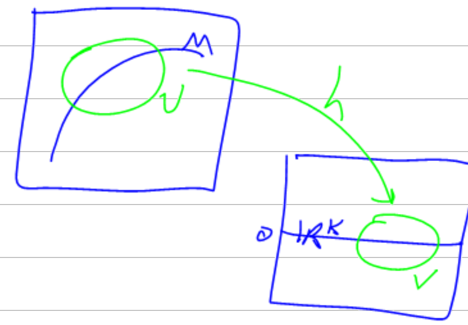
Exercise Given $c: I^k \to A \subset \mathbb{R}^n$, $\omega \in \Omega^k(A)$ and $r: I^k \to I^k$ 1-1, onto, w/ $\det r' > 0$, $\int_c \omega = \int_{c\circ r} \omega$

So integration of forms depends only on the images of cubes. Can we extend it to



?

Thm Given $k \le n$, $M \subset \mathbb{R}^n$, $x \in M$, TFAE:



(M) $\exists$ open $U \ni p$, open $V \subset \mathbb{R}^n_{\text{other}}$ & a diffeomorphism (smooth w/ smooth inverse) $h: U \to V$ s.t.

$h(U\cap M) = V \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\})$

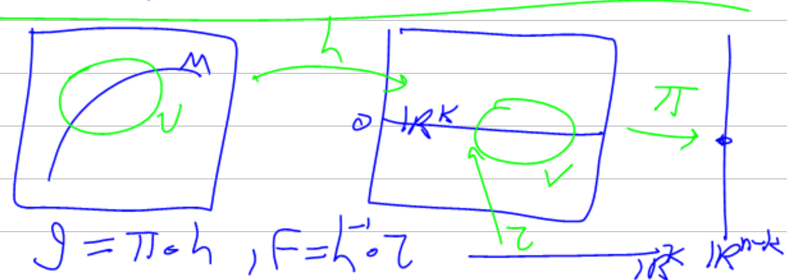
(Z) $\exists$ open $U \ni p$ & smooth $g: U \to \mathbb{R}^{n-k}$ s.t. $U\cap M = U \cap g^{-1}(0)$ & $\operatorname{rank}(g') = n-k$

(C) $\exists$ open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth 1-1 $f: W \to \mathbb{R}^n$ s.t. ① $f(W) = M\cap U$ ② $f^{-1}: M \cap U \to W$ is cont. ③ $\forall a \in W$, $\operatorname{rank} f'(a) = k$.

Def of a manifold, S^1 in two ways, $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$,

$T^2 =$  $= \left\{ \begin{pmatrix} (2+\cos\phi)\cos\theta \\ (2+\cos\phi)\sin\theta \\ \sin\phi \end{pmatrix} \right\}$

Pf of thm



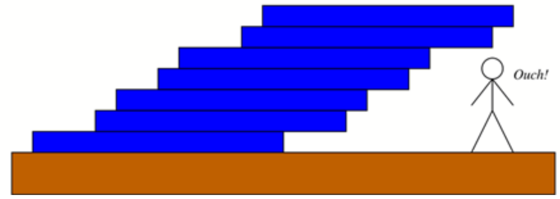
$M \Rightarrow Z, C$: Trivial

$C, Z \Rightarrow M$: Next class.

$g = \pi \circ h$, $f = h^{-1} \circ \iota$

MAT257 Analysis II on March 10, 2021: Manifolds.
Read Along: Spivak 109-115.
HW16 is on the web!
Our final assessment: Tuesday April 20 9-12 (morning TO time).

How far sideways can you reach by stacking up n identical blue domino pieces, before your tower will lean over and fall? What if n goes to infinity?
(no glue is allowed, and as shown, the stacking isn't necessarily "even")

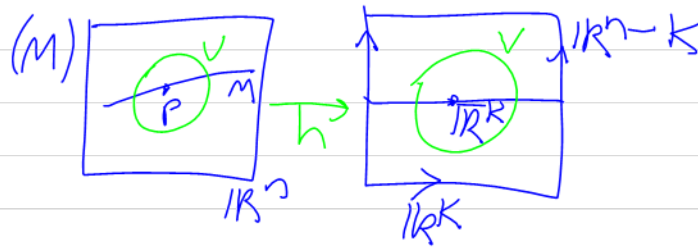


Within 10 minutes of the start of the test yesterday, the questions were posted on a question-sharing web site. Somebody thinks they are smart! We'll work to prove them wrong.

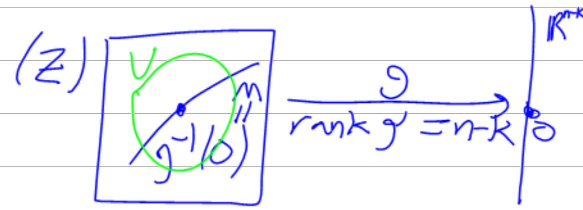
There was an issue with Q4; it will be managed after I know how many students were affected.

Thm Given $k\leq n$, $M\subset\mathbb{R}^n$, $x\in M$, TFAE:

(M) $\exists$ open $U\ni p$, open $V\subset\mathbb{R}^n_{\text{other}}$ & a diffeomorphism $h:U\to V$ s.t. $h(U\cap M)=V\cap(\mathbb{R}^k\times\{0_{\mathbb{R}^{n-k}}\})$.



(Z) $\exists$ open $U\ni p$ & smooth $g:U\to\mathbb{R}^{n-k}$ s.t. $U\cap M=U\cap g^{-1}(0)$ & $\operatorname{rank}(g')=n-k$



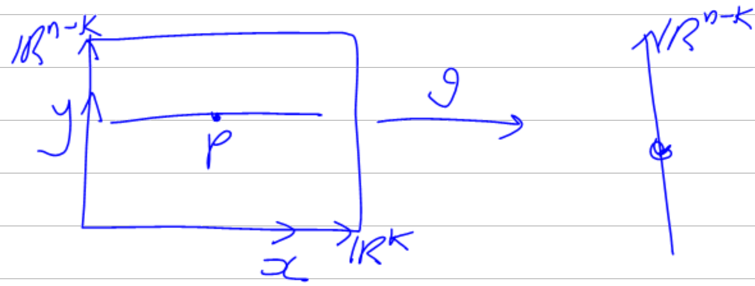
(C) $\exists$ open $U\ni p$, open $W\subset\mathbb{R}^k$ & smooth 1-1 $f:W\to\mathbb{R}^n$ s.t. ① $f(W)=M\cap U$ ② $f^{-1}:M\cap U\to W$ is cont. ③ $\forall a\in W$, $\operatorname{rank} f'(a)=k$.

