Pensieve header: Testing and implementing lemmas 1,2,3 of the DoPeGDO handouts. Continues pensieve://2019-12/, continued pensieve://2020-03/.

$$[F:\mathcal{E}]_B := \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad \text{and} \quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

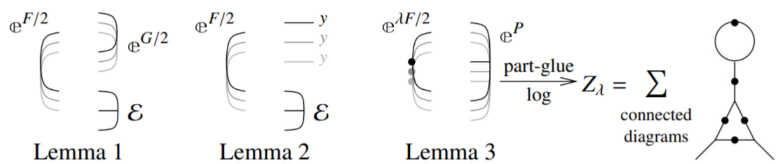
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\mathbb{e}^{\sum_{i\in B} y_iz_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Goals:

Implement the container $\mathbb{E}[\omega, Q, P] := \omega\, \mathbb{e}^{Q+P}$.

Implement the containers $|F, \mathcal{E}|_B := [F:\mathcal{E}]_B$ and $\langle F, \mathcal{E}\rangle_B := \langle F:\mathcal{E}\rangle_B$, their evaluator Ev_k as power series in $\hbar$ to degree k, and verify lemmas 1, 2, and 3. Inserting $\hbar$ in the appropriate places is user responsibility.

Implement DaGauss, DeLin, and a Lemma 3 evaluator, PEv.

Utilities

In[ ]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Generic Polynomials:

In[ ]:=
```
GenericPolynomial[d_Integer, vars_List, gc_] := Total[Map[
    gc_Sequence@@# Times @@ (vars^#) &,
    Join @@ (Permutations /@ IntegerPartitions[d + Length@vars, {Length@vars}]) - 1
   ]];
GenericPolynomial[specs_List, vars_List, gc_] :=
 Sum[GenericPolynomial[specs⟦1⟧, vars, gc], specs]
```

Preliminary Definitions

In[∘]:=
```
Unprotect[SeriesData];
Expand[sd_SeriesData] ^:= MapAt[Expand, sd, 3];
Protect[SeriesData];
```

In[∘]:=
```
CF[⟨F_, ℰ_⟩_B_] := ⟨Simplify@F, Simplify@ℰ⟩_B;
```

Variables and their duals:

