MAT 257Y Analysis II
Dror's Open Private Notebook

August 31 Social Meeting Agenda

At https://gather.town/vUrHft2R9fBjSraE/mat257

We'll start at 9, and then at around 9:25 I will take the podium and make a welcome announcement. If you don't hear anything from me by 9:30, or if the whole thing is a failure from the start, check your mailbox. I will make another email announcement and redirect you to a zoom meeting room, where (only if necessary) we will reconvene at 9:35.

Why are we having a social meeting? I hope to have more!
What classes will look like.
What tutorials will look like.
The textbook.
Evaluation scheme.

August 31 TA Meeting Agenda

3 Tutorial TAs?
AI TA?
Crowdmark.
Office hours.
Emails.
Group photo.
Weekly meeting.

September 7 TA Meeting Agenda

Add TAs to Quercus
Filming: Start Sebastian, only for first class.
Integrity: Peter.
Tutorials: No tutorials this week!
  W5-6: Sebastian.
  R4-5: Shuyang.
  F12-1: Petr.
  Locations?
TA Office Hours: (only 2!) Starting week 2.
  Peter: M1-2.
  Petr can M9-3, W10-3.
  Sebastian: W10-11:30.
  Shuyang: Fri 1PM
  Locations?
  Decision: Peter M1 Petr F1.

Hour 1, Friday September 11

1617 H1: Brief intro, LinAlg review
H2: "About", more LinAlg.
H3-10: Topology of $\mathbb{R}^n$

On board:

MAT257Y w/ DROR BAR-NATAN
http://drorbn.net/20-257 $\longrightarrow$ About

Math today. Admin on web. Yet,

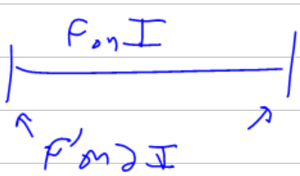
1. This year will be hard. We will all work more than in an ordinary year, but achieve less. Expect mishaps!
2. For student privacy reasons, you are NOT allowed to record classes, tutorial, and office hours. "Official" class recordings showing only me will be made available promptly [but the first may be a bit delayed]
3. Asking questions: By voice, or on chat (less reliable)
4. Onliners: If possible, video on, mic muted.

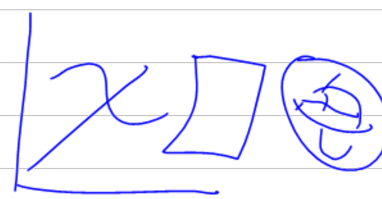
Also send this as an announcement, and add "Video volunteer needed!"

Math Intro: $\mathbb{R}^1 \nearrow \mathbb{R}^n$: LinAlg, cont., diff'ability, $\int$, ...

$$\int_a^b f'(t)\,dt = f(b) - f(a) \quad \nearrow \quad \int_C d\omega = \int_{\partial C} \omega$$

"Stokes' Thm"





Ambition: E&M.

<u>LinAlg review</u> <u>Def</u> For $x, y \in \mathbb{R}^n$,

$$\langle x, y\rangle = \sum x_i y_i \qquad |x|^2 = \langle x, x\rangle \qquad |x| = \sqrt{\sum x_i^2} \geq 0$$

<u>Thm</u> If $x, y \in \mathbb{R}^n$ & $a \in \mathbb{R}$

0. $\langle x, y\rangle$ is bilinear & symmetric; $|x|$ is semi-linear.
1. $|x| \geq 0$ & $|x| = 0$ iff $x = 0$.

"Cauchy-Schwarz"

2. $|\langle x,y\rangle| \leq |x|\cdot|y|$, equality iff x, y are lin-dep.

Pf $0 \leq \underbrace{\left| |y|^2 x - \langle x,y\rangle y \right|}_{\text{what's that ?}} = |y|^4 |x|^2 - 2|y|^2 \langle x,y\rangle^2$
$+ |y|^2 \langle x,y\rangle^2$

3. $|x+y| \leq |x| + |y|$ "Triangle ineq".

4. $\langle x,y\rangle = \dfrac{|x+y|^2 - |x-y|^2}{4}$ "Polarization".

Def $d(x,y) = |x-y|$

Thm Symmetric, Pos. def., Δ-ineq

Prep main projector!

On board: 1. Next class on Zoom!
2. Social today @ 4PM.
3. Office hours & tutorials starting!
4. Today: "About", more LinAlg.
5. Read Along: Spivak p 1–5.
6. Riddle Along: Can $\mathbb{R}^2$ be covered by a set of disjoint, non-degenerate circles? How about $\mathbb{R}^3$? $\mathbb{R}^4$?

Last class — Go over "About" here.

$$\langle x,y\rangle=\sum x_i y_i \qquad |x|^2=\langle x,x\rangle \qquad |x|=\sqrt{\sum x_i^2}\geq 0$$

Thm If $x,y\in\mathbb{R}^n$ & $a\in\mathbb{R}$
0. $\langle x,y\rangle$ is bilinear & symmetric; $|x|$ is semi-linear.
1. $|x|\geq 0$ & $|x|=0$ iff $x=0$.
2. $|\langle x,y\rangle|\leq|x|\cdot|y|$, "Cauchy–Schwarz" equality iff x,y are lin-dep.

review end

Pf $0\leq\Big|\underbrace{|y|^2x-\langle x,y\rangle y}_{\text{what's that?}}\Big|=|y|^4|x|^2-2|y|^2\langle x,y\rangle^2+|y|^2\langle x,y\rangle^2$

3. $|x+y|\leq|x|+|y|$ "Triangle Ineq"
4. $\langle x,y\rangle=\dfrac{|x+y|^2-|x-y|^2}{4}$ "Polarization"

done line

Def $d(x,y)=|x-y|$

Skipped

Thm Symmetric, Pos. def., Δ-ineq

Linear $\mathbb{R}^n\to\mathbb{R}^m$

$$\left\{\begin{matrix}T:\mathbb{R}^n\to\mathbb{R}^m \text{ s.t.}\\ T(ax+by)=aT(x)+bT(y)\end{matrix}\right\}\ \underset{L_A \leftarrow A}{\overset{T\to M_T}{\rightleftarrows}}\ M_{m\times n}(\mathbb{R})=\left\{\begin{pmatrix}a_{11}&\cdots&a_{1n}\\ \vdots&&\\ a_{m1}&\cdots&a_{mn}\end{pmatrix}\right\}$$

If A is a matrix, $L_A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\left(\quad\right)$

If T is a Lin Trans,

$$M_T = \left(Te_1 \,\middle|\, \cdots \,\middle|\, Te_n \right) \quad \text{where } e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th entry}$$

<u>Claim</u>: $M_{L_A} = A$ & $L_{M_T} = T$

2. If $T : \mathbb{R}^n \to \mathbb{R}^m$ & $S : \mathbb{R}^m \to \mathbb{R}^p$, namely $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^p$, then

$$M_{S \circ T} = M_S \cdot M_T$$

3. $M_{T+S} = M_T + M_S$, $M_{aT} = aM_T$

$L_{A+B} = L_A + L_B$, $L_{aA} = aL_A$

Week 2 Tutorials: 0 Introduce yourself

1 Why $\underbrace{\big|\, |x|^2 y - \langle x, y \rangle x \,\big|^2}$?

2 What is needed on $|\ \ |$ to make

$\langle x, y \rangle = \frac{1}{4}\left(|x+y|^2 - |x-y|^2\right)$ $\quad y - \frac{\langle x, y \rangle}{|x|^2} x$

an "inner product" ?

3 Whatever you want to add

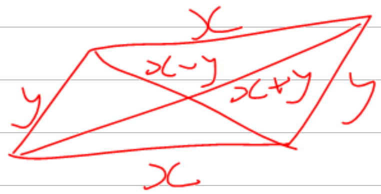
Q Piazza ?

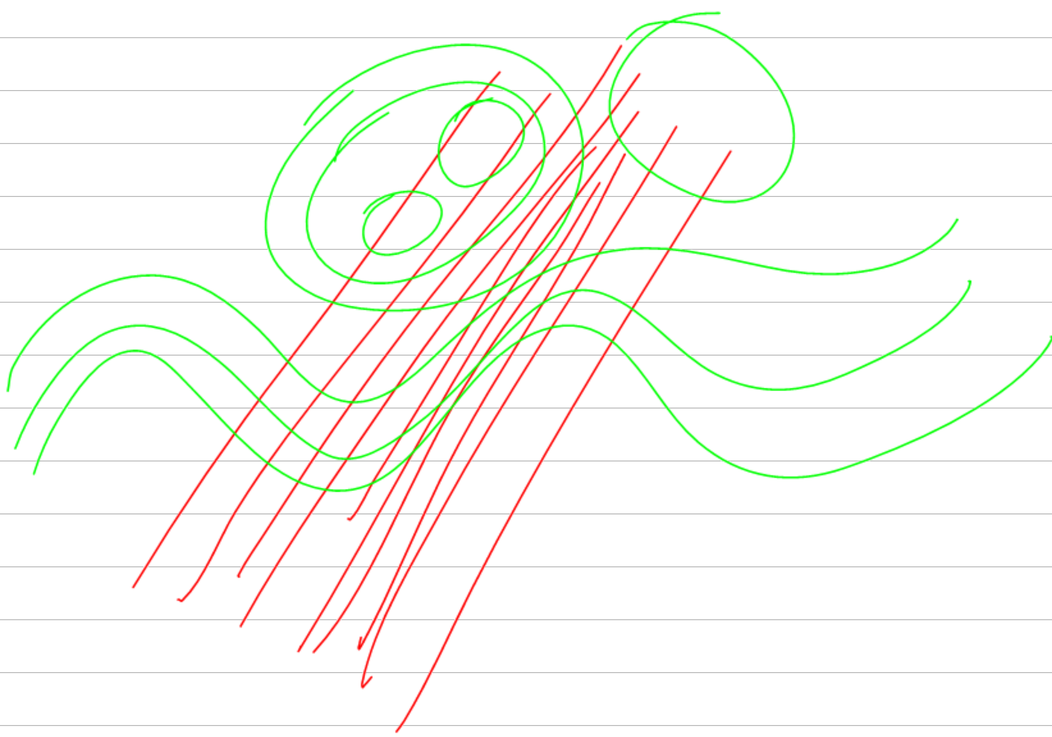
Announce the above? $\quad$ * examples of norms.

Duties $\quad |e_i|$

If $|\cdot|$ satisfies the axioms of a $|\cdot|$ then $\langle x,y\rangle = \frac{1}{4}\left(|x+y|^2 - |x-y|^2\right)$ then $\langle x,y\rangle$ satisfies the axioms of an inner product iff the parallelogram identity holds

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$$





Board: 1. Today: A bit more LinAlg, open & closed in $\mathbb{R}^n$.
2. Read Along: Spivak p1-10.
3. Riddle Along: A unit cube in $\mathbb{R}^3$, the area of its projection on any plane is equal to the length of its projection on the perpendicular line to that plane.
4. HW on web by midnight.

Linear $\mathbb{R}^n \to \mathbb{R}^m$

$$\left\{\begin{array}{l} T:\mathbb{R}^n \to \mathbb{R}^m \text{ s.t.} \\ T(ax+by)=aT(x)+bT(y) \end{array}\right\} \underset{L_A \leftarrow A}{\overset{T \to M_T}{\rightleftarrows}} M_{m\times n}(\mathbb{R})=\left\{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn}\end{pmatrix}\right\}$$

If A is a matrix, $L_A\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = A\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix} \text{---} \\ \text{---} \\ \text{---} \end{pmatrix}$

If T is a Lin Trans,

$$M_T = \left(Te_1 \middle| \cdots \middle| Te_n\right) \quad \text{where } e_i = \begin{pmatrix}0\\ \vdots \\ 1 \\ \vdots \\ 0\end{pmatrix} \leftarrow i\text{'th entry}$$

Claim! $M_{L_A}=A$ & $L_{M_T}=T$

2. If $T:\mathbb{R}^n \to \mathbb{R}^m$ & $S:\mathbb{R}^m \to \mathbb{R}^p$, namely $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^p$, then

$$M_{S\circ T} = M_S \cdot M_T$$

3. $M_{T+S}=M_T+M_S$, $M_{aT}=aM_T$

$L_{A+B}=L_A+L_B$, $L_{aA}=aL_A$

$X\times Y$, closed rectangles $R=\prod_{i=1}^{n}[a_i,b_i]$, open rectangles, open sets, closed sets, open rects are open, closed rects are closed, unions, intersections, $\emptyset$, $\mathbb{R}^n$.

Given $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, then exactly one of the following holds:

1. $\exists$ open rect. R s.t. $x \in R \subset A$. "x is in $\text{int}(A)$" (an open set!)
2. $\exists$ open rect R s.t. $x \in R \subset A^c$. "x is in $\text{ext}(A)$". (an open set!)
3. Every open rect containing x intersects both A & A^c "x is in $\text{Bd}(A)$". (A closed set!)

Open covers & compactness.

Board: 1. Today: open & closed in $\mathbb{R}^n$, compactness.
2. Read Along: Spivak p1-10.
3. Riddle Along: You owe me!
4. HW1 on web!

<u>Def</u> $A \subset \mathbb{R}^n$ "open" means $\forall x \in A$ $\exists$ open rect R s.t. $x \in R \subset A$.
$B \subset \mathbb{R}^n$ "closed" means B^c is open.

<u>Thm</u> 1. $\emptyset, \mathbb{R}^n$ are "clopen".
2. Any union of opens is open, any intersection of closed is closed.
3. A finite intersection of opens is open.
A finite union of closed is closed

Given $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, then exactly one of the following holds:

1. $\exists$ open rect. R s.t. $x \in R \subset A$. "x is in $\text{int}(A)$" (an open set!)
2. $\exists$ open rect R s.t. $x \in R \subset A^c$. "x is in $\text{ext}(A)$". (an open set!)
3. Every open rect containing x intersects both A & A^c "x is in $\text{Bd}(A)$". (A closed set!)

---

open covers & compactness.

<u>Heine-Borel</u> $[a,b]$ is compact.

<u>Pf</u> $G := \{g \in [a,b] : [a,g] \text{ has a finite cover}\}$

$\gamma = \sup(G)$ (makes sense!)

$\gamma \in G$; $\gamma = b$.

<u>Thm</u> If $A \subset \mathbb{R}^m$ is compact & $B \subset \mathbb{R}^n$ is compact, then $A \times B \subset \mathbb{R}^{m+n}$ is compact.

Cor. closed rect. are compact.

Thm A closed subset of a compact set is compact.

Cor $A \subset \mathbb{R}^n$ is compact iff it is closed & bndd.

possibly prove only $\Leftarrow$.

On board: 1. Today: compactness in $\mathbb{R}^n$.

2. Read Along: still Spivak p 1–10.

3. Riddle Along: Is there a compact uncountable subset of $\mathbb{R}$ that contains no rational numbers?

Def "$\mathcal{C}$ compact" means every open cover of $\mathcal{C}$ has a finite subcover.

Thm (Heine-Borel) $[a,b]$ is compact.

Pf Assume $\mathcal{U}$ is an open cover of $[a,b]$, set

$$G = \{g \in [a,b] : \text{some finite subset of } \mathcal{U} \text{ covers } [a,g]\}$$

$$\gamma = \sup(G)$$

Claim 1. $\gamma \in G$ 2. $\gamma = b$

Thm If $A \subset \mathbb{R}^m$ is compact & $B \subset \mathbb{R}^n$ is compact, then $A \times B \subset \mathbb{R}^{m+n}$ is compact.