Pf of thm

$M\Rightarrow Z, C$: Trivial

$g=\pi\circ h$, $f=h^{-1}\circ\iota$

$Z\Rightarrow M$:

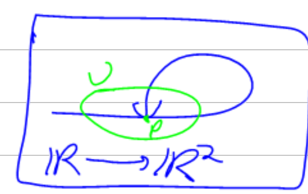


WLOG, $\operatorname{rank}\frac{\partial g}{\partial y}=n-k$

set $h(x,y)=(x,g(x,y))$ $\quad h'(p)=\begin{pmatrix} I & 0 \\ * & \frac{\partial g}{\partial y}\end{pmatrix}$

So use the IFT.

Aside: In C, cond ② is necessary:



$C \Rightarrow M$

wlog, $(\pi_1 \circ f)'$ is rank k.

Set $l(x,y) = f(x) + \begin{pmatrix} 0 \\ y \end{pmatrix}$ $\qquad l' = \begin{pmatrix} \frac{\partial f_1}{\partial x} & 0 \\ \frac{\partial f_2}{\partial x} & I \end{pmatrix}$

NTS. For smaller U'', V''

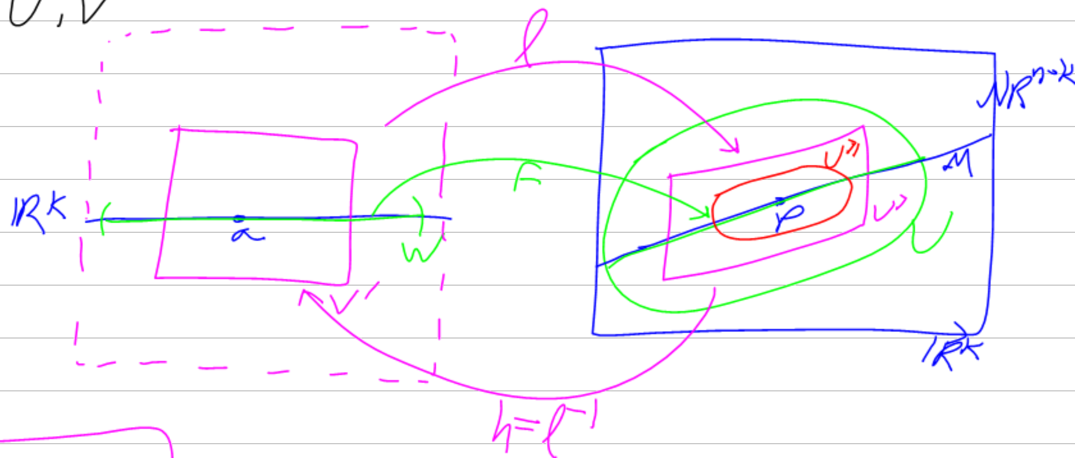
$h(U'' \cap M) = V'' \cap (\mathbb{R}^k \times \{0\})$

$\Updownarrow$

$l(V'' \cap \mathbb{R}^k) = U'' \cap M$

$\Updownarrow$

$f(V'' \cap \mathbb{R}^k) = U'' \cap M$



Aside $\quad f: \underset{\mathbb{R}^n}{\cup\!\!\!\!\!\;A} \longrightarrow \underset{\mathbb{R}^m}{\cup\!\!\!\!\!\;B}$ cont

$\Leftrightarrow \tilde{V} \subset \mathbb{R}^m$ open $\Rightarrow \exists$ open $\tilde{U}$ in $\mathbb{R}^m$ s.t. $f^{-1}(\tilde{V}) = A \cap \tilde{U}$

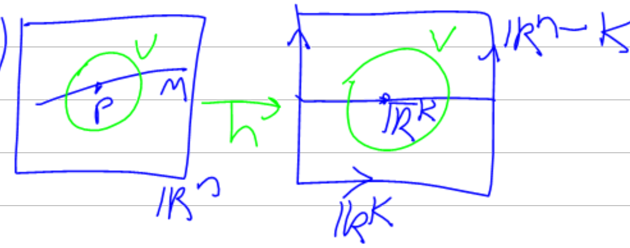
$f^{-1}: M \cap U \to W$ cont

$f: W \to M \cap U$ is open

$f(V'' \cap \mathbb{R}^k)$ is open in $M \cap U$

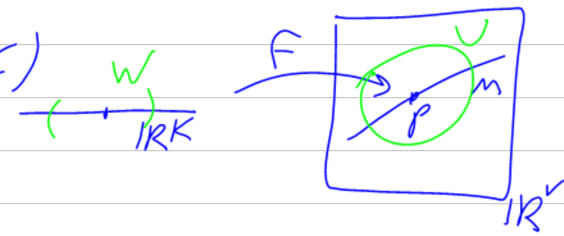
so it is $M \cap U''$

MAT257 Analysis II on March 12, 2021: Manifolds, manifolds with boundary.
Read Along: Spivak 109-115.
No news yet on TT3.
Our final assessment: Tuesday April 20 9-12 (morning TO time).
Old riddle (sol'n at end): Can you fit 125 4x2x1 boxes in one 10x10x10 cube?

(M) 

$\frac{1}{2}$Thm Given $k \leq n$, $M \subset \mathbb{R}^n$, $p \in M$, (C) $\Rightarrow$ (M)

(M) $\exists$ open $U \ni p$, open $V \subset \mathbb{R}_{\text{other}}$ & a diffeomorphism $h: U \to V$ s.t. $h(U \cap M) = V \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\})$.

(C) 

(C) $\exists$ open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth 1-1 $f: W \to \mathbb{R}^n$ s.t.
① $f(W) = M \cap U$ ② $f^{-1}: M \cap U \to W$ is cont. ③ $\forall a \in W$, rank $f'(a) = k$.

