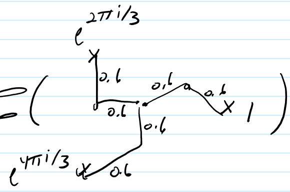
1617-257 Wed Mar 1, hour 57: Forms and d

February 15, 2017 12:58 PM

HW15 due, HW16 on web by midnight.
Read Along: Sections 29-31.

Today: $\displaystyle\int_M d\omega = \int_{\partial M} \omega$

Riddle Along. $\mathcal{E}\Big($



$\Big) = ?$

k-form on M: $\omega : M \to \bigcup_{p\in M} A^k(T_pM)$ s.t. $\forall p\in M$ $\omega(p)\in A^k(T_pM)$

on $\mathbb{R}^n$, $\omega = \sum_{I\in\binom{n}{k}} a_I(x)\psi_I = \sum_{I\in\binom{n}{k}} a_I(x)\,\phi_{i_1}\wedge\ldots\wedge\phi_{i_k}$

C^r means $\forall I$ a_I is C^r $\Leftrightarrow$ $\forall C^r$ v.f. $V_1\ldots V_k$, $\omega(V_1\ldots V_k)$ is C^r

Forms pull back! $f:\mathbb{R}^n\to\mathbb{R}^m$, ω on $\mathbb{R}^m$, $\xi_1\ldots\xi_k\in T_x\mathbb{R}^n$

$$(f^*\omega)(\xi_1\ldots\xi_k) := \omega(f_*\xi_1\ldots f_*\xi_k)$$

on board

The wedge product

0-forms

Everything also works on manifolds.

$\Omega^k(\mathbb{R}^n)$ / $\Omega^k(M)$ (use C^∞ coeffs)

The "exterior derivative"/"differential" operator $d:\Omega^k(\mathbb{R}^n)\to\Omega^{k+1}(\mathbb{R}^n)$

"Right" but hard definition:

$$d\omega(\xi_1\ldots\xi_{k+1}) = \lim_{\epsilon\to 0}\frac{1}{\epsilon^{k+1}}\,\omega\left(\begin{array}{l}\text{the boundary of the parallelepiped}\\ \text{spanned by } \epsilon\xi_1\ldots\ \epsilon\xi_{k+1}\end{array}\right)$$

philosophically
"wrong" but easier definition:

done fine. Also done A below.

1. On functions, $df(\xi) = D_\xi f = Df_x\cdot v$ $\quad \xi = (x,v)$

Aside: $x_j:\mathbb{R}^n\to\mathbb{R}$, $dx_j = \phi_j$ hence from this point on, on $\mathbb{R}^n$, dx_I will always replace ϕ_I.

$$df = \sum_i \frac{\partial f}{\partial x_i}dx_i \qquad [\text{indeed, this works for } (x,e_i)]$$

2. Theorem $\exists!$ linear operator $d:\Omega^k(\mathbb{R}^n)\to\Omega^{k+1}(\mathbb{R}^n)$ s.t.

1. If f is a 0-form, df is as above.
2. $\omega\in\Omega^k$, $\eta\in\Omega^l$ $\Rightarrow$ $d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^k\omega\wedge d\eta$

3. $d^2=0$; more precisely, $d(d\omega)=0$.

PF 1-3 imply

$$d(\underbrace{\sum a_I\, dx_I}_{\omega}) = \sum_{j=1}^{n} \sum_I \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I \overset{\text{loosely}}{=} \sum_{j=1}^{n} dx_j \wedge \frac{\partial \omega}{\partial x_j}$$

So d is unique, if it exists. As for existence, take the above as the def of d, and verify 1-3.

The $\mathbb{R}^3$ example.

A: In $\mathbb{R}^3$:

$\Omega^0(\mathbb{R}^3) \longrightarrow$	$\Omega^1(\mathbb{R}^3)$	$\Omega^2(\mathbb{R}^3)$	$\Omega^3(\mathbb{R}^3)$
f	$a_1(x)\phi_1 + a_2(x)\phi_2 + a_3(x)\phi_3$	$b_1(x)\phi_2\wedge\phi_3 + b_2(x)\phi_3\wedge\phi_1 + b_3(x)\phi_1\wedge\phi_2$	$c\,\phi_1\wedge\phi_2\wedge\phi_3$
Functions	V.F.	V.F.	Functions.