Cor. closed rect. are compact.

Thm A closed subset of a compact set is compact.

Cor $A \subset \mathbb{R}^n$ is cmpct iff it is closed & bndd.

possibly prove only $\Leftarrow$.

On board: 1. Today: Continuity.
2. Read Along: Spivak 11–14
3. HW1 due midnight, HW2 is on web.
4. Today: Last day to painlessly add classes.
5. Riddle Along: Is there a continuous surjective $\phi:[0,1]\to[0,1]$, which is constant on disjoint intervals whose lengths sum to 1?

"open with rects is equiv to open w/ balls"

Not discussed: $f:\mathbb{R}^n\to\mathbb{R}^m$, $f:A\to B$, $f(C)$, $f^{-1}(D)$, graphs, compositions, component functions, coordinate projections

the better notion!

$\lim_{x\to a} f(x)$ [Spivak is annoying]

continuity at a; continuity.

<u>Thm 1</u>. $f:\mathbb{R}^n\to\mathbb{R}^m$ is cont. iff whenever $U\subset\mathbb{R}^m$ is open, $f^{-1}(U)\subset\mathbb{R}^n$ is open too.

done line

2. $f: A\subset\mathbb{R}^n\to\mathbb{R}^m$ is cont. iff whenever $U\subset\mathbb{R}^m$ is open, there is an open $V\subset\mathbb{R}^n$ s.t. $f^{-1}(U)=V\cap A$.

<u>Thm 2</u> If $f:A\to\mathbb{R}^m$ is cont. and A is compact, then $f(A)$ is compact.

On board: 1. Today: More Continuity.

2. Read Along: Spivak 11–14

3. Riddle along: $\exists$? cont. $f:\mathbb{R}\to\mathbb{R}$ s.t. $f\circ f=\cos$?

Thm1. $f:\mathbb{R}^n\to\mathbb{R}^m$ is cont. iff $V\subset\mathbb{R}^m$ open $\Rightarrow f^{-1}(V)$ is open.

2. $f:A\to\mathbb{R}^m$ is cont iff whenever $V\subset\mathbb{R}^m$ is open, there is $U\subset\mathbb{R}^n$ open s.t. $f^{-1}(V)=U\cap A$. "open in A".

Cor. The composition of cont. fnctns is cont.

Thm 2 If $f:A\to\mathbb{R}^m$ is cont. and A is cmpct, then $f(A)$ is cmpct.

Take $f:A\subset\mathbb{R}^n\to\mathbb{R}$ bndd.

$M(a,f,\delta)$ $M(a,f)$

$m(a,f,\delta)$ $m(a,f)$

$o(f,a)=M(f,a)-m(f,a)$

Skipped

Thm f is cont. at a iff $o(f,a)=0$.

Thm If A is closed and $\epsilon>0$, $\{a\in A: o(f,a)\geq\epsilon\}$ is closed.

Decide if o/o? Yes.

Today: The differential.

Read Along: Spivak 15–25 (warning: philosophical differences)

Riddle Along: 1. $\exists$? $f:\mathbb{R}\to\mathbb{R}$ cont. s.t. $f\circ f=\cos$?

2. Can you put uncountably many disjoint Y shapes in $\mathbb{R}^2$?

Def $f:\mathbb{R}^n\to\mathbb{R}^m$ is diff'ble at $a\in\mathbb{R}^n$ if there is a lin trans $L:\mathbb{R}^n\to\mathbb{R}^m$ s.t. $f(a+h)=f(a)+L\cdot h+e(h)$, where $\lim_{h\to 0}\frac{e(h)}{|h|}=0$.

Def $o(h)$, abuse of notation: $f(a+h)=f(a)+L\cdot h+o(h)$.

Comment: $o(h)$ is a vector space.

Thm If f is diff'ble at a, then L is unique.

Pf NTS that if $L\in o(h)$ then $L=0$ ...

Def $Df(a)$ E.g. $f:\mathbb{R}\to\mathbb{R}$, $Df(a)=(f'(a))$

Comments 1. Def makes sense for $f:A\subset\mathbb{R}^n\to\mathbb{R}^m$, A open.

2. Extend to non-open sets.

3. Diff'ability on A.

Thm The chain rule: If $f:\mathbb{R}^n\to\mathbb{R}^m$ is diff'ble at $a\in\mathbb{R}^n$, and $g:\mathbb{R}^m\to\mathbb{R}^p$ is diff'ble at $f(a)$, the $g\circ f$ is diff'ble at a and

$$D(g\circ f)(a)=(Dg)(f(a))\cdot(Df)(a)$$

This generalizes the 1D chain rule?

Pf $(g\circ f)(a+h)=g(f(a+h))=g(\underbrace{f(a)}_{a}+\underbrace{Df(a)\cdot h+e_1(h)}_{h})$

$$=g(f(a))+(Dg)(f(a))(Df(a)h+e_1(h))+e_2(Df(a)\cdot h+e_1(h))$$

---

done line

NTS: 1. $Dg(f(a))\cdot e_1(h)\in o(h)$ easy!

2. $e_2(Df(a)\cdot h+e_1(h))\in o(h)$.

Lemma If $e\in o(h)$ & $|\lambda(h)|\leq C\cdot|h|$ then $e\circ\lambda\in o(h)$

Pf $\frac{e(\lambda(h))}{|h|}=\begin{cases}\frac{e(\lambda(h))}{|\lambda(h)|}\frac{|\lambda(h)|}{|h|} & \text{if } |\lambda(h)|\neq 0\\ 0 & \text{otherwise.}\end{cases}$

Better: NTS $|e(\lambda(h))|<\epsilon|h|$ for $|h|<\delta$, some δ. write $|e(\lambda(h))|<\frac{\epsilon}{C}|\lambda(h)|<\epsilon|h|$, Provided $h<\min(\frac{\delta_1}{C},\delta_2)$ where $|\lambda(h)|\leq C|h|$ on $B_{\delta_2}(0)$, & $|e(y)|<\frac{\epsilon}{C}|y|$ on $B_{\delta_1}(0)$.

Also NTS $(e\circ\lambda)(0)=0$

Continue w/ Spivak's from 2-3.

Team meeting Sep 28

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ f is diff-able at a

$$f(a+h) = f(a) + Df(a)\cdot h + \cancel{e(h)}$$

$o(h)$

$$o(h) = \left\{ e : \frac{e(h)}{|h|} \xrightarrow[h\to 0]{} 0 \right\}$$

a v.s.

$$(g\circ f)(a+h) = g(\overbrace{f(a) + Df(a)\cdot h + e_1(h)})$$

$$= \underline{\underline{g(f(a))}} + \underline{Dg(f(a))}\cdot(\underline{Df(a)}\cdot h + e_1(h)) + e_2(Df(a)\cdot h + e_1(h))$$

$=$

3.2174512529525675232 5

↑ ↑ ↑ ↑ ↑

Today: The chain rule & more on Df.

Read Along: Spivak 15–25.

Riddle Along: $(x^x)' = ?$

Silly A: Use $(x^n)' = nx^{n-1}$ w/ $n=x$, get $x x^{x-1} = x^x$

Silly B: Use $(a^x)' = a^x \log a$ w/ $a=x$, get $x^x \log x$

Silly A + Silly B $= x^x(1+\log x)$ = correct! Why?

HW3 on web, HW2 due by midnight.

Reminder: f diff'able at $a \iff f(a+h) = f(a) + Df(a)\,h + o(h) = f(a) + f'(a)h + o(h)$

Thm The chain rule: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff'able at $a \in \mathbb{R}^n$, and $g: \mathbb{R}^m \to \mathbb{R}^p$ is diff'able at $f(a)$, the $g \circ f$ is diff'able at a and

$$D(g\circ f)(a) = (Dg)(f(a)) \cdot (Df)(a)$$

This generalizes the 1D chain rule!

PF $(g\circ f)(a+h) = g(f(a+h)) = g(\underbrace{f(a)}_{a} + \underbrace{Df(a)\cdot h + e_1(h)}_{h})$

$= g(f(a)) + (Dg)(f(a))(Df(a)h + e_1(h)) + e_2(Df(a)\cdot h + e_1(h))$

start line

NTS: 1. $Dg(f(a))\cdot e_1(h) \in o(h)$ (easy!)

2. $e_2(Df(a)\cdot h + e_1(h)) \in o(h)$

Lemma If $e \in o(h)$ & $|\lambda(h)| \leq C\cdot|h|$ then $e\circ\lambda \in o(h)$

PF $\dfrac{e(\lambda(h))}{|h|} = \begin{cases} \dfrac{e(\lambda(h))}{|\lambda(h)|}\dfrac{|\lambda(h)|}{|h|} & \text{if } |\lambda(h)| \neq 0 \\ 0 & \text{otherwise} \end{cases}$

Continue w/ Spivak's from 2-3.

done line

Also stated $D+ = +$, $(+)' = (1,1)$

2-3 Theorem

(1) *If $f: \mathbf{R}^n \to \mathbf{R}^m$ is a constant function (that is, if for some $y \in \mathbf{R}^m$ we have $f(x) = y$ for all $x \in \mathbf{R}^n$), then*

$$Df(a) = 0.$$

(2) *If $f: \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then*

$$Df(a) = f.$$

(3) *If $f: \mathbf{R}^n \to \mathbf{R}^m$, then f is differentiable at $a \in \mathbf{R}^n$ if and only if each f^i is, and*

$$Df(a) = (Df^1(a), \ldots, Df^m(a)).$$

Thus $f'(a)$ is the $m \times n$ matrix whose ith row is $(f^i)'(a)$.

(4) *If $s: \mathbf{R}^2 \to \mathbf{R}$ is defined by $s(x,y) = x + y$, then*

$$Ds(a,b) = s.$$

(5) *If $p: \mathbf{R}^2 \to \mathbf{R}$ is defined by $p(x,y) = x \cdot y$, then*

$$Dp(a,b)(x,y) = bx + ay.$$

Thus $p'(a,b) = (b,a)$.

Hour 10, Friday October 2: More differentials!
Read Along: Spivak 15-25.
Riddle Along: Your turn!

$$f(a+h)=f(a)+Df(a)h+o(h)=f(a)+f'(a)h+o(h)$$

Facts on Df/f':

1. f const $\Rightarrow f'=0$
2. f a lin-trans $\Rightarrow Df(a)=f$
3. If $s:\mathbb{R}^2\to\mathbb{R}$ is $s(a,b)=a+b$, then $Ds(a,b)(x,y)=x+y$

$$D+=+ \qquad (+)'=(1\ 1)$$

3' $d(a,b)=a-b$.

4 $f=\begin{pmatrix}f_1\\ \vdots\\ f_m\end{pmatrix} \Rightarrow Df(a)(h)=\begin{pmatrix}Df_1(a)h\\ \vdots\\ Df_m(a)h\end{pmatrix} \qquad f'(a)=\begin{pmatrix}\text{—}\ f_1'(a)\ \text{—}\\ \\ \text{—}\ f_m'(a)\ \text{—}\end{pmatrix}$

5. If $p(a,b)=ab$ $\quad Dp(a,b)(x,y)=bx+ay$

$$p'(a,b)=(b\ a)$$

6. If $q(a,b)=\frac{a}{b}$ $\quad q'=\left(\frac{1}{b},-\frac{a}{b^2}\right)$ Ex. Give a direct proof!

$\begin{pmatrix}a\\b\end{pmatrix}\mapsto\begin{pmatrix}\log a\\ \log b\end{pmatrix}$ $\qquad \frac{a+x}{b+y}-\frac{a}{b}=\frac{\cancel{ba}+xb-\cancel{ba}-ay}{b(b+y)}=\frac{xb-ay}{b(b+y)}$

$e^{\log a-\log b}$ $\qquad (e^{\log a-\log b})(1\ -1)\begin{pmatrix}1/a & \\ & 1/b\end{pmatrix}=\frac{a}{b}\left(\frac{1}{a}\ -\frac{1}{b}\right)$

Example $(f/g)'=\frac{gf'-fg'}{g^2}$ $\quad f,g:\mathbb{R}\to\mathbb{R}$.

D_i, min/max, $D_{i,j}$, $D_{ij}=D_{ji}$

Hour 11, Monday October 5: Differentials and Partial Derivatives.
Read Along: Spivak 25-34.
Riddle Along: Can you fit 31 2x1 domino pieces on an 8x8 chessboard with two diagonally opposite squares removed?
Can you fit 21 3x1 tromino pieces on an 8x8 chessboard with one square removed (anywhere)?

computing the differential of $\frac{x}{y} = e^{\log x - \log y}$.

$$\mathbb{R}^2_{x,y} \xrightarrow[D:\ (1/x \quad 1/y)\ !]{f_1 : (x,y) \mapsto (\log x, \log y)} \mathbb{R}^2_{\xi,\eta} \xrightarrow[D:\ (1 \quad -1)]{f_2 : (\xi,\eta) \mapsto (\xi - \eta)} \mathbb{R}_\gamma \xrightarrow[D:\ e^\gamma]{f_3 : \gamma \mapsto e^\gamma} \mathbb{R}$$

$$x, y \mapsto x/y$$

$$D:\ e^{\log x - \log y} \cdot (1 \quad -1) \cdot \begin{pmatrix} 1/x & 1/y \end{pmatrix} = \left(y \quad -x/y^2\right)$$

Example: compute $(f/g)'$

Partial Derivatives: D_i, min, max.

Thm If $f : \mathbb{R}^n \to \mathbb{R}$ is diff-able at a, then all it's partial derivatives exist at a and

$$Df(a) = (D_1 f(a), \ldots, D_n f)(a)$$

Thm If $D_i f$ exist and are cont. near a, then f is diff-able at a.

$$\frac{x}{y} = e^{\log x - \log y}$$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \log x \\ \log y \end{pmatrix}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \longmapsto \xi - \eta$$

$$\gamma \longmapsto e^{\gamma}$$

$$\begin{pmatrix} 1/x & 0 \\ 0 & 1/y \end{pmatrix}$$

$$\overbrace{\underbrace{Df(a)}}^{e}(h)$$

Hour 12, Wednesday October 7: Differentials and Partial Derivatives.
HW4 on web, HW3 due at midnight!
Read Along: Spivak 25-40.
Riddle Along: Can you fit 125 4x2x1 boxes in one 10x10x10 cube?