Pf For convenience, assume $(\pi_1 \circ f)' = (\pi_{1-k} \circ f)'$ is rank k

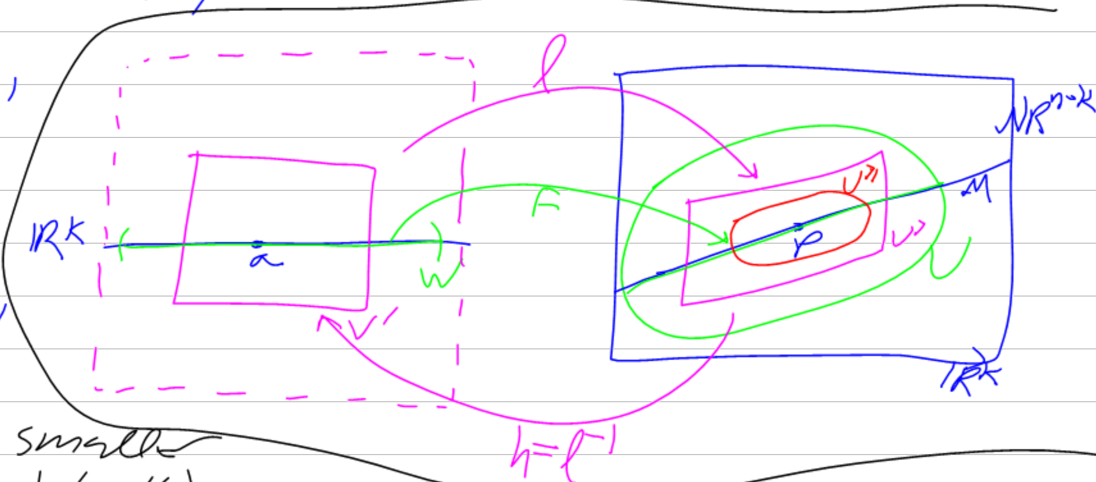
Let $l: W_x^{\subset \mathbb{R}^k} \times \mathbb{R}^{n-k}_y \to \mathbb{R}^n$ be $(x,y) \mapsto f(x) + \begin{pmatrix} 0 \\ y \end{pmatrix}$

then $l' = \begin{pmatrix} \partial f_1/\partial x & 0 \\ \partial f_2/\partial x & I \end{pmatrix}$ is invertible, so so is

$l|_{V'}: V' \to U'$,

where $V' \subset W \times \mathbb{R}^{n-k}$ & $U' \subset U$.

Let $h = l^{-1}: U' \to V'$



NTS. For an even smaller $U'' \subset U'$, w/ $V'' = h(U'')$

$h(U'' \cap M) = V'' \cap (\mathbb{R}^k \times \{0\}) \Leftrightarrow l(V'' \cap \mathbb{R}^k) = U'' \cap M \Leftrightarrow f(V'' \cap \mathbb{R}^k) = U'' \cap M$

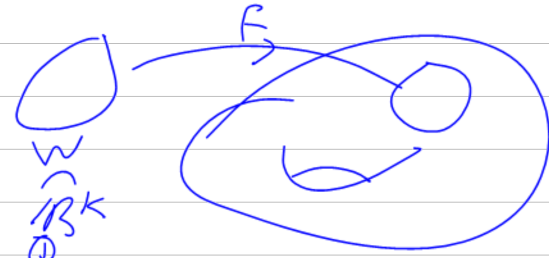
Aside $f: \underset{\subset \mathbb{R}^n}{A} \to \underset{\subset \mathbb{R}^m}{B}$ cont

$\Leftrightarrow$ $\tilde{V} \subset \mathbb{R}^m$ open $\Rightarrow \exists$ open $\tilde{U}$ in $\mathbb{R}^m$ s.t. $f^{-1}(\tilde{V}) = A \cap \tilde{U}$

$f^{-1}: M \cap U \to W$ cont
$f: W \to M \cap U$ is open
$f(V' \cap \mathbb{R}^k)$ is open in $M \cap U$
so it is $M \cap U''$ $\square$

On to coord patches, Mfld w/ ∂, ∂M, ∂M is a mfld.

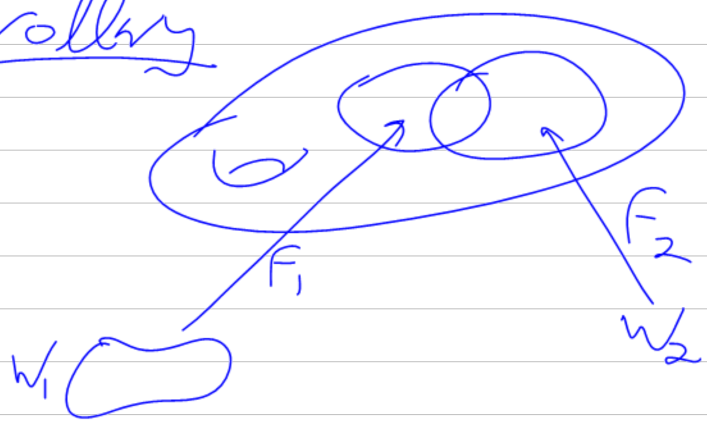
Def If $M^k \subset \mathbb{R}^n$ is a mfld, the f's of (C) are called "coord. patches"



W
$\cap$
$\mathbb{R}^k$

①
② $\forall a \in W$ $f'(a)$ is 1-1.
③

Corollary



$f_2^{-1} \circ f_1 : f_1^{-1} f_2(W_2) \to \dots$

is a diffeo.

(smooth w/ smooth inverse)

"transition functions"

Some philosophy on mflds as seen in 367

"An object is lots of partial views, along w/ the knowledge of how to move between them"

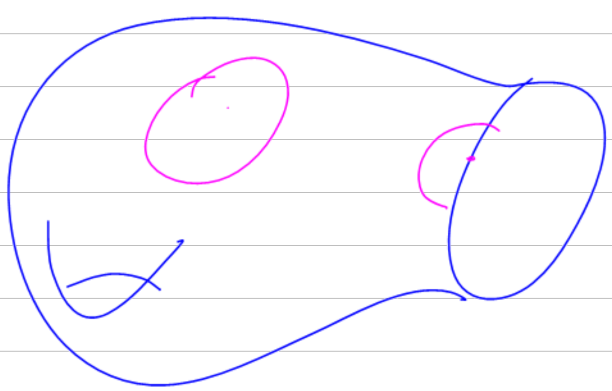
Manifold w/ bndry

$\mathbb{R}^k_{x_k \geq 0}$ $\qquad$ $\mathbb{R}^n$

$\cup$ $\qquad\qquad$ $\cup$

$f : W \longrightarrow M$

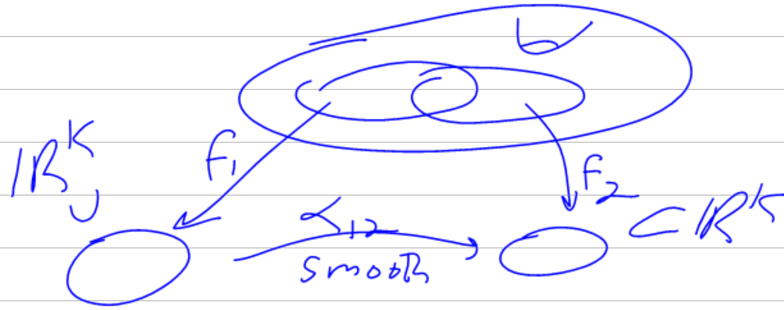
1-1 & smooth*

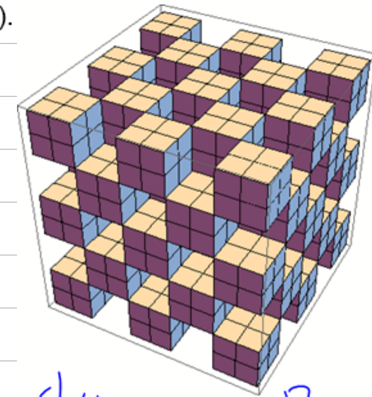


①
②
③

1. a pt is either "interior" or "bndry"
2. ∂M is a mfld.

MAT257 Analysis II on March 15, 2021: Manifolds with boundary, forms on manifolds (weight on defs).
Read Along: Spivak 113-122. HW17 on web by midnight!
No news yet on TT3.
Old riddle (sol'n at end): Can you fit 4 axb rectangles in one (a+b)^2 square? Can you fit 27 axbxc boxes in one (a+b+c)^3 cube? Why do I care?

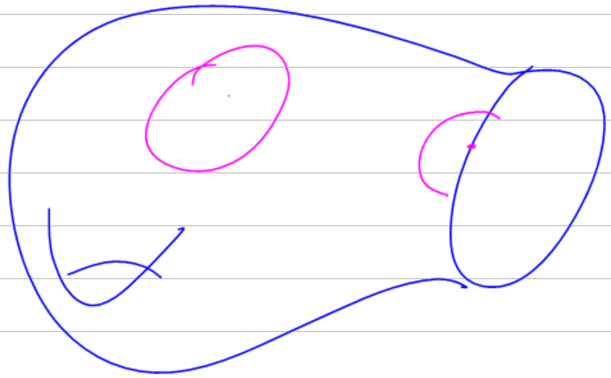




shame on me!
#black = 504

Manifold w/ bndry $\quad \mathbb{R}^k_{x_k \geq 0} \supset W \xrightarrow{F} M \subset \mathbb{R}^n$

$F: W \to M$ 1-1 & smooth*



① ② ③

1. a pt is either "interior" or "bndry"
2. ∂M is a mfld.

Tangent space M_p, T_pM [image from a patch, indep of the patch]

Vector Fields [smoothness]

p-forms

d

orientation $\quad$ <u>Examples</u> S^1, S^2, pt
Möb

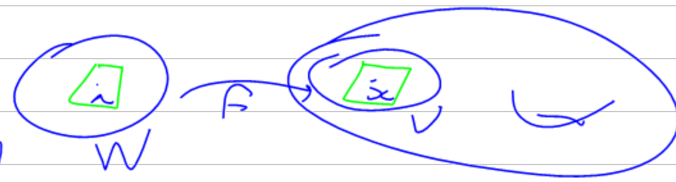
orientable, oriented

orienting the bndry by prepending & outward pointing normal

MAT257 Analysis II on March 17, 2021: Forms on manifolds, orientations.
Read Along: Spivak 113-122.
TT3 Report.
Old riddle (sol'n at end). The Moscow Subway Problem: Can you fit a box of dimensions a'xb'xc' inside a box of dimensions axbxc, if a'+b'+c'>a+b+c?

$M_x := T_xM := f_* T_a W$

indep of the patch!



There is $f_*: T_aW \to T_xM$, so there is $f^\#: T_xM \to T_aW$

Vector Fields [smoothness]

p-forms. $f^*: \Omega^p(U) \to \Omega^p(W)$ an iso!

$+, \wedge, d$ make sense & obey the same rules!

Example on S^2 $\quad x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$

$x\,dx + y\,dy + z\,dz$

orientation Examples S^1, S^2, pt

Möb

orientable, oriented.

orienting the bndry by prepending & outward pointing

$$\left(\frac{x\,dy - y\,dx}{x^2+y^2}\right)\wedge dz = \frac{\frac{x^2}{z}+\frac{y^2}{z}}{x^2+y^2}\,dx\,dy = \frac{1}{z}\,dx\,dy$$