Thm If $f:\mathbb{R}^n \to \mathbb{R}$ is diffable at a, then all it's partial derivatives exist at a and

$$Df(a) = (D_1 f(a) \ldots\ldots D_n f)(a)$$

board line.

main Thm For $f:\mathbb{R}^n \to \mathbb{R}$,

if $D_i f$ exist and are cont. near a,

then f is diffable at a & $f'(a) = (D_1 f(a) \ldots D_n f(a))$
$= \left(\frac{\partial f}{\partial x_1}(a) \ldots \frac{\partial f}{\partial x_n}(a)\right)$

Cor For $f:\mathbb{R}^n \to \mathbb{R}^m$,

if $D_i f_j$ exist and are cont. near a,

then f is diffable at a & $f'(a) = \begin{pmatrix} D_1 f_1(a) & \cdots & D_n f_1(a) \\ \vdots & & \\ D_n f_1(a) & \cdots & D_n f_m(a) \end{pmatrix}$

"The Jacobian matrix of f at a"

done line

Aside/Lemma Let $R \subset \mathbb{R}^n$ be a rectangle, and $f: R \to \mathbb{R}^m$ be ~~cont.~~ diffable. Suppose $|D_i f_j(x)| \leq M$ in $\mathrm{int}(R)$. Then $\forall x, y \in R$,

$$|f(x) - f(y)| \leq \cancel{\text{const}}\ n \cdot M \cdot M \cdot |x - y|$$

Q. Can you deduce "main thm" from "Aside/Lemma" w/o going through telescopic summation/MVT once again?

Higher partials, partials commute, C^∞

Hour 13, Friday October 9: A bit more on differentials and partials, the IFT.
Read Along: Spivak 25-40.
Riddle Along: Can you fit 4 axb rectangles in one (a+b)^2 square? Can you fit 27 axbxc boxes in one (a+b+c)^3 cube? Why do I care?

Thm $f:\mathbb{R}^n\to\mathbb{R}^m$, partials exist & cont. near $a \Rightarrow$ f diffable @ a & $f'(a)=(D_j f_i(a))_{ij}$

Proof Use axis crawl / telescopic summation & the MVT.

board line

Aside/Lemma Let $R\subset\mathbb{R}^n$ be a rectangle, and $f:R\to\mathbb{R}^m$ be ~~cont.~~ diffable. Suppose $|D_i f_j(x)|\leq M$ in int(R). Then $\forall x,y\in R$,

$$|f(x)-f(y)|\leq \cancel{\text{Const}}\ n\cdot M\cdot M\cdot|x-y|$$

Q. Can you deduce "main thm" from "Aside/Lemma" w/o going through telescopic summation/MVT once again?

Higher partials, partials commute, C^∞.

Thm (IFT) $f:\mathbb{R}^n\to\mathbb{R}^n$ cont. diffable in an open set containing $a\in\mathbb{R}^n$, $f'(a)$ invertible $\Rightarrow \exists$ open $V\ni a$, open $W\ni b=f(a)$ s.t. $f|_V:V\to W$ is invertible with $f^{-1}=(f|_V)^{-1}$ is cont., diffable, and with

$$* \quad (f^{-1})'(y)=\left[f'(f^{-1}(y))\right]^{-1}$$

Well known as hard... My goal: Convince you that it isn't!

1. Dispatch $*$.
2. WLOG, $f'(a)=I$ [also wlog $a=b=0$, but we don't care)
3. Strategy:

MAT257 Analysis II on October 14, 2020: The Inverse Function Theorem.
Read Along: Spivak 34-40.
Riddle Along: Can you fit a box of dimensions a'xb'xc' inside a box of dimensions axbxc, if a'+b'+c'>a+b+c?
HW5 is on web, HW4 is due by 11:59PM.

Thm (IFT) $f:\mathbb{R}^n \to \mathbb{R}^n$ cont. diffable in an open set containing $a \in \mathbb{R}^n$, $f'(a)$ invertible $\Rightarrow \exists$ open $V \ni a$, open $W \ni b = f(a)$ s.t. $f|_V : V \to W$ is invertible with $f^{-1} := (f|_V)^{-1}$ is cont., diffable, and with

$$* \quad (f^{-1})'(y) = \left[f'(f^{-1}(y))\right]^{-1}$$

board line

Well known as hard... My goal: Convince you that it isn't!

1. Dispatch $*$.
2. WLOG, $f'(a) = I$ [also wlog $a=b=0$, but we don't care)
3. Strategy: a. $|(x_1 - x_2) - (f(x_1) - f(x_2))| < (\text{tiny}) \cdot |x-y|$ "All-Scale Fidelity"
   b. Pictorial pf of $\exists f^{-1}$
4. $U = B_r(a)$ s.t. $|D_i f_j| < \frac{1}{257 n^2}$ on U; $V = B_{1/3 r}(a)$ $W = f(V)$
   So f is 257-ASF.
5. f^{-1} exists.
6. f^{-1} is cont. $[|\alpha - \beta| \leq \epsilon |\alpha| \Rightarrow |\alpha - \beta| \leq \epsilon(|\beta| + |\alpha - \beta|)$
   $\Rightarrow |\alpha - \beta| \leq \frac{\epsilon}{1-\epsilon} |\beta|$
7. f^{-1} is diffable at a.

MAT257 Analysis II on October 16, 2020: The Inverse Function Theorem.
Read Along: Spivak 34-40.
Riddle Along: Your turn!

Thm (The Inverse Function Theorem, IFT)

$F:\mathbb{R}^n\to\mathbb{R}^n$ cont. diff'ble in an open set A containing $a\in\mathbb{R}^n$, $F'(a)$ invertible $\Rightarrow \exists$ open $V\ni a$, open $W\ni b=F(a)$ s.t. $F|_V: V\to W$ is invertible with $F^{-1}:=(F|_V)^{-1}$ is cont., diff'ble, and with

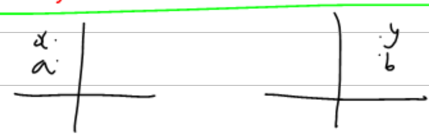
$$\cancel{* \quad (F^{-1})'(y) = \left[F'(F^{-1}(y))\right]^{-1}}$$ done!

WLOG, $F'(a)=I$. Given that, "All Scale Fidelity";

$$\left|(F(x_1)-F(x_2))-(x_1-x_2)\right| < \frac{1}{257}|x_1-x_2|$$

where $x_1,x_2\in B_r(a)$

Idea of the proof



1. Let $W=B_{r/2}(b)$. $\forall y\in W \quad \exists x\in B_r(a)$ s.t. $F(x)=y$.

$x_1 = a+(y-b)$

$x_2 = x_1+(y-F(x_1))$

$x_3 = x_2+(y-F(x_2))$

$|x_n-x_{n-1}| = |(x_{n-1}-x_{n-2})-(F(x_{n-1})-F(x_{n-2}))|$

$\leq \frac{1}{257}|x_{n-1}-x_{n-2}|$

So $x_n\in B_r(a)$, (x_n) is Cauchy, $\lim x_n = x$ exists, & $F(x)=y$.

Now let $V=F^{-1}(W)$; $F|_V: V\to W$ is onto & 1-1 !

2. F^{-1} is cont. Indeed,

$$\Big|\underbrace{(x_1-x_2)}_{\alpha}-\underbrace{(F(x_1)-F(x_2))}_{\beta}\Big| \leq \frac{1}{257}\underbrace{|x_1-x_2|}_{\alpha}$$ done line

$$|\alpha-\beta| \leq \frac{1}{257}|\alpha| \leq \frac{1}{257}\left(|\beta|+|\alpha-\beta|\right)$$

So $\left|F^{-1}(y_1)-F^{-1}(y_2)-(y_1-y_2)\right| \leq \frac{1}{256}|y_1-y_2|$ So $|\alpha-\beta|\leq\frac{1}{256}|\beta|$

So $\left| |f^{-1}(y_1) - f^{-1}(y_2)| - |y_1 - y_2| \right| \leqslant \frac{1}{256} |y_1 - y_2|$

So $|f^{-1}(y_1) - f^{-1}(y_2)| \leqslant \frac{257}{256} |y_1 - y_2|$

So f^{-1} is cont.

3 f^{-1} is diff'able at b:

$$f^{-1}(\underbrace{b+h}_{y_2}) = f^{-1}(\underbrace{b}_{y_1}) + I h + e(h)$$

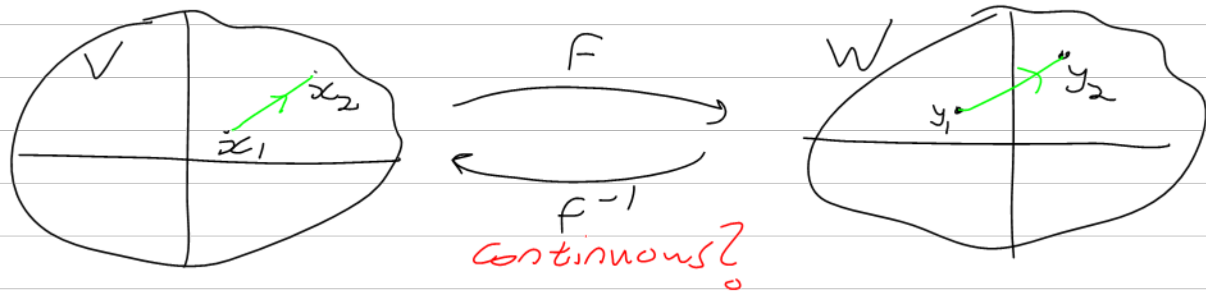
$$e(h) = f^{-1}(y_2) - f^{-1}(y_1) - (y_2 - y_1)$$

So $|e(h)| \leqslant \frac{1}{256} |h|$.

So f^{-1} is diff'able at b, hence everywhere.

$\square$

MAT257 Analysis II on October 19, 2020: The Inverse / Implicit Function Theorems.
Read Along: Spivak 34-45.
Riddle Along: On any pair of potatos, can you draw a pair of 3D congruent curves?



ASF:

$$\left|\underbrace{(x_1-x_2)}_{\alpha}-\underbrace{(y_1-y_2)}_{\beta}\right| \leq \frac{1}{257}\underbrace{|x_1-x_2|}_{\alpha}$$ Trouble!

board line

$$|\alpha-\beta| \leq \frac{1}{257}|\alpha| = \frac{1}{257}|\beta+\alpha-\beta| \leq \frac{1}{257}|\beta| + \frac{1}{257}|\alpha-\beta|$$

$$\Rightarrow |\alpha-\beta| \leq \frac{1}{256}|\beta| \Rightarrow |(x_1-x_2)-(y_1-y_2)| \leq \frac{1}{256}|y_1-y_2|$$

$$\Rightarrow |x_1-x_2| \leq \frac{257}{256}|y_1-y_2| \Rightarrow \text{cont.!}$$

F^{-1} is diffable at b:

$$F^{-1}(\underbrace{b+h}_{y_2}) = F^{-1}(\underbrace{b}_{y_1}) + I\,h + e(h)$$

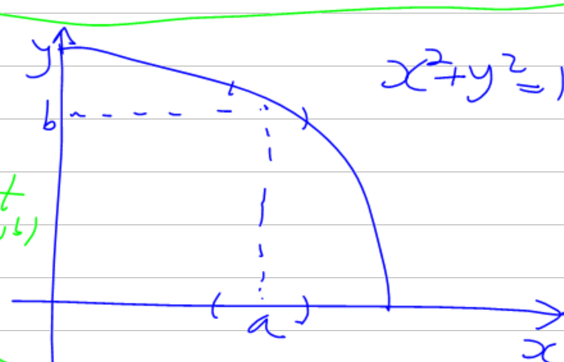
$$e(h) = F^{-1}(y_2) - F^{-1}(y_1) - (y_2-y_1)$$

So $|e(h)| \leq \frac{1}{256}|h|$.

But why is F^{-1} cont. diffable?

So F^{-1} is diffable at b, hence everywhere. $\square$

The Implicit Function Thm



Thm Given a cont. diffable (at least near (a,b)) $f: \mathbb{R}^n_{x_1\dots x_n} \times \mathbb{R}^k_{y_1\dots y_k} \longrightarrow \mathbb{R}^k$ and $(a,b) \in \mathbb{R}^n \times \mathbb{R}^k$ s.t. $f(a,b)=0$, such that

$\underbrace{\qquad\qquad\qquad\qquad}$, $\exists$ nbd A of a, nbd B of b, & $\exists!$ $g: A \to B$ s.t. $g(a)=b$ & $\forall z \in A$ $f(z,g(z))=0$. Furthermore,

g is cont. diff'able & $g' = \underbrace{\qquad\qquad}$.

PF $F(z, y) = 0 \iff \begin{cases} x = z \\ F(x,y) = 0 \end{cases}$ so w/ $H(x,y) = \begin{pmatrix} x \\ F(x,y) \end{pmatrix}$

this is $H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$ where $H\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. If $H'\begin{pmatrix} a \\ b \end{pmatrix}$ is non-singular, H^{-1} exists near $\begin{pmatrix} a \\ b \end{pmatrix}$, so for z near a $\exists!$ (x, y) s.t. $H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$; so set

$$g(z) = \pi_2 \circ H^{-1}\begin{pmatrix} z \\ 0 \end{pmatrix}.$$

* When is H' invertible?
* what is g'?

JA meeting:

IFT: 1. $(F^{-1})' = (F')^{-1}$

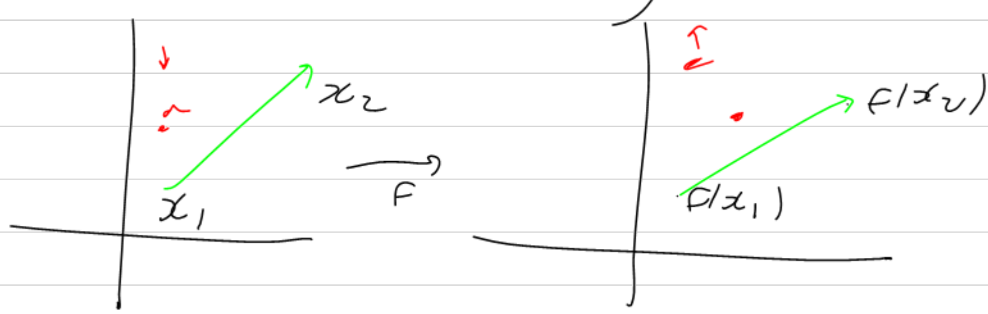
2. $F'(a) = I$ WLOG.

3. use lemma w/ $g = F - I$, get.

Lemma: if (g') is small. then $|g(x_1) - g(x_2)| \leq \epsilon |x_1 - x_2|$

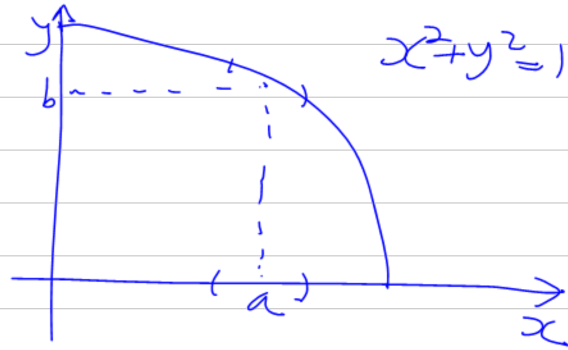
$$\|(F(x_1) - F(x_2)) - (x_1 - x_2)\| < \frac{1}{257} |x_1 - x_2|$$

"All scale Fidelity" ASF



MAT257 Analysis II on October 21, 2020: The Implicit Function Theorem.
Read Along: Spivak 40-45. Read Ahead: Spivak 46-52, "Integration".
Riddle Along: Can you find uncountably many sets of natural numbers, such that the intersection of any two of them is finite?