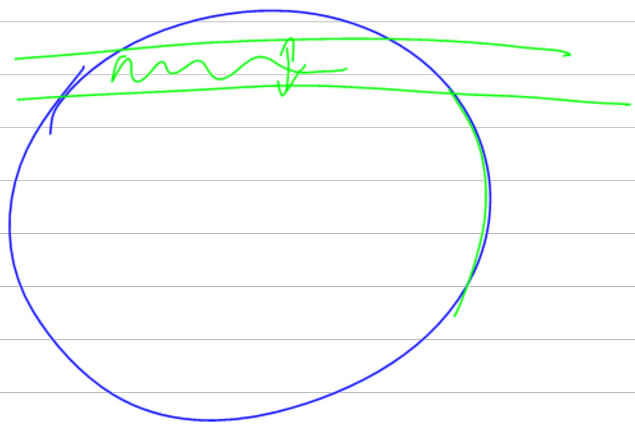
$$x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \left(\frac{x^2+y^2}{z} + \frac{z^2}{z}\right)dx\,dy$$

$$x\,dx + y\,dy + z\,dz = 0 \implies$$

$$x\,dz\,dx = y\,dy\,dz$$

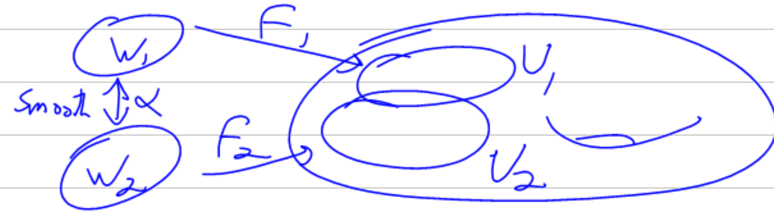
$$y\,dx\,dy = z\,dz\,dx$$

$$z\,dy\,dz = x\,dx\,dy$$



MAT257 Analysis II on March 19, 2021: Orientations.
Read Along: Spivak 113-122.
Old riddle (sol'n at end). On any pair of potatos, can you draw a pair of 3D congruent curves?

Work patch by patch, use the smooth α to show that patches agree: T_xM, $\Omega^l(M)$, $+$, $\wedge$, d.

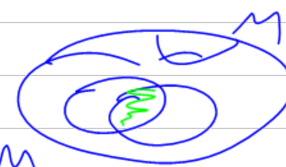


Sometimes hard to tell if $W_1 = W_2$.

Comments 1. Smooth maps $M^k \to N^l$ make sense, everything pushes & pulls under the usual rules.

2. Cubes, chains, $C^l(M)$, ∂, $\int_{\text{chains}}$ & Stokes thm all make sense.

One further ingredient is necessary to integrate on manifolds.



Can we choose coord. patches for the whole M (an "atlas") s.t. all trans. functions have $\det \alpha' > 0$?

Reminder orientation on V^k:

$$\frac{\text{ordered basis}}{\text{pos det changes}} \sim \frac{\eta \in \Lambda^k(V)}{\text{mult by pos scalars}}$$

Orientation of M   Examples S^1, S^2, pt
Möb

orientable, oriented.

Orienting the bdry by prepending (outward normal))

$$\eta_{\partial M}(x) = \iota^*_{\partial M \to M} \, \iota_\nu \eta_M(x)$$

MAT257 Analysis II on March 22, 2021: Orientation, integration.
Read Along: Spivak 117-125.
HW17 due on Wednesday, HW18 on web by midnight.
Old riddle (sol'n at end). Can you find uncountably many sets of natural numbers, such that the intersection of any two of them is finite?

Orientation of M^k: A choice of an orientation for each T_xM, which can be presented in a locally ~~continuous~~ constant manner.
"orientable": has an orientation. "oriented": comes with one.

Comment. If M^k is connected, it has 0 or 2 orientations.

Orienting the bdry by prepending outward normal (ν)

Examples: D^2, D^3, I^k, D^1 roughly

$$\eta_{\partial M}(x) = i^*_{\partial M \to M}\, \iota_\nu \eta_M(x)$$

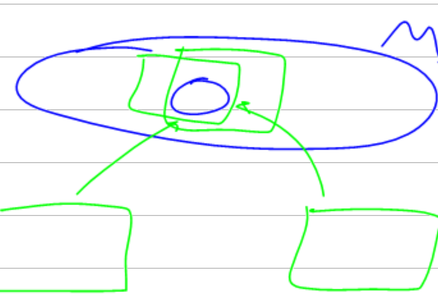
D^1 precisely

Proposition Let C_i $(i=1,2)$ be smooth injective orientation preserving k-cubes in an oriented M^k, and let $\omega \in \Omega^k(M)$ be s.t.

$$\operatorname{supp}\omega \subset C_1(I^k) \cap C_2(I^k)$$

then $\int_{C_1} \omega = \int_{C_2} \omega$; Def Call this $\int_M \omega$

Proof & injectivity warning



Now general integration using POI, assuming $\sum \int |\varphi_i \omega|$ is absolutely convergent

Indep. of the POI.

Stokes' Theorem

MAT257 Analysis II on March 24, 2021: Orientation, integration.
Read Along: Spivak 117-125.
HW17 is due tonight, HW18 is on web.
Old riddle (sol'n at end). Can you find uncountably many sets of natural numbers, such that for any two of them, one contains the other?

Suppose $x\in\partial M\subset M$, and $\nu\in T_xM$ is the outward-pointing normal to $T_x\partial M$

then $\mathcal{O}_{T_xM}=[\nu,u_1,\dots,u_{k-1}]$ $\quad\mathcal{O}_{T_x\partial M}=[u_1,\dots,u_{k-1}]$

Examples. D^2, D^3, I^k, now D^1 first roughly, then

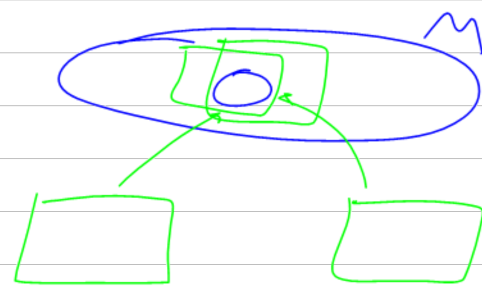
$\eta_{\partial M}(x)=i^*_{\partial M\to M}\,\iota_\nu\,\eta_M(x)$, then D^1 precisely.

__Proposition__ Let C_i $(i=1,2)$ be smooth injective orientation preserving k-cubes in an oriented M^k, and let $\omega\in\Omega^k(M)$ be s.t.

$$\operatorname{supp}\omega\subset C_1(I^k)\cap C_2(I^k)$$

then $\int_{C_1}\omega=\int_{C_2}\omega$; __Def__ Call this $\int_M\omega$

Proof & injectivity warning



Now general integration using POU, assuming $\sum\int|\varphi_i\omega|$ is absolutely convergent

Indep. of the POU.

Stokes' Theorem

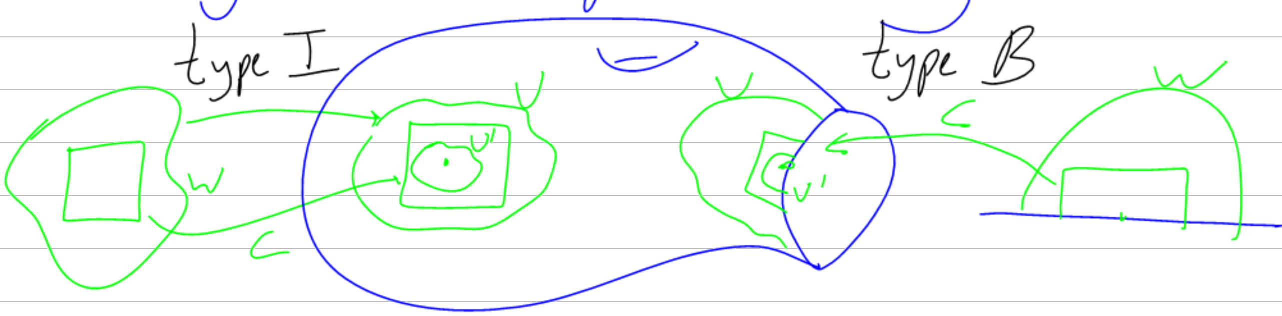
MAT257 Analysis II on March 26, 2021: Integration, Stokes' Theorem.
Read Along: Spivak 122-126.
Old riddle (sol'n at end). n prisoners. Each wears a tower of infinitely-many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right? Can they do better than 1/2^n?

Proposition Let C_i $(i=1,2)$ be smooth injective "good cubes" orientation preserving k-cubes in an oriented M^k, and let $\omega\in\Omega^k(M)$ be s.t. $\operatorname{supp}\omega\subset C_1(I^k)\cap C_2(I^k)$ then $\int_{C_1}\omega=\int_{C_2}\omega$; **Def** Call this $\int_M\omega$

works near ∂M too!

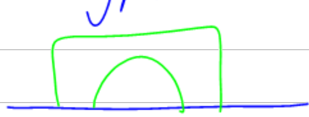
Note: 1. where it makes sense, $\int\omega$ is linear in ω.
2. $\int_{-M}\omega=-\int_M\omega$.

Now general integration using PO1.

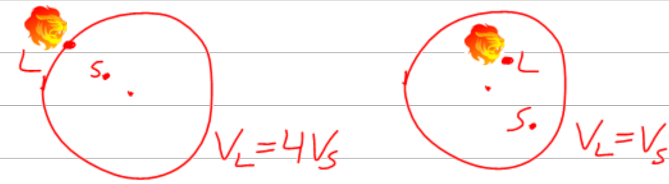


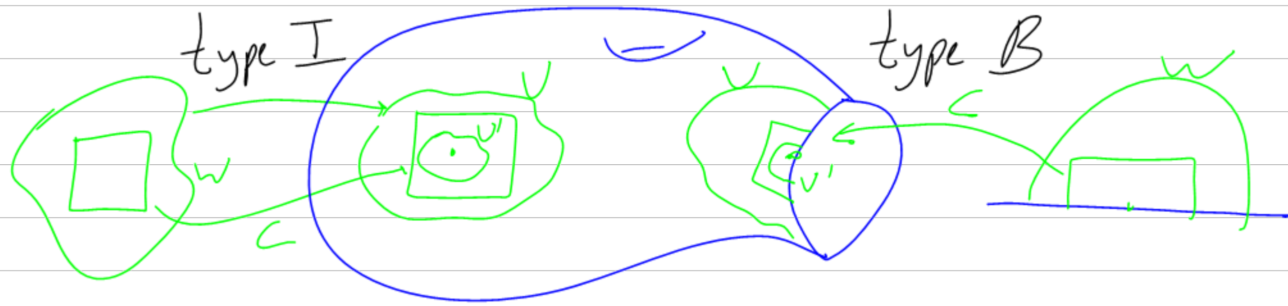
1. Assume $\sum\int|\varphi_i\omega|$ is absolutely convergent [automatic if M is compact]
2. Indep. of the PO1.
3. The type B nbds, restricted, give cubes & PO1 for ∂M.

Stokes' Thm If M is compact and oriented & $\omega\in\Omega^{k-1}(M)$, then $\int_M d\omega=\int_{\partial M}\omega$

Type I  Type B  Then merge ......

MAT257 Analysis II on March 29, 2021: Stokes' Theorem.
Read Along: Spivak 122-126.
HW18 due and HW19 assigned by 11:59PM.
Old riddles (sol'n at end). Survive!





Stokes' Thm If M is compact and oriented & $\omega\in\Omega^{k-1}(M)$, then $\int_M d\omega=\int_{\partial M}\omega$

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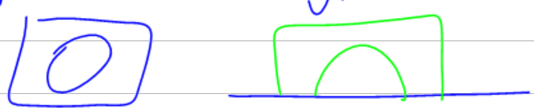
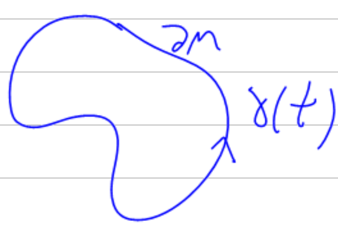


Then merge $\int_{\partial M}\omega=\sum\int_{\partial M}\varphi_i\omega$

$$=\sum\int_M d(\varphi_i\omega)=\sum\int_M(d\varphi_i)\cdot\omega+\sum\int_M\varphi_i\,d\omega$$

Example. The Fundamental Thm of Calculus.

Example. Green's theorem. $\omega=P\,dx+Q\,dy$

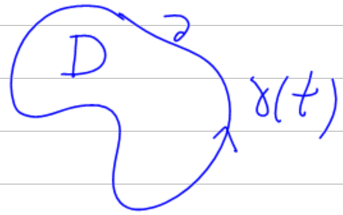
$M=$ 

$$d\omega=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\wedge dy$$

Two intuitions!