The Implicit Function Thm



Thm Given $f: \mathbb{R}^n_{x_1\dots x_n} \times \mathbb{R}^k_{y_1\dots y_k} \longrightarrow \mathbb{R}^k$ cont. diffable near $(a,b) \in \mathbb{R}^n \times \mathbb{R}^k$ and s.t. $f(a,b)=0$ and $\underbrace{\qquad\qquad\qquad}$,
$\exists$ nbd A of a, nbd B of b, & $\exists !\, g: A \to B$ s.t. $g(a)=b$ & $\forall z \in A$ $f(z, g(z)) = 0$. Furthermore, g is cont. diffable & $g' = \underbrace{\qquad\qquad}$.

PF $f(z,y)=0 \iff \begin{cases} x = z \\ f(x,y)=0 \end{cases}$ so w/ $H(x,y) = \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$

this is $H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$ where $H\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$. If $H'\begin{pmatrix} a \\ b \end{pmatrix}$ is non-singular, H^{-1} exists near $\begin{pmatrix} a \\ b \end{pmatrix}$, so for z near a $\exists !\, (x,y)$ s.t. $H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$; so set

$$g(z) = \pi_2 \circ H^{-1}\begin{pmatrix} z \\ 0 \end{pmatrix}.$$

* When is H' invertible?
* what is g'?

To do: * complete

* uniqueness.

MAT257 Analysis II on October 23, 2020: The Implicit Function Theorem (a bit), and integration (boot up).
Read Along: Spivak 40-52.
TT1 discussion on Wednesday October 29. Until then, just keep on top of things!
Riddle Along: Can you find uncountably many sets of natural numbers, such that for any two of them, one contains the other?

Thm Given $F:\mathbb{R}^n_{x_1\dots x_n}\times\mathbb{R}^k_{y_1\dots y_k}\to\mathbb{R}^k$ cont. diffable near $(a,b)\in\mathbb{R}^n\times\mathbb{R}^k$ and s.t. $F(a,b)=0$ and $\frac{\partial F}{\partial y}$ is invertible, $\exists$ nbd A of a, nbd B of b, & $\exists!\, g:A\to B$ s.t. $g(a)=b$ & $\forall z\in A$ $F(z,g(z))=0$. Furthermore, g is cont. diffable & $g' =$ ______.

Pf For $z\in A, y\in B$

$$F(z,y)=0 \iff \begin{matrix} x=z \\ F(x,y)=0\end{matrix} \iff \begin{matrix}\text{w/ } H(x,y)=\begin{pmatrix}x\\F(x,y)\end{pmatrix} \\ H(x,y)=\begin{pmatrix}z\\0\end{pmatrix}\end{matrix} \iff \begin{pmatrix}x\\y\end{pmatrix}=H^{-1}\begin{pmatrix}z\\0\end{pmatrix} \iff \begin{matrix}x=z\\ y=\pi_2 H^{-1}\begin{pmatrix}z\\0\end{pmatrix}\end{matrix}$$

So y is unique! set $g(z)=\pi_2 H^{-1}\begin{pmatrix}z\\0\end{pmatrix}$

$0=F(x,g(x))$ so

$$0=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial g}{\partial z} \quad \text{so} \quad g' = -\frac{\partial F}{\partial y}\cdot\frac{\partial F}{\partial x}$$

Aside. If $T:\mathbb{R}^n\to\mathbb{R}^k$, $k<n$, $\operatorname{rank} T=k$, $A=M_T=\underbrace{\begin{pmatrix} & \end{pmatrix}}_{n}{}_k$

then $\exists$ invertible $P\in M_{n\times n}$ s.t. $AP=(I_{k\times k}|0)$.

True for functions! If $F:\mathbb{R}^n\to\mathbb{R}^k$ cont diffable at 0, $F(0)=0$ & $\operatorname{rank} F'(0)=k$, then $\exists$ invertible, near 0 $\phi:\mathbb{R}^n\to\mathbb{R}^n$ s.t. $F(\phi(x_1\dots x_n))=(x_1\dots x_k)$

Pf WLOG, $\frac{\partial F}{\partial(x_1\dots x_k)}$ is invertible. Let $H:\mathbb{R}^n\to\mathbb{R}^n$ be $H\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}F(x_1\dots x_n)\\ x_{k+1}\\ \vdots \\ x_n\end{pmatrix}$. Now $\phi=H^{-1}$.

$$\begin{matrix} & (x_1\dots x_n) & \\ \swarrow^{H} & & \searrow^{F} \\ \begin{pmatrix}F(x_1\dots x_n)\\ x_{k+1}\\ \vdots\end{pmatrix} & & F(x_1\dots x_n)\end{matrix}$$

Postpone

Post Mortem added March 8, 2021:
Perhaps the chapter on manifolds, Spivak pp 109–115, should be done here following the notes of March 2021, while the inverse/implicit function thms are still fresh.

Recall $\int_M dw = \int_{\partial M} w$

infrastructure placed

Now this.

Partition P of $[a,b]$: $a=t_0\le t_1 \le \dots \le t_N = b$

$R = \prod [a_i, b_i]$ $\quad P: (P_i)$ where P_i partitions $[a_i, b_i]$

$P_i = (a_i = t_{i0} \le \dots \le t_{iN_i} = b_i)$

R is divided into a union of nearly-disjoint subrectangles $\prod [t_{i,j_i-1}, t_{i j_i}]$ $\quad \prod N_i$ of them.

$V(R) := \prod (b_i - a_i)$

Claim (Given R, P) $\quad V(R) = \sum_{\substack{\text{subrectangles} \\ S\in P}} V(S)$ $\qquad$ PF in 1D, in 2D ...

$f: R \to \mathbb{R}$ bdd.

$m_S(f)$ $\quad M_S(f)$ $\quad L(f,P)$, $U(f,P)$ $\quad L(f,P)\le U(f,P)$

defn!

Lemma If P' refines P, $L(f,P) \le L(f,P')$

$U(f,P) \ge U(f,P')$

Corollary If P & P' are any two partitions.

$L(f,P) \le U(f,P')$

def $\quad U(f) := \inf_P U(f,P) \qquad L(f) = \sup_P L(f,P)$

def integrable

MAT257 Analysis II on October 26, 2020: The definition of the integral.
Read Along: Spivak 46-56.
TT1 discussion on Wednesday. Until then, just keep on top of things!
Riddle Along: A spherical loaf of bread is put into a bread-cutting machine. Which slice gets the most crust?

$$\int_M dw = \int_{\partial M} w$$

$R = \prod [a_i, b_i]$ $\quad P: (P_i)$ where P_i partitions $[a_i, b_i]$

$$P_i = (a_i = t_{i0} < \dots < t_{iN_i} = b_i)$$

R is divided into a union of nearly-disjoint subrectangles $\prod [t_{i,j_i-1}, t_{ij_i}]$ $\quad \prod N_i$ of them.

$v(R) := \prod (b_i - a_i)$ $\quad$ Claim $\quad v(R) = \sum_{S \in P} v(S)$

$f: R \to \mathbb{R}$ bdd. $\quad$ E.g. $f_1 \equiv c$, $\quad f_2(x) = \begin{cases} 1 & \forall i\ x_i \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

$m_S(f) \quad M_S(f) \quad L(f,P),\ U(f,P) \quad L(f,P) \le U(f,P)$

def!

Lemma If P' refines P, $L(f,P) \le L(f,P')$
$U(f,P) \ge U(f,P')$

Corollary If P & P' are any two partitions.

$$L(f,P) \le U(f,P')$$

def

$$U(f) := \inf_P U(f,P) \qquad L(f) = \sup_P L(f,P)$$

def integrable

---

done live.

Thm f is integrable $\iff \forall \epsilon > 0\ \exists P$ s.t.

$$U(f,P) - L(f,P) < \epsilon.$$

Goal: f is integrable iff f is cont. except on a tiny set.

Def Measure 0 [using either open or closed rectangles]

Finite sets, countable sets (define, e.g. $\mathbb{Q}$).

A countable union of meas-0.

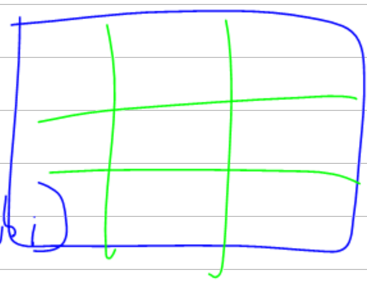
The Cantor set.

TA meeting
oct 26

$$x = z$$

$$F(z,y) = 0 \iff \underbrace{F(x,y)}_{} = 0$$

$$H(x,y) = \begin{pmatrix} x \\ F(x,y) \end{pmatrix}$$



$$R = \prod [a_i, b_i]$$

$$P = (P_1 \ldots P_n)$$

$$P_i = (a_i = t_{i0} < \ldots < t_{iN_i} = b_i)$$

$$U(F,P) \quad L(F,P)$$

$$L(F,P) \leq U(F,P')$$

$$U(F) = \inf U(F,P)$$

$$\vee\!/$$

$$L(F) = \sup L(F,P)$$

MAT257 Analysis II on October 28, 2020: Integrability and continuity.
Read Along: Spivak 46-56.
Riddle Along: Your turn!

On TT1:
* Tuesday November 3, 5-7PM (Toronto time), on Crowdmark. Other than documented accessibility matters, no exceptions!
* I will be available to answer questions throught the exam, at my usual office (http://drorbn.net/vchat, but I'll add a waiting room).
* There will be mishaps! I just hope that not too many. If you encounter one, document everything with specific details, times, and screen shots, and send me a message by Wednesday November 4 at 7PM. I will deal with these situations on a case by case basis.
* Material: Everything up to but not including integration.
* Open book(s), open notes, but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor/TAs to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.
* You will be required to copy in your handwriting and sign an academic integrity statement and submit it to Crowdmark along with the rest of your exam. You will be given an extra 15 minutes for this purpose.
* The vast majority of students will do honest work, and I appreciate that. Out of respect for the honest students I will do my best to pursue and punish any cheating that may occur. I'm more experienced than you! If you plan to be dishonest, think again.
* It is not the exam I want! Class material and HW are important, but there won't be questions straight from class/HW. Many things in 2020 are not as we want them.

Corollary If P & P' are any two partitions.

$$L(f,P) \leq U(f,P')$$

Def $U(f) := \inf_P U(f,P)$ $\quad L(f) = \sup_P L(f,P)$

Def integrable $\iff U(f) = L(f)$

Thm f is integrable $\iff \forall \epsilon > 0\ \exists P$ s.t.

$$U(f,P) - L(f,P) < \epsilon.$$

I should have proven here that cont. functions are integrable, following Spivak's 3-7, before the measure non-sense.

Goal: f is integrable iff f is cont. except on a tiny set.

Def measure 0 [using either open or closed rectangles]

Finite sets, countable sets (define, e.g. $\mathbb{Q}$).

The Cantor set. $\quad + \mathbb{R}$ isn't countable

done later.

Subsets.

A countable union of meas-0.

MAT257 Analysis II on October 30, 2020: Measure 0 and content 0, integrability and continuity.
Read Along: Spivak 46-56.
Riddle Along: Players A and B alternate placing 1x2, 1x3, and 1x4 lego pieces (as they choose) on a 19 x 21 board, with no layering and no overlaps. If you cannot place a piece, you lose. Who would you rather be A or B? What if the overall size was 20 x 20?

Goal: ($f: \mathbb{R} \to \mathbb{R}$ integrable) $\Leftrightarrow$ (f is cont. except on a set of measure 0)

Def A is meas-0 means $\forall \epsilon > 0$ $\exists$ open rectangles (R_i) s.t.

1. $A \subset \bigcup R_i$ 2. $\sum v(R_i) < \epsilon$

could say "closed".

Example Finite & countable sets are meas-0.

A line in $\mathbb{R}^2$

The Cantor set

Subsets.

A countable union of meas-0.

Def content 0

Thm Compact + measure 0 $\Rightarrow$ content 0.

Thm $[a,b]$ does not have content 0,

(hence not measure 0, hence not countable)

Skipped Thm Same for $\prod [a_i, b_i]$.

Thm Cont. $\Rightarrow$ Integrable.

MAT257 Analysis II on November 2, 2020: Integrability and continuity.
Read Along: Spivak 46-56.
Read Together: https://q.utoronto.ca/courses/183900/discussion_topics/894005
Riddle Along: Cars A,B,C,D drive in the Sahara Desert on generic straight lines and at constant speed; it is known that A meets B (they arrive at the same place at the same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

Goal: ($f: R \to \mathbb{R}$ integrable) $\Leftrightarrow$ f is cont. except on meas-0.

Thm $[a,b]$ does not have content 0,
(hence not measure 0, hence not countable)

Skipped Thm Same for $\prod [a_i, b_i]$.

Thm Cont. $\Rightarrow$ Integrable.

done online

Take $f: A \subset \mathbb{R}^n \to \mathbb{R}$ bndd.

$o(f,a,\delta)$, $o(f,a)$

Thm f is cont. at a iff $o(f,a)=0$.

Thm If A is closed and $\epsilon>0$, $\{a \in A : o(f,a) \geq \epsilon\}$ is closed.

Aside So disc(f) is F_σ and cont(f) is G_δ

Riddle Is every F_σ set disc(f) for some f?
Is every G_δ set cont(f) for some f?

$$|(x_1 - x_2) - (f(x_1) - f(x_2))|$$

$$< \epsilon |x_1 - x_2|$$

$$x_0 = 0$$

$$x_n = x_{n-1} - (y - f(x_{n-1}))$$

$$x_{n+1} = x_n - (y - f(x_n))$$

$$|x_n - x_{n+1}| = |x_{n-1} - x_n - (f(x_{n-1}) - f(x_n))|$$

MAT257 Analysis II on November 4, 2020: Integrability, continuity, measure 0.
Read Along: Spivak 46-56 (warning: pf of Thm 3-8 is buggy; see page 145).
Riddle Along: Cars A,B,C,D drive in the Sahara Desert on generic straight lines and at constant speed; it is known that A meets B (they arrive at the same place at the same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

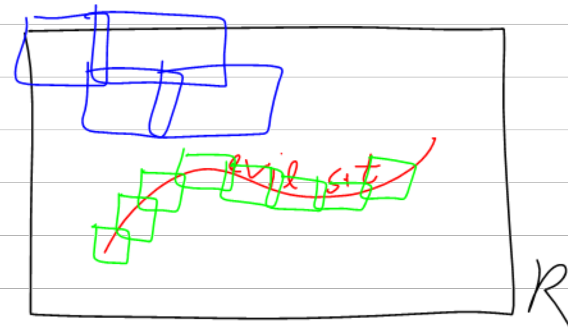
Goal: $(f: R \to \mathbb{R}$ integrable$) \Leftrightarrow f$ is cont. except on meas-0.

Done $(f: R \to \mathbb{R}$ integrable$) \Leftarrow f$ is cont.

Strategy

oscillation

$$U(f,P)-L(f,P)=\sum_{S\in P} v(S)\cdot o(f,S)$$



Take $f: A\subset\mathbb{R}^n \to \mathbb{R}$ bndd.

$o(f,a,\delta)$, $o(f,a)$

Thm f is cont. at a iff $o(f,a)=0$.

Aside If A is closed and $\epsilon>0$, $\{a\in A: o(f,a)\geq\epsilon\}$ is closed.

Aside So disc(f) is F_σ and cont(f) is G_δ

Riddle Is every F_σ set disc(f) for some f?

done
done

Is every G_δ set cont(f) for some f?

Pf of Goal, $\Leftarrow$.

MAT257 Analysis II on November 6, 2020: Integrability, continuity, measure 0.
Read Along: Spivak 46-56 (warning: pf of Thm 3-8 is buggy; see page 145).
Riddle Along: Cars A,B,C,D drive in the Sahara Desert on generic straight lines and at constant speed; it is known that A meets B (they arrive at the same place at the same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

Goal: ($f: R \to \mathbb{R}$ integrable) $\Leftrightarrow$ f is cont. except on meas-0.

Claim R closed, $E \subset R$ s.t. $\forall x \in R \setminus E$ $\exists \epsilon > 0$ $B_\epsilon(x) \cap R \subset R \setminus E$
$\Rightarrow$ E is closed $\quad$ [For $E = R \cap \left(\bigcup \left(\begin{smallmatrix}\text{all these}\\ \text{balls}\end{smallmatrix}\right)\right)^c$]

Pf of $\Leftarrow$: Given ϵ,

Let $E = \{a \in R : o(f,a) \geq \epsilon_1\}$ $\quad \epsilon_1$: TBD

Cover E with open rects $\mathcal{A} = \{A_i\}$ s.t. $\sum V(A_i) < \epsilon_2$ (TBD)

Cover $R \setminus E$ with open rects $\mathcal{B}$.

$\mathcal{A} \cup \mathcal{B}$ covers R; by compactness, find some finite subcover $\mathcal{C}$. Let P be a partition s.t. every $C \in \mathcal{C}$ is a union of $S \in P$.

$$U(f,P) - L(f,P) = \sum_{S \in P} V(S)\, o(f,S)$$
$$\leq \sum_{\substack{S \in P,\\ \exists i\, S \subset A_i}} V(S)\, o(f,S) + \sum_{\substack{S \in P\\ \exists B \in \mathcal{B}\, S \subset B}} V(S)\, o(f,S)$$
$$\leq M \cdot \epsilon_2 + V(R)\, \epsilon_1 \ \ldots$$

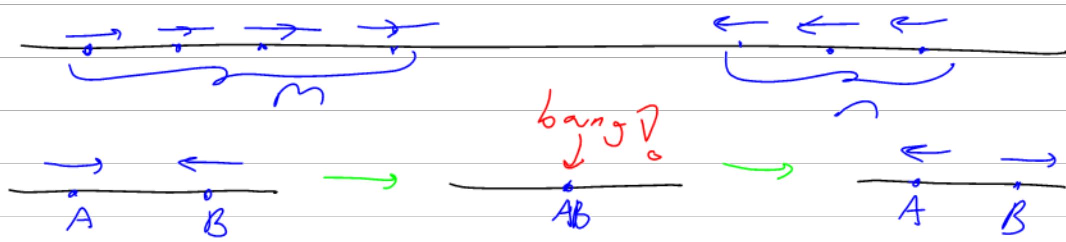
$\Longrightarrow$ Let $E_n = \{a \in R : o(f,a) \geq \frac{1}{n}\}$; Let $\epsilon > 0$ be given

Find P s.t. $\sum_{S \in P} V(S)\, o(f,S) < \epsilon_1$ $\quad (\epsilon_1$ TBD)

$$\frac{1}{n} \sum_{\substack{S \in P\\ (\operatorname{int} S) \cap E_n \neq \emptyset}} V(S) \leq \ldots \qquad \text{so } \sum_{\substack{S \in P\\ (\operatorname{int} S) \cap E_n \neq \emptyset}} V(S) < n \epsilon_1$$

- - - -

MAT257 Analysis II on November 16, 2020: Fubini's Theorem.
Read Along: Spivak 55-56, 56-62.
TT1 Comments: Solution sets, appeals.
Riddle Along: How many bangs?



$\chi_C(x) = 1_C(x)$

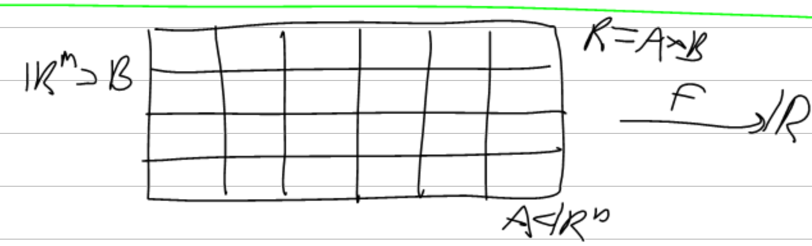
For $C \subset \mathbb{R}$, $\mathrm{vol}(C)$ Aka "content", area, length.

Claim χ_C integrable $\iff$ bd C has meas 0.

Def $\int_C f := \int f \chi_C$ [may not make sense even if C is open!]

$L\int = \underline{\int} = \underline{\int} := \sup L(f,P)$

$U\int = \overline{\int} = \overline{\int} := \inf U(f,P)$



Given $f: R = A\times B \to \mathbb{R}$, set $\underline{F}(x) = \underline{\int_B} f(x,y)\,dy$ $\quad \overline{F}(x) = \overline{\int_B} f(x,y)\,dy$

Thm (not really Fubini)
If f is integrable, then $\int_{A\times B} f = \int_A \underline{F} = \int_A \overline{F}$

Comments 1. f cont. $\Rightarrow \ldots$ 2. $\int_{[0,1]^2} x\cdot y\, dx\,dy = \frac{1}{4}$

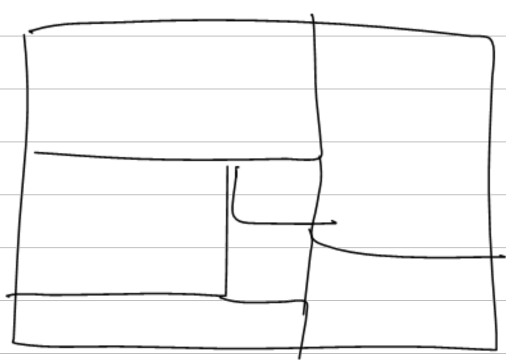
3. $f(x,-)$ integrable except on a finite set.

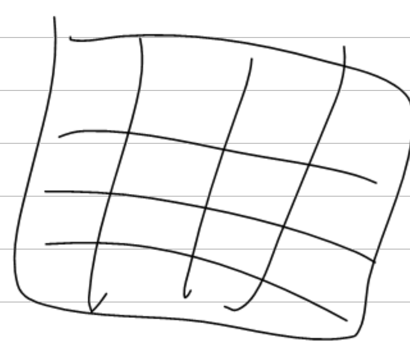
4. $f(x,y) = \begin{cases} 1+\frac{1}{q} & x,y\in\mathbb{Q},\ x = p/q \\ 0 & \text{otherwise} \end{cases} =$

Pf of Fubini: P is $P_A \times P_B$

$$L(f,P) \overset{\text{work here}}{\leq} L(\underline{F}, P_A) \begin{matrix} \leq L(\overline{F}, P_A) \leq \\ \leq U(\underline{F}, P_A) \leq \end{matrix} U(\overline{F}, P_A) \overset{\text{same work}}{\leq} U(f,P)$$

$\square$





$$\int_R e^{2\pi i(x+y)}$$

MAT257 Analysis II on November 18, 2020: Fubini's Theorem.
Read Along: Spivak 56-62. HW7 is due at 11:59PM, HW8 is on web.
Riddle Along: Your turn!

Guido?

Thm $f:(R=A\times B\subset\mathbb{R}^n\times\mathbb{R}^m)\to\mathbb{R}$ integrable,

$$\underline{g}(x):=\underline{\int_B} f(x,y)\,dy \qquad \overline{g}(x):=\overline{\int_B} f(x,y)\,dy$$

then $$\int_R f\,dx\,dy=\int_A \underline{g}(x)\,dx=\int_A \overline{g}(x)\,dx$$

Comments 1. makes integrals computable?

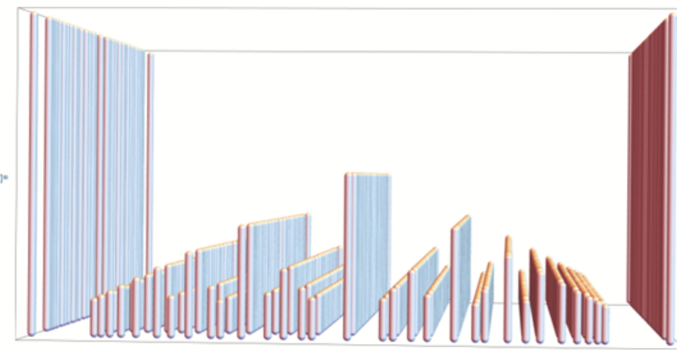
2. $f(x,-)$ integrable except on a finite set?

3. $f(x,y)=1+\begin{cases}\frac{1}{q} & x,y\in\mathbb{Q},\ x=p/q\\ 0 & \text{otherwise}\end{cases}$

on $[0,1]^2$

```
In[1]:= Q = Union @@ Table[p / q, {q, 1, 10}, {p, 0, q}];
Graphics3D[{Table[Tube[{{x, y, 0}, {x, y, 1/(2 Denominator[x])}}], {x, Q}, {y, Q}]},
 {ImageSize → {671.5, Automatic}, ViewPoint → {0.239392, -3.37531, -0.000439167},
  ViewVertical → {0.0384962, -0.589943, 0.806526}}]
```

Out[2]=



Let $g(x)=\begin{cases}\int_{[0,1]} f(x,y)\,dy & \text{if exists}\\ 0 & \text{otherwise}\end{cases}$

$\overline{g}$, $\underline{g}$ as before.

Compare $\int f$, $\int\overline{g}$, $\int\underline{g}$, $\int g$.

Pf of Fubini: P is $P_A\times P_B$

$$L(f,P)\overset{\text{work here}}{\leq} L(\underline{g},P_A)\ \begin{matrix}\leq L(\overline{g},P_A)\leq\\ \leq U(\underline{g},P_A)\leq\end{matrix}\ U(\overline{g},P_A)\overset{\text{same work}}{\leq} U(f,P)$$

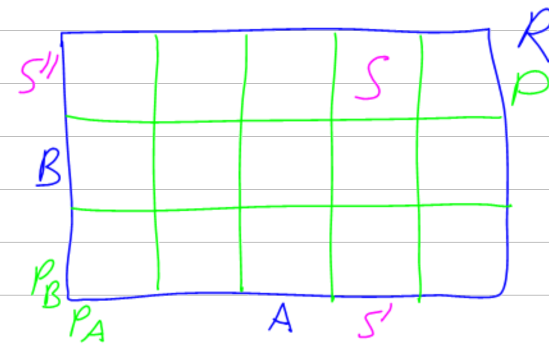
$\square$

MAT257 Analysis II on November 20, 2020: Fubini's Theorem, partitions of unity.
Read Along: Spivak 56-62 and 63-66.
Riddle Along: n prisoners. Each wears a tower of infinitely-many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right? Can they do better than 1/2^n?

Thm $f:(R=A\times B\subset\mathbb{R}^n\times\mathbb{R}^m)\longrightarrow\mathbb{R}$ integrable, $\underline{g}(x):=\underline{\int_B} f(x,y)\,dy$

$$\Rightarrow \int_R f\,dx\,dy=\int_A \underline{g}(x)\,dx=\int_A \overline{g}(x)\,dx \qquad \overline{g}(x):=\overline{\int_B} f(x,y)\,dy$$

Proof Given $P=P_A\times P_B$, write each $S\in P$ as $S=S'\times S''$, $S'\in P_A$, $S''\in P_B$.



$$L(f,P)=\sum_{S'\in P_A} v(S')\sum_{S''\in P_B} v(S'')\inf_{S'\times S''} f$$

$$=\sum_{S'\in P_A} v(S')\sum_{S''\in P_B}\inf_{x\in S'} v(S'')\cdot\inf_{y\in S''} f(x,y)$$

Aside
$\inf_x h_k(x)\le h_k(x)$
so
$\sum \inf_x h_k(x)\le \sum h_k(x)$
so
$\sum \inf h_k(x)\le \inf \sum h_k(x)$

$$\le\sum_{S'\in P_A} v(S')\inf_{x\in S'}\underbrace{\sum_{S''\in P_B} v(S'')\inf_{y\in S''} f(x,y)}_{L(f(x,-),P_B)\le \underline{g}(x)}$$

$$\le\sum_{S'\in P_A} v(S')\inf_{x\in S'}\underline{g}(x)=L(\underline{g},P_A)$$

Similarly $U(f,P)\ge U(\overline{g},P_A)$, so

$$L(f,P)\le L(\underline{g},P_A)\le L(\overline{g},P_A)\le U(\overline{g},P_A)\le U(f,P)$$
$$L(\underline{g},P_A)\le U(\underline{g},P_A)\le U(\overline{g},P_A)$$

..... $\square$

Thm (POI) $\mathcal{U}=\{U\}$ an open cover of $A\subset\mathbb{R}^n\Rightarrow$ $\exists$ countable $\Phi=\{\varphi_i:W\to[0,1]\}$ C^∞ on an open set $W\supset A$ s.t.

1. Φ is "locally finite"

2. $\forall x \in A \quad \sum_{\varphi \in \Phi} \varphi(x) = 1$

3. $\forall \varphi \in \Phi \ \exists U \in \mathcal{U} \quad \operatorname{supp} \varphi \subset U$.

Φ is "a partition of unity for A subordinate to $\mathcal{U}$"

Philosophy about why care.

Indeed, suppose $f : \underset{\text{not nec. bdd.}}{A^{\text{open}}} \to \mathbb{R}$ locally bdd but not necessarily bdd

and with disc(f) of meas-0.

Let $\mathcal{U} = \{U\}$ be a cover of A by bdd open sets contained in A and let $\Phi = \{\varphi\}$ be a POI of A sub to $\mathcal{U}$. Then $\forall \varphi \in \Phi$, $\int \varphi |f|$ makes sense. Call f "integrable(NT)" if $\sum_i \int \varphi_i |f|$ converges. Then $\sum_i \int \varphi_i f$

is absolutely convergent. Define

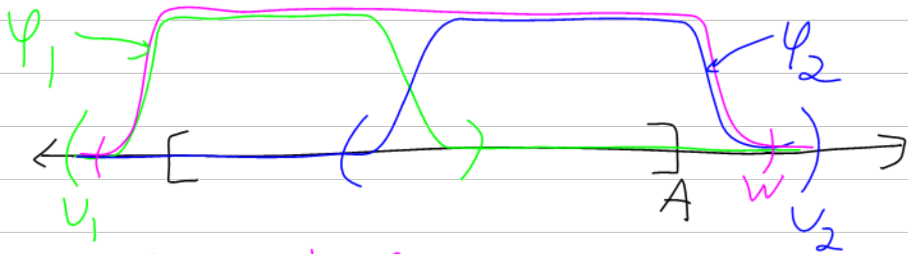
$$\int_A^{(\mathcal{U},\Phi)} f = \sum_i \int \varphi_i f$$

<u>Thm</u> 1. $\displaystyle\int_A^{(\mathcal{U},\Phi)} f = \int_A^{(\mathcal{U}',\Phi')} f$

2. If A & f are bdd, then f is integ (NT)

3. If also A is Jordan-meas, then $\int_A^{NT} f = \int_A f$.

MAT257 Analysis II on November 23, 2020: Partitions of unity.
Read Along: Spivak 63-66. Note the change in schedule!
Riddle Along: n b/w-hat-wearing prisoners stand in a row; each one sees the hats ahead of them but not their own or the ones behind. They each must guess and shout the colour on their head, going from the back forward. If more than one is wrong, all are executed. Could they have devised a strategy in advance, to save themselves?