MAT257 Analysis II on March 31, 2021: Green's Theorem, the planimeter.
Read Along: Spivak 122-126.
No class Friday!
I'm giving a talk in Moscow (ha) tomorrow. Wish me luck!
Old riddles (sol'n at end). Four points form a perfect square in the plane. Can you turn that square into a larger one by a sequence of moves of the form (x,y)->(x,2x-y) performed on pairs from within our four points? (Namely, by reflections of y about x?)

Green's theorem.



$$W = P\,dx + Q\,dy$$

$$dW = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy$$

(If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, measures area)

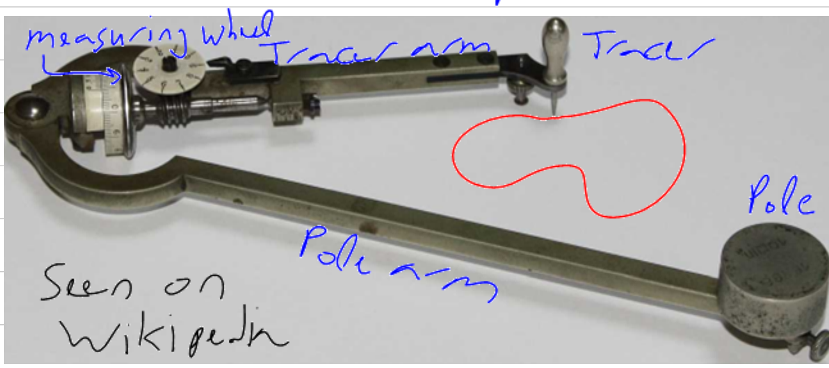
$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_0^1 \dot\gamma\cdot\begin{pmatrix}P\\Q\end{pmatrix} dt$$

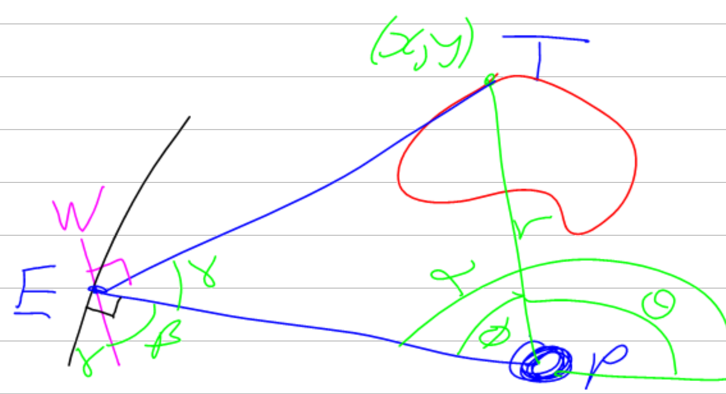
2nd interpretation $Q\to P$, $P\to -Q$

$$\int_D \operatorname{div}\begin{pmatrix}P\\Q\end{pmatrix} = \int_D\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) = \int_0^1 \begin{pmatrix}\dot\gamma_1\\ \dot\gamma_2\end{pmatrix}\begin{pmatrix}-Q\\P\end{pmatrix} dt = \int_0^1 \begin{pmatrix}P\\Q\end{pmatrix}\cdot\begin{pmatrix}\dot\gamma_2\\ -\dot\gamma_1\end{pmatrix} dt$$

$$= \int_0^1 \begin{pmatrix}P\\Q\end{pmatrix}\cdot\vec n\cdot dt$$

Example. The planimeter.





$$w = (d\alpha)\cdot\cos\gamma$$

$$= \cos(\pi - 2\phi)\, d(\phi+\theta) = -\cos 2\phi\, d(\phi+\theta)$$

$$dw = 2\sin 2\phi\, d\phi\wedge d\theta = \underbrace{2\cos\phi}_{r}\,\underbrace{2\sin\phi\, d\phi}_{-dr}\wedge d\theta$$

$$= -r\,dr\wedge d\theta = dx\wedge dy$$

MAT257 Analysis II on April 5, 2021: Volume forms, the 3D theorems.
Read Along: Spivak 126 to \infty.
HW19 is due tonight, HW20 (last!) will be online by midnight.
Old riddles (sol'n at end). Can you fold a rectangular piece of paper (perhaps many times) so that the result will have a longer perimeter than the original? If f:A->R^2 is a distance-non-increasing map from a rectangle to the plane, is it always the case that the length of the boundary of f(A) is less than the length of the boundary of A?

Blame COVID!

Next tasks: If $M^3 \subset \mathbb{R}^3$ and F is a vector field,

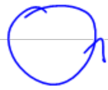
$$\int_M \operatorname{div} F \, dV = \int_{\partial M} F \cdot n \, dA$$

n: unit normal to ∂M

If $M^2 \subset \mathbb{R}^3$ is compact orient,

$$\int_M (\operatorname{curl} F) \cdot n \, dA = \int_{\partial M} F \cdot T \, ds$$

n: unit normal to M; T: unit tangent to ∂M

Integration on  ? on  ?

From Monkres' Analysis on Manifolds:

***Theorem 35.2.** *Let M be a compact oriented k-manifold in $\mathbf{R}^n$. Let ω be a k-form defined in an open set of $\mathbf{R}^n$ containing M. Suppose that $\alpha_i : A_i \to M_i$, for $i = 1, \ldots, N$, is a coordinate patch on M belonging to the orientation of M, such that A_i is open in $\mathbf{R}^k$ and M is the disjoint union of the open sets $M_1, \ldots, M_N$ of M and a set K of measure zero in M. Then*

$$\int_M \omega = \sum_{i=1}^N \left[\int_{A_i} \alpha_i^* \omega\right].$$

But first, what are dV, dA, ds?

M^k oriented. dV: That multiple of the orientation k-form for which $dV(u_1, \ldots, u_k) = 1$ if (u_i) make an o.n. basis of $T_x M$

E.g. $ds(T) = 1$. $dA_{(0,0,1)} = dx \wedge dy$

In general, if $M^2 \subset \mathbb{R}^3$ & n is the positive unit normal,

$$dA(u,v) = \begin{vmatrix} - u - \\ - v - \\ - n - \end{vmatrix} = (u \times v) \cdot n = \begin{matrix} n_1 \, dy \wedge dz \\ n_3 \, dx \wedge dy \end{matrix}$$

reminder: 1. Perp to u, v 2. Area 3. direction

Example: S^2

$$= \pm |u \times v| = \pm |u| \cdot |v| \sin\theta \stackrel{\pm}{=} \sqrt{|u|^2 |v|^2 - \langle u, v \rangle}$$

Formulas for arc length & area.

Recall in $\mathbb{R}^3$

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3 \\
\underset{\text{fnctns}}{\{f\}} \arrow[u] \arrow[r, "\text{grad}"] & \underset{\text{V.F.}}{\{F\}} \arrow[u, "W^1_F = \cdots"'] \arrow[r, "\text{curl}"] & \underset{\text{V.F}}{\{G\}} \arrow[u, "W^2_F"'] \arrow[r, "\text{div}"] & \underset{\text{fnctns.}}{\{g\}} \arrow[u, "W^3_g ="']
\end{tikzcd}$$

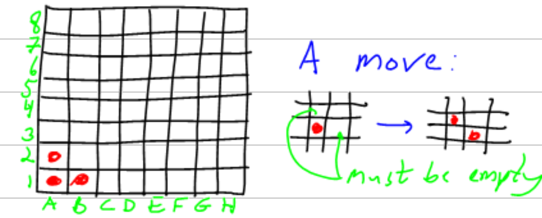
Claim 1. On $N^1 \subset \mathbb{R}^3$, $W^1_F = (T \cdot F)\,ds$

2. On $M^2 \subset \mathbb{R}^3$, $W^2_G = (G \cdot n)\,dA$

Pf 1 Enough to compute on T.

Pf 2 Enough to compute on u, v, an ON basis of TM.

MAT257 Analysis II on April 7, 2021: The 3D theorems.
Read Along: Spivak 126 to \infty.
Old riddle (sol'n at end). On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you clear the original 3 pawns?



Next tasks: If $M^3 \subset \mathbb{R}^3$ and F is a vector field, $\int_M \operatorname{div} F\, dV = \int_{\partial M} F \cdot n\, dA$ — unit normal to ∂M

If $M^2 \subset \mathbb{R}^3$ is compact orient, $\int_M (\operatorname{curl} F) \cdot n\, dA = \int_{\partial M} F \cdot T\, ds$ — unit normal to M; unit tangent to ∂M

But first, what are dV, dA, ds?

If $M^2 \subset \mathbb{R}^3$ & n is the positive unit normal,

$$dA(u,v) = \begin{vmatrix} - u - \\ - v - \\ - n - \end{vmatrix} = (u \times v) \cdot n = (n_1\, dy\, dz + n_2\, dz\, dx + n_3\, dx\, dy)(u,v)$$

$$= \pm |u \times v| = \pm |u| \cdot |v| \sin\theta \overset{\pm}{=} \sqrt{|u|^2 |v|^2 - \langle u,v\rangle}$$

Formulas for arc length & area

Recall in $\mathbb{R}^3$

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3 \\
\{f\} \arrow[u] \arrow[r, "\text{grad}"] & \{F\} \arrow[u, "W^1_F = \ldots"'] \arrow[r, "\text{curl}"] & \{G\} \arrow[u, "W^2_F"'] \arrow[r, "\text{div}"] & \{g\} \arrow[u, "W^3_g ="']
\end{tikzcd}$$

Fnctns — V.F. — V.F — Fnctns.

Claim 1 On $N^1 \subset \mathbb{R}^3$, $W^1_F = (T \cdot F)\, ds$

2 On $M^2 \subset \mathbb{R}^3$, $W^2_G = (G \cdot n)\, dA$

PF 1 Enough to compute on T.

PF 2 Enough to compute on u, v, a positive ON basis of TM.

$W^2_G(u,v) = G \cdot (u \times v) = G \cdot n\, |u \times v| = G \cdot n\, dA(u,v)$ $\square$

Thms & Geometric interpretations...

MAT257 Analysis II on April 9, 2021: The 3D theorems, towards the Maxwell equations.
Read Along: Spivak 126 to \infty and beyond.
Note that we have a class on Monday as usual! But no Petr tutorial.
Old riddle (sol'n at end). A unit cube in R^3, the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

In $\mathbb{R}^3$:

1. On $N^1 \subset \mathbb{R}^3$, $W^1_F = (T \cdot F)\,ds$
2. On $M^2 \subset \mathbb{R}^3$, $W^2_G = (G \cdot n)\,dA$

$$\int_M dw = \int_{\partial M} w$$

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] \arrow[d, shift right, "F"'] & \Omega^1 \arrow[r, "d"] \arrow[d, shift right, "F"'] & \Omega^2 \arrow[r, "d"] \arrow[d, shift right, "G"'] & \Omega^3 \arrow[d, shift right, "g"'] \\
\mathcal{F} \arrow[r, "\text{grad}"] \arrow[u, shift right, "F"'] & \mathcal{V} \arrow[r, "\text{curl}"] \arrow[u, shift right, "W^1_F"'] & \mathcal{V} \arrow[r, "\text{div}"] \arrow[u, shift right, "W^2_G"'] & \mathcal{F} \arrow[u, shift right, "W^3_g"']
\end{tikzcd}$$

Fnctns V.F. V.F Fnctns.

M^3, $w = W^2_G$ $dw = W^3_{\text{div}\,G} = (\text{div}\,G)\,dx\,dy\,dz$

$$\int_M (\text{div}\,G) = \int_{\partial M} W^2_G = \int_{\partial M} (G \cdot n)\,dA$$

Interpretation...

M^2 $w = W^1_F$ $dw = W^2_{\text{curl}\,F}$

$$\int_M dw = \int_{\partial M} w$$

$$\int_M (\text{curl}\,F) \cdot n\,dA = \int_{\partial M} (F \cdot T)\,ds$$

Interpretation...

Towards Maxwell

As at Maxwell-Slides@210409.pdf

The Monday April 12 class is at Maxwell-Slides@210412.pdf