A way to divide labour: $1_A \le \varphi_1 + \varphi_2 \le 1_W$

φ_i smooth, $\operatorname{supp}\varphi_i \subset U_i$

Later: $F: A \to \mathbb{R}$

$$\int_A^{NT} F := \int_{U_1} \varphi_1 F + \int_{U_2} \varphi_2 F$$

Thm (PO1) Given $A \subset \mathbb{R}^n$ & $\mathcal{U}$ an open cover thereof, we can find a countable collection $\Phi = \{\varphi_i : W \to [0,1]\}$ of C^∞ functions defined on some open $W \supset A$, s.t.

1. Φ is locally finite: Each $x \in W$ has some open neighborhood $V \ni x$, s.t. "loc. fin."
$$|\{i : \operatorname{supp}(\varphi_i) \cap V \neq \emptyset\}| < \infty$$
2. $\forall x \in A \quad \sum \varphi_i(x) = 1$. "Sum=1"
3. $\forall \varphi_i \in \Phi \ \exists U \in \mathcal{U}$ s.t. $\operatorname{supp}\varphi_i \subset U$. "subordinate"

Prel. 1 $\exists$ smooth flat-top mountains:

Mt. Conner, Aus

If $\underset{\text{compact}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists f \in C^\infty(\mathbb{R}^n)$ s.t.

$$f|_C \equiv 1, \quad \operatorname{supp} f \subset U$$

Steps 1. $\exists$ smooth 1D slashes: $\sigma(x) = 0 \quad x \le 0$, $\sigma(x) > 0 \quad x > 0$ $\qquad \sigma(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}$

2. $\exists$ smooth 1D bumps: $\beta_\epsilon(x) \ge 0$, $\beta(0) > 0$, $\operatorname{supp}\beta_\epsilon \subset [-\epsilon, \epsilon]$ $\qquad \beta_\epsilon(x) = \sigma(\epsilon + x) \cdot \sigma(\epsilon - x)$

3. $\exists$ smooth nD bumps: $\beta(a) > 0$, $\beta(x) = 0 \quad |x - a| > \epsilon$ $\qquad \beta_{a,\epsilon} = \beta_{\epsilon^2}\left(\sum (x_i - a_i)^2\right)$

4. $\exists$ smooth 1D steps: $\theta(x) = 0 \quad x \le 0$, $\theta(x) = 1 \quad x \ge 1$ $\qquad \theta(x) = \frac{1}{Z} \int_0^x \beta_{\frac{1}{2},\frac{1}{2}}(t)\,dt$

5. Finish the proof. done live

<u>Prel 2</u> If $\underset{\text{cmpct}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists$ cmpct D s.t.

$$C \subset \operatorname{int} D \subset D \subset U.$$

Back to PO1: Given $A, \mathcal{U}$ $\exists\, \varphi_i$ loc fin, sum $=1$, subordinate.

<u>Case I</u> A is cmpct.

<u>Pf</u> WLOG $\mathcal{U} = \{U_i\}_1^n$ is finite. Shrink U_i

to a cmpct $C_i \subset U_i$ s.t. $\{\operatorname{int} C_i\}$ covers A.

Find ψ_i on $\bigcup U_i$ w/ $\psi_i|_{C_i} \equiv 1$, $\operatorname{supp} \psi_i \subset U_i$ &

f s.t. $f|_A \equiv 1$, $\operatorname{supp} f \subset \bigcup \operatorname{int} C_i$

& s.t.

$$\varphi_i(x) = \begin{cases} f(x) \dfrac{\psi_i(x)}{\sum \psi_i(x)} & x \in \bigcup \operatorname{int} C_i \\ 0 \text{ otherwise} \end{cases}$$

MAT257 Analysis II on November 25, 2020: Partitions of unity.
Read Along: Spivak 63-66. Note the newer change in schedule!
Riddle Along: Me? Riddles?

PO1: Given $A, \mathcal{U}$ $\exists\, \varphi_i$ loc fin, sum $=1$, subordinate.

Prel 1 $\exists$ smooth flat-top mountains:

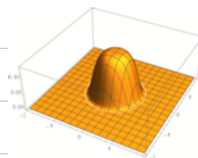
Mt. Conner, Aus

If $\underset{\text{compact}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists\, f \in C^\infty(\mathbb{R}^n)$ s.t.

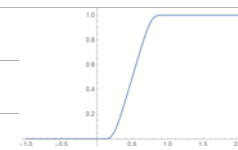
$f|_C \equiv 1$, $\operatorname{supp} f \subset U$

Steps

3. $\exists$ smooth nD bumps $\beta_{a,\epsilon}(a) > 0$, $\beta_{a,\epsilon}(x) = 0$ $|x-a| > \epsilon$



4. $\exists$ smooth 1D steps $\theta(x) = 0$ $x \leq 0$, $\theta(x) = 1$ $x \geq 1$



5. Finish the proof.

Prel 2 If $\underset{\text{cmpct}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists$ cmpct D s.t.

$C \subset \operatorname{int} D \subset D \subset U$.

Back to PO1: Given $A, \mathcal{U}$ $\exists\, \varphi_i$ loc fin, sum $=1$, subordinate.

Case I A is cmpct.

Pf WLOG $\mathcal{U} = \{U_i\}_1^n$ is finite. Shrink U_i to a cmpct $C_i \subset U_i$ s.t. $\{\operatorname{int} C_i\}$ covers A.

Find ψ_i on U_i w/ $\psi_i|_{C_i} \equiv 1$, $\operatorname{supp} \psi_i \subset U_i$ &

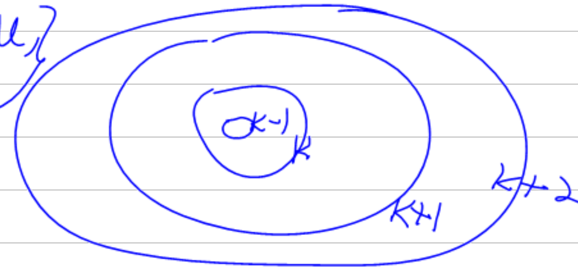
F s.t. $F|_A \equiv 1$, $\operatorname{supp} F \subset \bigcup \operatorname{int} C_i$

& set

$$\varphi_i(x) = \begin{cases} F(x) \dfrac{\psi_i(x)}{\sum \psi_i(x)} & x \in \bigcup \operatorname{int} C_i \\ 0 & \text{otherwise} \end{cases}$$

Case II $A = \bigcup_{k=0}^{\infty} A_k$, A_k cmpct, $A_k \subset \operatorname{int} A_{k+1}$; $A_0 = \emptyset$

$\mathcal{U}_k = \{ U \cap \operatorname{int} A_{k+2} \cap (\mathcal{U}_{k-1})^c : U \in \mathcal{U} \}$ $k \geq 1$



Find Φ_k for $A_{k+1} \setminus \operatorname{int} A_k$, let

$\{\bar{\varphi}_i\} = \bar{\Phi} = \bigcup \Phi_k$ (still countable!)

and set $\varphi_i(x) = \dfrac{\bar{\varphi}_i(x)}{\sum \bar{\varphi}_i(x)}$

Case III A open. Take $A_k = \{ x : |x| \leq k \ \& \ \operatorname{dist}(x, A^c) \geq \frac{1}{k} \}$.

Case IV Any A.

MAT257 Analysis II on November 27, 2020: Partitions of unity, integration NT.
Read Along: Spivak 63-66.
Riddle Along: Can you find irrational x and y such that x^y is rational?

POI: Given A, $\mathcal{U}$ $\exists \varphi_i$ loc fin, sum$=1$, subordinate.

Case I A is cmpct.

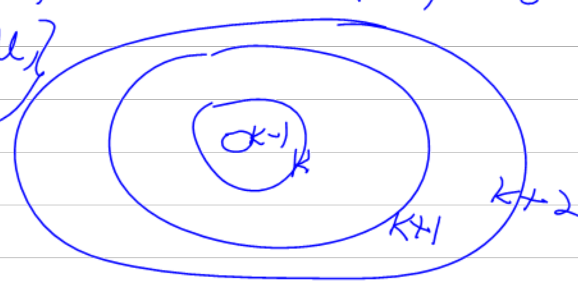
PF WLOG $\mathcal{U}=\{U_i\}_1^n$ is finite. Shrink U_i to a cmpct $D_i \subset U_i$ s.t. $\{\operatorname{int} D_i\}$ covers A.

Find ψ_i on $\bigcup U_i$ w/ $\psi_i|_{D_i}\equiv 1$, supp $\psi_i \subset U_i$ & f s.t. $f|_A \equiv 1$, supp $f \subset \bigcup \operatorname{int} D_i$

& set
$$\varphi_i(x)=\begin{cases} f(x)\dfrac{\psi_i(x)}{\sum \psi_i(x)} & x\in \bigcup \operatorname{int} D_i \\ 0 & \text{otherwise}\end{cases}$$

Case II $A=\bigcup_{k=0}^{\infty} A_k$, A_k cmpct, $A_k \subset \operatorname{int} A_{k+1}$; $A_0=\emptyset$

$\mathcal{U}_k=\{U\cap \operatorname{int} A_{k+2} \cap (A_{k-1})^c : U\in\mathcal{U}, k\geq 1\}$



Find Φ_k for $A_{k+1}\setminus \operatorname{int} A_k$, let $\{\bar\varphi_j\}=\bar\Phi=\bigcup \Phi_k$ (still countable!)

and set $$\varphi_j(x)=\frac{\bar\varphi_j(x)}{\sum \bar\varphi_i(x)}$$

Case III A open. Take $A_k=\{x: |x|\leq k \text{ \& } \operatorname{dist}(x,A^c)\geq \frac{1}{k}\}$.

Case IV Any A.

Suppose $f: A \xrightarrow{\text{open}} \mathbb{R}$ locally bdd but not necessarily bdd
(A not nec. bdd.)
and with disc(f) of meas-0.

Let $\mathcal{U}=\{U\}$ be a cover of A by bndd open sets contained in A and let $\Phi=\{\varphi\}$ be a POI of A sub to $\mathcal{U}$. Then $\forall \varphi\in\Phi$, $\int \varphi|f|$ makes sense. Call f "integrable(NT)" if $\sum_i \int \varphi_i|f|$ converges. Then $\sum_i \int \varphi_i f$ is absolutely convergent. Define

$$\int_A^{(\mathcal{U},\Phi)} f = \sum_i \int \varphi_i f$$

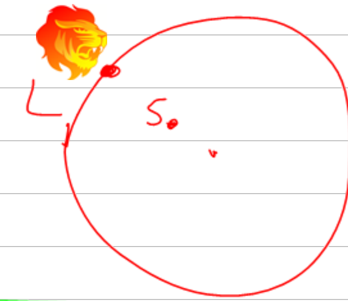
<u>Thm</u> 1. $\int_A^{(\mathcal{U},\Phi)} f = \int_A^{(\mathcal{U}',\Phi')} f$ Review proof!

2. If A & f are bndd, then f is intg (NT)

3. If also A is Jordan-meas, then $\int_A^{NT} f = \int_A f$.

MAT257 Analysis II on November 30, 2020: Integration NT.
Read Along: Spivak 63-66 and 66-74. HW9 deadline extended to Friday!
Riddle Along: Survive!

$V_L = 4V_S$



POI: Given $A, \mathcal{U}$ $\exists\, \varphi_i$ loc fin, sum $=1$ subordinate.

Suppose $F : \underset{\text{not nec. bndd}}{A} \xrightarrow{\text{open}} \mathbb{R}$ locally bndd but not necessarily bndd and with disc(F) of meas-0.

Let $\mathcal{U} = \{U\}$ be a cover of A by bndd open sets contained in A and let $\Phi = \{\varphi\}$ be a POI of $\underbrace{A \text{ sub to } \mathcal{U}}_{A}$. Then $\forall \varphi \in \Phi$, $\int \varphi |F|$ makes sense. Call F "integrable(NT)" if $\sum_i \int_{\text{where?}} \varphi_i |F|$ converges. Then $\sum_i \int \varphi_i F$ is absolutely convergent. Define

A: Wrong def, $\sum a_i$ when $\sum |a_i| < \infty$, a tale of good & evil

$$\int_A^{(\mathcal{U},\Phi)} F = \sum_i \int \varphi_i F$$

Thm

1. $\int_A^{(\mathcal{U},\Phi)} F = \int_A^{(\mathcal{U}',\Phi')} F$ (so $\int^{NT}$ makes sense)

2. If A & F are bndd, then F is integ (NT)

3. If also A is Jordan-meas, then $\int_A^{NT} F = \int_A F$.

Pf 1.

$$\int_A^{(\mathcal{U},\Phi)} g \underset{(1)}{=} \sum_i \int \varphi_i g \underset{(2)}{=} \sum_i \int \Big(\sum_j \varphi'_j\Big) \varphi_i g \underset{(3)}{=} \sum_i \sum_j \int \varphi'_j \varphi_i g$$

$$\underset{(4)}{=} \sum_j \sum_i \int \varphi_i \varphi'_j g \underset{(3)}{=} \sum_j \int \Big(\sum_i \varphi_i\Big) \varphi'_j g \underset{(2)}{=} \sum_j \int \varphi'_j g \underset{(1)}{=} \int_A^{(\mathcal{U}',\Phi')} g$$

For $g = |f|$:

(1): ignore.
(2): sum $= 1$ ✓
(3): A finite sum ✓
(4): all ≥ 0

For $g = f$:

(1) def
(2) sum $= 1$
(3) a finite sum
(4) absolute convergence.

2. If $|f| \leq M$ & $A \subset R$ rect, & if $F \subset \Phi$ is finite,

$$\sum_{\varphi \in F} \int_A \varphi |f| = \int_A \Big(\sum_{\varphi \in F} \varphi\Big) |f| \leq 1 \cdot M \cdot \mathrm{vol}(R) \ldots$$

3. If also A is Jordan-measurable, find a compact $C \subset A$ s.t. $\mathrm{vol}(A - C) < \epsilon$. For only finitely many i's, $\mathrm{supp}\, \varphi_i \cap C \neq 0$; let N be bigger than the biggest of those. Then

$$\Big| \int_A f - \sum_{i=1}^{N} \varphi_i f \Big| \leq \int_A \Big| f - \sum \varphi_i f \Big|$$

$$\leq M \int_A \Big(1 - \sum_{i=1}^{N} \varphi_i\Big) \leq M \int_{A-C} 1 \leq M\epsilon. \qquad \square$$

MAT257 Analysis II on December 2, 2020: Integration NT, @9:50: changes of variable.
Read Along: Spivak 63-66 and 66-74.
Riddle Along: Save Vidhu!

$V_L = V_V$

$f : A \xrightarrow{\text{open}} \mathbb{R}$ (A not nec. bdd.) locally bdd but not necessarily bdd; disc(f) of meas-0

$\mathcal{U}$: Cover A by bdd open sets contained in A. $\Phi = \{\varphi_i\}$: PO1 for A subordinate to $\mathcal{U}$

f "$(\mathcal{U},\Phi)$-integrable" means $\sum \varphi_i |f| < \infty$; $\int_A^{(\mathcal{U},\Phi)} f := \sum \int \varphi_i f$

Thm $(\mathcal{U},\Phi)$-integrable $\Leftrightarrow$ $(\mathcal{U}',\Phi')$-integrable & $\int_A^{(\mathcal{U},\Phi)} f = \int_A^{(\mathcal{U}',\Phi')} f$.

PF 1.
$$\int_A^{(\mathcal{U},\Phi)} g \overset{(1)}{=} \sum_i \int \varphi_i g \overset{(2)}{=} \sum_i \int \Big(\sum_j \varphi_j'\Big)\varphi_i g \overset{(3)}{=} \sum_i \sum_j \int \varphi_j' \varphi_i g$$
$$\overset{(4)}{=} \sum_j \sum_i \int \varphi_i \varphi_j' g \overset{(3)}{=} \sum_j \int \Big(\sum_i \varphi_i\Big)\varphi_j' g \overset{(2)}{=} \sum_j \int \varphi_j' g \overset{(1)}{=} \int_A^{(\mathcal{U}',\Phi')} g$$

For $g=|f|$:
(1): ignore.
(2): sum=1 !
(3): A finite sum!
(4): all ≥ 0

For $g=f$:
(1) def
(2) sum=1
(3) a finite sum
(4) absolute convergence.

Thm 1. If A & f are bdd, then f is integ (NT)
2. If also A is Jordan-meas, then $\int_A^{NT} f = \int_A f$.

PF 1. If $|f| \leq M$ & $A \subset R$ rect, & if $F \subset \Phi$ is finite,
$$\sum_{\varphi \in F} \int_A \varphi |f| = \int_A \Big(\sum_{\varphi\in F}\varphi\Big)|f| < 1 \cdot M \cdot \mathrm{vol}(R) \ldots$$

2. If also A is Jordan-measurable, find a compact $C \subset A$ s.t. $\mathrm{vol}(A-C) < \epsilon$. For only finitely

many i's, $\operatorname{supp} \varphi_i \cap C \neq \emptyset$; let N be bigger than the biggest of those. Then

$$\left| \int_A f - \sum_{i=1}^{N} \int_A \varphi_i f \right| \le \int_A \left| f - \sum \varphi_i f \right|$$

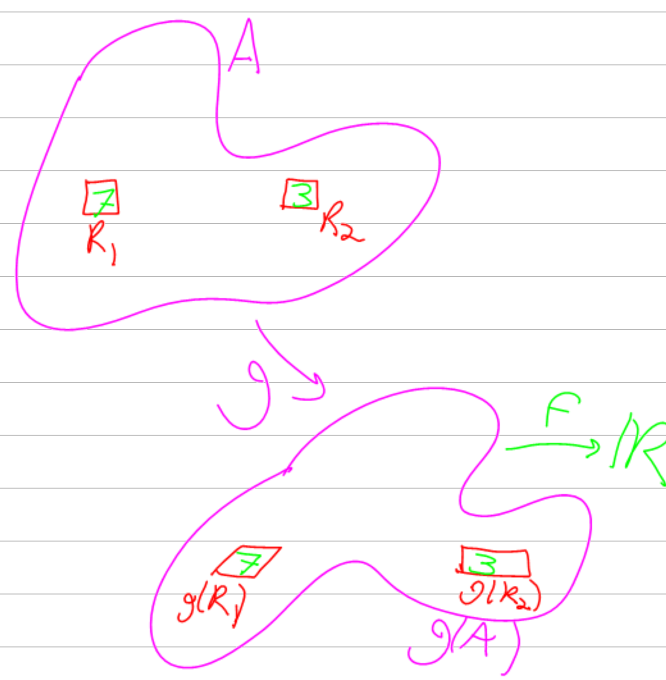
$$\le M \int_A \left(1 - \sum_{i=1}^{N} \varphi_i\right) \le M \int_{A-C} 1 \le M\epsilon.$$

$\square$

<u>Thm</u> (Change of Variables, "COV")

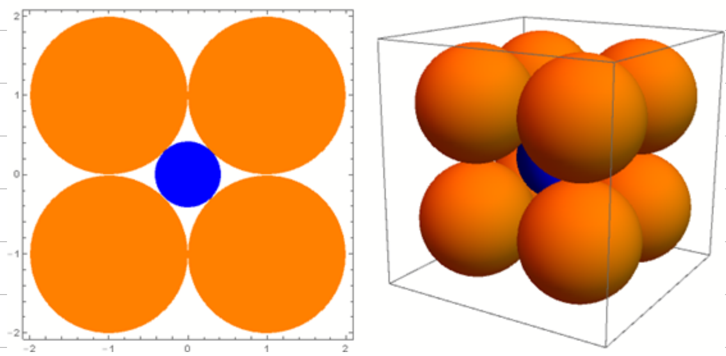
Let $A \subset \mathbb{R}^n$ be open, $g : A \to \mathbb{R}^n$ cont. diffable, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $f : g(A) \to \mathbb{R}$ is integrable, then

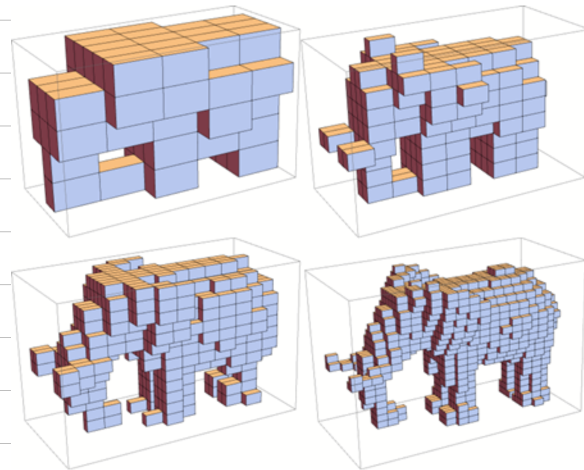
$$\int_{g(A)} f = \int_A (f \circ g) |\det g'|$$



MAT257 Analysis II on December 4, 2020: Changes of variable, Gaussian integration, volumes of spheres and balls.
Read Along: Spivak 66-74.
Riddle Along: Compute lim_{n\to\infty}Vol(B_n)/Vol(C_n), where B_n is the largest ball bounded by 2^n balls of radius ones with centers at {-1,1}^n and C_n is the smallest cubes bounding same balls. Promise: You will learn something very surprising if you solve this riddle.

```
GraphicsRow[{
  Graphics[{Orange, Disk /@ Tuples[{1, -1}, 2], Blue, Disk[{0, 0}, √2 - 1]}, Frame -> True],
  Graphics3D[{Orange, Ball /@ Tuples[{1, -1}, 3], Blue, Ball[{0, 0, 0}, √3 - 1]}]
 }, ImageSize → 720]
```

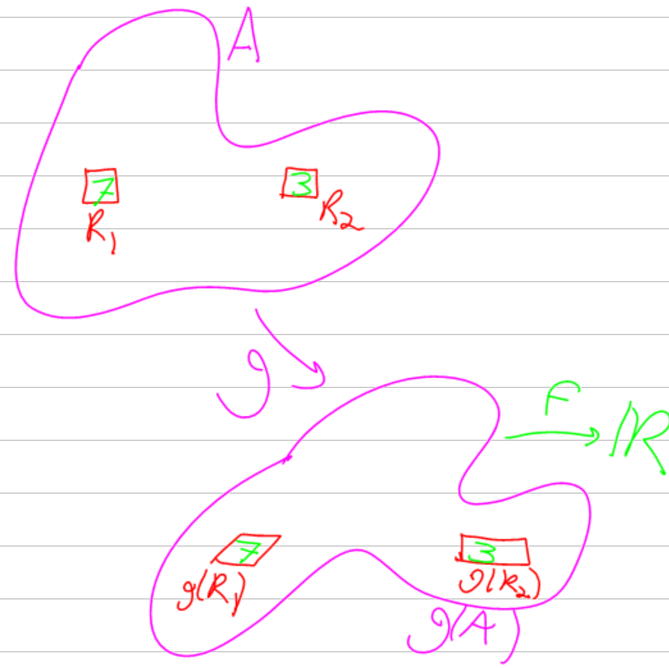




Thm (Change of Variables, "COV")

Let $A\subset\mathbb{R}^n$ be open, $g: A\to\mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x\in A$ $g'(x)$ is invertible. If $F: g(A)\to\mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F\circ g)|\det g'|$$



Compute $I_1=\int_{\mathbb{R}} e^{-x^2/2}dx$, "the most important integral in math".

$$I_2=\int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}dx\,dy \overset{(1)}{=} \int_{\mathbb{R}}dx\int_{\mathbb{R}}dy\, e^{-x^2/2}e^{-y^2/2} \overset{(2)}{=} \int dx\, e^{-x^2/2}\int dy\, e^{-y^2/2} = I_1^2$$

$(3)\ \Big\|\ \begin{matrix} x=r\cos\theta \\ y=r\sin\theta \end{matrix}$

$$\int dr\,d\theta\, r e^{-r^2/2} = 2\pi\int dr\, r e^{-r^2/2} \overset{(4)}{=} 2\pi\cdot\left(-e^{-r^2/2}\Big|_0^\infty\right) = 2\pi$$

So $I_1=\sqrt{2\pi}$

Let's compute like physicists!

$\sigma_n : \mathrm{Vol}(S^n)$ $S^n = \{z \in \mathbb{R}^{n+1} : |z| = 1\}$

$$(2\pi)^{\frac{n+1}{2}} = I_{n+1} = \int_{\mathbb{R}^{n+1}} e^{-|z|^2/2} dz = \sigma_n \int_0^\infty r^n e^{-r^2/2} dr = \sigma_n \tau_n$$

$$\tau_{n-2} = \int_0^\infty r^{n-2} e^{-r^2/2} = \frac{1}{n-1} \int_0^\infty r^n e^{-r^2/2} = \frac{1}{n-1} \tau_n \quad \text{so}$$

$$\sigma_n = \frac{(2\pi)^{(n+1)/2}}{\tau_n} = 2\pi \frac{(2\pi)^{(n-1)/2}}{(n-1)\tau_{n-2}} = \frac{2\pi}{n-1} \sigma_{n-2}$$

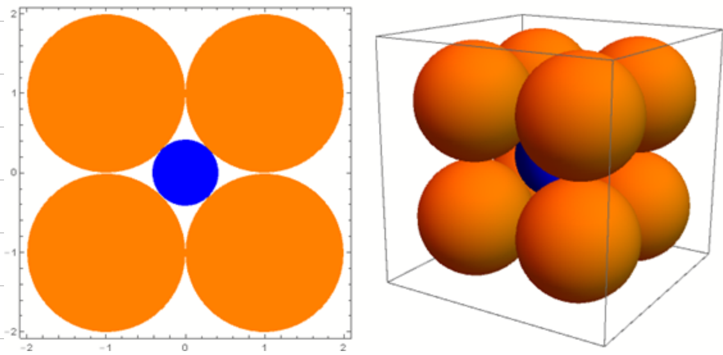
$\sigma_0 = 2$ $\quad \beta_0 = \emptyset$
$\sigma_1 = 2\pi$ $\quad \beta_1 = 2$
$\sigma_2 = 4\pi$ $\quad \beta_2 = \pi$
$\sigma_3 = 2\pi^2$ $\quad \beta_3 = 4\pi/3$
$\vdots$

And $\beta_n = \mathrm{Vol}(B_n) = \dfrac{\sigma_{n-1}}{n}$

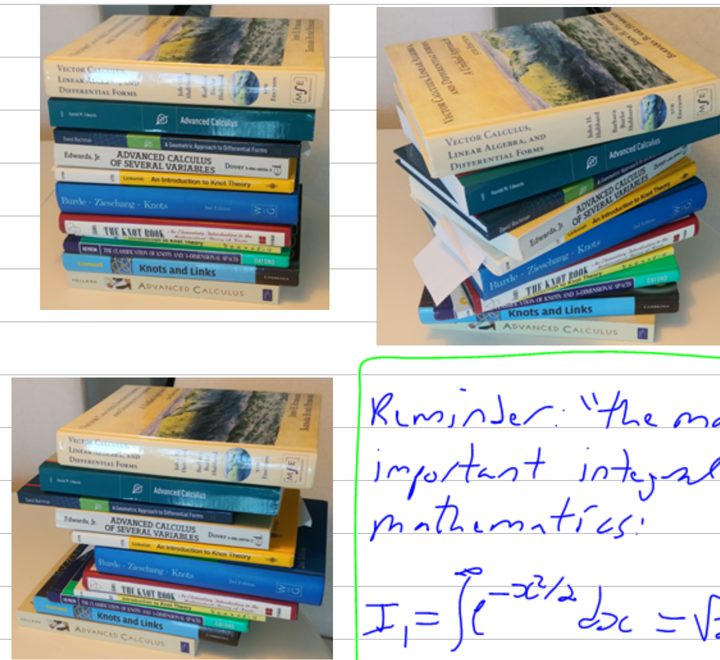
MAT257 Analysis II on December 7, 2020: Volumes of spheres and balls, proof of the COV formula (1).
Read Along: Spivak 66-74.

Riddle: Compute lim_{n\to\infty}Vol(B_n)/Vol(C_n), where B_n is the largest ball bounded by 2^n balls of radius ones with centers at {-1,1}^n and C_n is the smallest cubes bounding same balls. Promise: You will learn something very surprising if you solve this riddle.

```
GraphicsRow[{
  Graphics[{Orange, Disk /@ Tuples[{1, -1}, 2], Blue, Disk[{0, 0}, √2 - 1]}, Frame -> True],
  Graphics3D[{Orange, Ball /@ Tuples[{1, -1}, 3], Blue, Ball[{0, 0, 0}, √3 - 1]}]
 }, ImageSize → 720]
```



Non-riddle (Cavalieri's principle): Which of the following piles of books weighs the most?



Reminder: "the most important integral in mathematics":

$$I_1=\int_{-\infty}^{\infty} e^{-x^2/2}\,dx=\sqrt{2\pi}$$

(also pre-copy next page)

Let's compute like physicists!

σ_n : $Vol(S^n)$ $\quad S^n=\{z\in\mathbb{R}^{n+1} : |z|=1\}$

$$(2\pi)^{\frac{n+1}{2}}=I_{n+1}=\int_{\mathbb{R}^{n+1}} e^{-|z|^2/2}\,dz=\sigma_n\int_0^\infty r^n e^{-r^2/2}\,dr=\sigma_n\tau_n$$

$$\tau_{n-2}=\int_0^\infty r^{n-2}e^{-r^2/2}=\frac{1}{n-1}\int_0^\infty r^n e^{-r^2/2}=\frac{1}{n-1}\tau_n \quad \text{so}$$

$$\sigma_n=\frac{(2\pi)^{(n+1)/2}}{\tau_n}=2\pi\frac{(2\pi)^{(n-1)/2}}{(n-1)\tau_{n-2}}=\frac{2\pi}{n-1}\sigma_{n-2}$$

$\sigma_0=2 \quad \beta_0=\emptyset$
$\sigma_1=2\pi \quad \beta_1=2$
$\sigma_2=4\pi \quad \beta_2=\pi$
$\sigma_3=2\pi^2 \quad \beta_3=4\pi/3$
$\vdots$

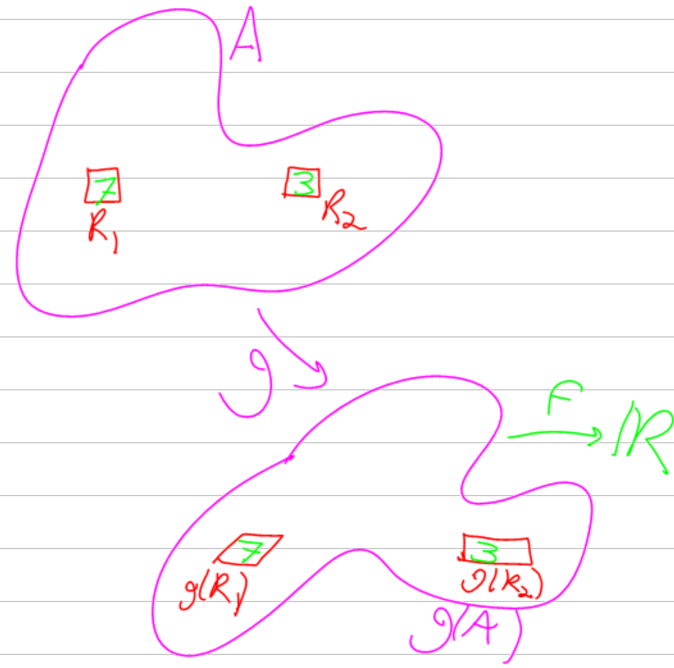
And $\beta_n=Vol(B_n)=\dfrac{\sigma_{n-1}}{n}$

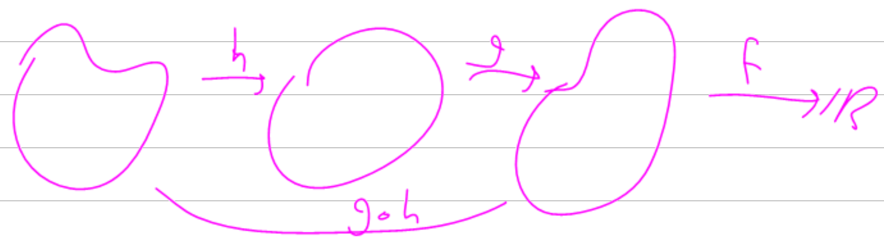
Go over CubeOfOranges.nb!

Thm (Change of Variables, "COV")

Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \to \mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F \circ g) |\det g'|$$





$y_1 = g_1(x_1 \dots x_n)$
$\vdots$
$y_n = g_n(x_1 \dots x_n)$

$$\begin{matrix} x_1 & & x_1 & & y_1 \\ \vdots & \mapsto & \vdots & \mapsto & \vdots \\ & & x_{n-1} & & \\ x_n & & y_n & & y_n \end{matrix}$$

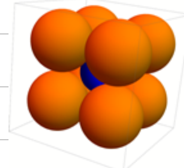
will work only locally!

will need to re-order variables!

Debts

1. $\text{cov}(g), \text{cov}(h) \Rightarrow \text{cov}(g \circ h)$
2. Pf that every g is a composition of layer-preserving maps.
3. Cov for small sets implies cov for all.
4. Cov for coordinate swaps.
5. $\text{Cov}(n-1) \Rightarrow \text{cov}(n)$ for layer-preserving functions.
6. Prove $\text{Cov}(1D)$
7. Maybe more.

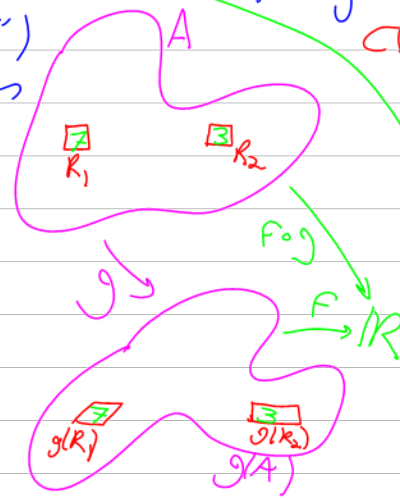
MAT257 Analysis II on December 9, 2020:
Proof of the COV formula (1).
Read Along: Spivak 66-74.



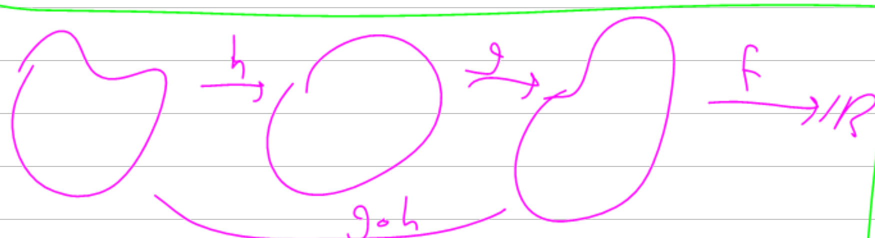
Aside Integrals don't change if you modify the integrand/integration domain on a closed set of meas-0.

Thm (Change of Variables, "COV") Let $A\subset\mathbb{R}^n$ be open, $g:A\to\mathbb{R}^n$ cont. diffable, 1-1, and s.t. $\forall x\in A$ $g'(x)$ is invertible. If $F:\overset{\text{open}}{g(A)}\to\mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F\circ g)\,|\det g'|$$



layer preserving map



Debts 1. $COV(g), COV(h) \Rightarrow COV(g\circ h)$

2. $COV(n-1) \Rightarrow COV(n)$ for layer-preserving maps.

3. Pf that every g is a composition of layer-preserving maps.

4. $COV(\text{small sets}) \Rightarrow COV(\text{large sets})$

5. Prove $COV(1D)$

6. maybe more.

$y_1 = g_1(x_1 \dots x_n)$
$\vdots$
$y_n = g_n(x_1 \dots x_n)$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_n \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Will work only locally!

All debt will be cancelled at the end of the term!
(only details will remain)

Lemma $COV(g), COV(h) \Rightarrow COV(g\circ h)$

Lemma Assume $COV(n-1)$. Let $g: A \xrightarrow{} \mathbb{R}^n$ (open) be layer preserving (namely $g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \equiv \\ \equiv \\ x_n \end{pmatrix}$, or $g_n(x_1 \ldots x_n) = x_n$. Then a restricted $COV(g)$ holds: If $R = R' \times [a,b] \subset A$ is a rectangle and $F: g(R) \to \mathbb{R}$ is cont., then

$$\int_{g(R)} F = \int_R (F \circ g) |\det g'|$$

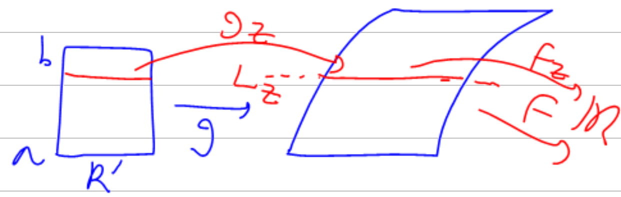
Def't 6: $g(R)$ is Jordan measurable
Def't 7: $RCOV(\text{cont.}) \Rightarrow RCOV(\text{integ})$.

Pf For $z \in [a,b]$, $L_z := \{x \in \mathbb{R}^n : x_n = z\}$

$g_z : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by $g_z(x) = \begin{pmatrix} g_1(x,z) \\ \vdots \\ g_{n-1}(x,z) \end{pmatrix}$ & $F_z : g_z(R') \to \mathbb{R}$ by $F_z(y) = F(y,z)$

then $g(R) = \bigcup_z \{z\} \times g_z(R')$

$$\int_{g(R)} F = \int_{[a,b]} dz \int_{g(R) \cap L_z} F = \int_{[a,b]} dz \int_{g_z(R')} F_z$$



$$= \int_{[a,b]} dz \int_{R'} (F_z \circ g_z) |\det(g_z')| = \int_R (F \circ g) |\det g'|$$

Lemma If $g: \mathbb{R}^n \to \mathbb{R}^n$ is cont. diff'able & $g'(a)$ is invertible, then in some nbd U of a g can be written as a composition of l.p. maps (and ....).

MAT257 Analysis II on December 8, 2020: Proof of the COV formula (2).
Next class on January 11, nothing until then!
Read Along: Spivak 66-74.
COV Strategy: Show that every g is a composition of layer-preserving maps, and use dimensional reduction on those.

Debts.

1. COV(n-1) $\Rightarrow$ COV(n) for l.p. maps
2. Every g is a composition of l.p. maps & coord. swaps.
3. ~~COV(g), COV(h) $\Rightarrow$ COV(g∘h)~~
4. COV holds for coordinate swaps
5. local COV $\Rightarrow$ global COV.
6. ~~Prove COV(1)!~~

7+ maybe more.

Lemma 1 Assume COV(n-1). Let $g: A \xrightarrow{(\text{open})} \mathbb{R}^n$ be layer preserving (namely $g\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix}\equiv \\ \equiv \\ x_n\end{pmatrix}$, or $g_n(x_1 \dots x_n) = x_n$. Then a restricted COV(g) holds: If $R = R' \times [a,b] \subset A$ is a rectangle and $F: g(R) \to \mathbb{R}$ is cont., then

$$\int_{g(R)} F = \int_R (F\circ g)\cdot|\det g'|$$

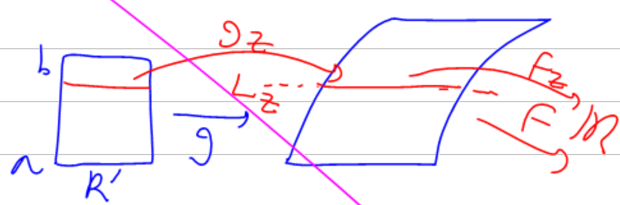
Debt 7: $g(R)$ is Jordan measurable

Debt 8: RCOV(cont.) $\Rightarrow$ RCOV(integ).

PF For $z \in [a,b]$, $L_z := \{x \in \mathbb{R}^n : x_n = z\}$

$g_z : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by $g_z(x) = \begin{pmatrix} g_1(x,z) \\ \vdots \\ g_{n-1}(x,z)\end{pmatrix}$ & $F_z : g_z(R') \to \mathbb{R}$ by $F_z(y) = F(y,z)$

then $g(R) = \bigcup_z \{z\} \times g_z(R')$



$$\int_{g(R)} F = \int_{[a,b]} dz \int_{g(R)\cap L_z} F = \int_{[a,b]} dz \int_{g_z(R')} F_z$$

$$= \int_{[a,b]} dz \int_{R'} (F_z \circ g_z)|\det(g_z')| = \int_R (F\circ g)|\det g'|$$

MAT257 Analysis II on December 8, 2020: Proof of the COV formula (2).
Next class on January 11, nothing until then!
Read Along: Spivak 66-74.
COV Strategy: Show that every g is a composition of layer-preserving maps, and use dimensional reduction on those.

Debts.

1. $COV(n-1) \Rightarrow COV(n)$ for l.p. maps
2. Every g is a composition of l.p. maps & coord. swaps.
3. ~~$COV(g), COV(h) \Rightarrow COV(g\circ h)$~~
4. COV holds for coordinate swaps
5. local COV $\Rightarrow$ global COV.
6. ~~Prove COV(1)!~~

7+ maybe more.

Lemma 1 Assume $COV(n-1)$. Let $g: U \to \mathbb{R}^n$ (open & bndd) be layer preserving (& s.t. $g(U)$ is bndd) (namely $g\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix}\equiv\\ x_n\end{pmatrix}$, or $g_n(x_1 \dots x_n) = x_n$. Then a restricted $COV(g)$ holds: If $F: g(U) \to \mathbb{R}$ is cont. (& bndd), and $\operatorname{supp}(F) \subset U$ then

$$\int_{\cancel{g(U)}\ \mathbb{R}^n} F = \int_{\cancel{U}\ \mathbb{R}^n} (F\circ g)\,|\det g'|$$

Debt 7: $RCOV(\text{cont.}) \Rightarrow RCOV(\text{integ})$.

For simplicity, write all integrals on $\mathbb{R}^n/\mathbb{R}^{n-1}$, extending the integrands by 0 as necessary.

Pf For $z \in \mathbb{R}$ define $g_z: \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by $g_z(x) = \begin{pmatrix} g_1(x,z) \\ \vdots \\ g_{n-1}(x,z)\end{pmatrix}$ & $F_z: \mathbb{R}^{n-1} \to \mathbb{R}$ by $F_z(y) = F(y,z)$.

$$g' = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n-1}} & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_{n-1}}{\partial x_1} & & \frac{\partial g_{n-1}}{\partial x_{n-1}} & \frac{\partial g_{n-1}}{\partial x_n} \\ 0 & & 0 & 1 \end{pmatrix}$$

$$\int_{\mathbb{R}^n} F = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} dx\, F(x,z) = \int_{\mathbb{R}^n} dz \int_{\mathbb{R}^{n-1}} F_z$$

$$= \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} (F_z \circ g_z)\,|\det(g_z')| = \int_{\mathbb{R}^n} (F\circ g)\,|\det g'|$$

Lemma 2 For every $a \in A$ there is some open $U \ni a$ s.t. on U g is a composition of l.p. maps & coordinate swaps.

Pf Let $\alpha_K : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ g_K(x_1 \ldots x_n) \end{pmatrix}$

$$y_i = g_i(x)$$

$$\begin{tikzcd} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \arrow[r, mapsto, "\alpha_K"'] \arrow[rr, bend left, "g"] & \begin{pmatrix} x_1 \\ x_{n-1} \\ y_K \end{pmatrix} \arrow[r, mapsto, "\beta_K"'] & \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \end{tikzcd}$$

then
$$\alpha_K' = \left(\begin{array}{ccc|c} 1 & 0 & & 0 \\ 0 & I_{n-1} & & 0 \\ 0 & & & 0 \\ \hline \frac{\partial g_K}{\partial x_1} & \cdots & \frac{\partial g_K}{\partial x_{n-1}} & \frac{\partial g_K}{\partial x_n} \end{array}\right).$$

For at least one k, $\frac{\partial g_K}{\partial x_n} \neq 0$, pick that k.

Near a, α_K is invertible, set $\beta_K = g \circ \alpha_K^{-1}$

Let T_{ij} be the (ij) swap. Then

$$g = \beta_K \circ \alpha_K = T_{Kn} \circ \underbrace{T_{Kn} \circ \beta_K} \circ T_{1n} \circ \underbrace{T_{1n} \circ \alpha_K \circ T_{1n}} \circ T_{1n}$$

Lemma 5 Local CoV $\Rightarrow$ global CoV.

Find a cover $\mathcal{V} = \{V\}$ of $g(A)$ by bndd open sets s.t. $\forall V \in \mathcal{V}$ $g^{-1}(V)$ is bndd & on it g is a composition of l.p. maps & coord. swaps.

here we use g is a bijection!

Let $\{\psi_i\}$ be a PO1 for $g(A)$ sub to $\mathcal{V}$.

Then $\{\varphi_i = \psi_i \circ g\}$ is a PO1 for A sub to $\mathcal{U} = \{g^{-1}(V)\}$

So

$$\int_{g(A)} F = \sum_i \int_{\mathbb{R}^n} \psi_i F = \sum_i \int_{\mathbb{R}^n} (\psi_i \circ g)(F \circ g) |\det g'| = \ldots$$