In[∘]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};  (u_i_)* := (u*)_i;  (B_List)* := #* & /@ B;
```

Act and Contract:

In[∘]:=
```
Ev_k_ @ ¦F_, ℰ_¦_B_ := Expand[Total[
    CoefficientRules[Normal@Series[ⅇ^(B*.F.B*/2), {ℏ, 0, k}], B*] /.
     (ps_ → c_) ⧴ c D[ℰ, Sequence @@ Thread[{B, ps}]]
   ] + O[ℏ]^(k+1)];
Ev_k_ @ ⟨F_, ℰ_⟩_B_ := Ev_k @ ¦F, ℰ¦_B /. Alternatives @@ B → 0
```

In[∘]:=
$$\left\{\mathrm{Ev}_2 @ \left|\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e^{x y}\right|_{\{x,y\}}, \mathrm{Ev}_3 @ \left\langle \hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e^{3 x y}\right\rangle_{\{x,y\}}\right\}$$

Out[∘]=
$$\left\{e^{x y} + \left(e^{x y} + e^{x y}\, x\, y\right)\hbar + \left(e^{x y} + 2\, e^{x y}\, x\, y + \frac{1}{2}\, e^{x y}\, x^2\, y^2\right)\hbar^2 + O[\hbar]^3,\ 1 + 3\,\hbar + 9\,\hbar^2 + 27\,\hbar^3 + O[\hbar]^4\right\}$$

Implementing / Testing Lemma 1

Lemma 1. With convergences left to the reader,
$$\left\langle F : \mathcal{E}\,e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1} : \mathcal{E}\right\rangle_B$$

In[∘]:=
```
{p = 10, B = {x}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ {{f}}, G = {{g}}, ℰ = GenericPolynomial[{d, 0, 4}, B, c]}
lhs = Ev_p@⟨F, ℰ e^(B.G.B/2)⟩_B
rhs = Ev_p@⟨F.Inverse[𝓘 - G.F], Det[𝓘 - G.F]^(-1/2) ℰ⟩_B;
HL[lhs == rhs]
```

Out[∘]= $\{10, \{x\}, \{\{1\}\}, \{\{f\,\hbar\}\}, \{\{g\}\}, c_0 + x\,c_1 + x^2\,c_2 + x^3\,c_3 + x^4\,c_4\}$

Out[∘]=
$$c_0 + \left(\frac{1}{2} f\,g\,c_0 + f\,c_2\right)\hbar + \left(\frac{3}{8} f^2 g^2 c_0 + \frac{3}{2} f^2 g\,c_2 + 3 f^2 c_4\right)\hbar^2 +$$
$$\left(\frac{5}{16} f^3 g^3 c_0 + \frac{15}{8} f^3 g^2 c_2 + \frac{15}{2} f^3 g\,c_4\right)\hbar^3 + \left(\frac{35}{128} f^4 g^4 c_0 + \frac{35}{16} f^4 g^3 c_2 + \frac{105}{8} f^4 g^2 c_4\right)\hbar^4 +$$
$$\left(\frac{63}{256} f^5 g^5 c_0 + \frac{315}{128} f^5 g^4 c_2 + \frac{315}{16} f^5 g^3 c_4\right)\hbar^5 + \left(\frac{231 f^6 g^6 c_0}{1024} + \frac{693}{256} f^6 g^5 c_2 + \frac{3465}{128} f^6 g^4 c_4\right)\hbar^6 +$$
$$\left(\frac{429 f^7 g^7 c_0}{2048} + \frac{3003 f^7 g^6 c_2}{1024} + \frac{9009}{256} f^7 g^5 c_4\right)\hbar^7 + \left(\frac{6435 f^8 g^8 c_0}{32\,768} + \frac{6435 f^8 g^7 c_2}{2048} + \frac{45\,045 f^8 g^6 c_4}{1024}\right)\hbar^8 +$$
$$\left(\frac{12\,155 f^9 g^9 c_0}{65\,536} + \frac{109\,395 f^9 g^8 c_2}{32\,768} + \frac{109\,395 f^9 g^7 c_4}{2048}\right)\hbar^9 +$$
$$\left(\frac{46\,189 f^{10} g^{10} c_0}{262\,144} + \frac{230\,945 f^{10} g^9 c_2}{65\,536} + \frac{2\,078\,505 f^{10} g^8 c_4}{32\,768}\right)\hbar^{10} + O[\hbar]^{11}$$

Out[∘]= True

In[∘]:=
```
{p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ (f11 f12; f12 f22), G = (g11 g12; g12 g22), 𝓔 = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Ev_p@⟨F, 𝓔 e^(B.G.B/2)⟩_B]
rhs = Ev_p@⟨F.Inverse[𝓘 - G.F], Det[𝓘 - G.F]^(-1/2) 𝓔⟩_B;
HL[lhs == rhs]
```

Out[∘]= $\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$

Out[∘]= $$\Big\{0.0625,\ c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right)\hbar + \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right)\hbar^2 + O[\hbar]^3\Big\}$$

Out[∘]= True

In[∘]:=
```
{p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ (f11 f12; f12 f22), G = (g11 g12; g12 g22), 𝓔 = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Ev_p@⟨F, 𝓔 e^(B.G.B/2)⟩_B]
rhs = Expand[Series[Det[𝓘 - G.F]^(-1/2), {ℏ, 0, p}] Ev_p[⟨F.Inverse[𝓘 - G.F], 𝓔⟩_B]]
HL[lhs == rhs]
```

Out[∘]= $\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$

Out[∘]=
$$\Big\{0.0625,\; c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right)\hbar + \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right)\hbar^2 + O[\hbar]^3\Big\}$$

Out[∘]=
$$c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right)\hbar + \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right)\hbar^2 + O[\hbar]^3$$

Out[∘]= True

$$\left[F\colon \mathcal{E}\, e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right]_B = \det(1-GF)^{-1/2} e^{\frac{1}{2}\sum_{i,j\in B} (G(I-FG)^{-1})_{ij} z_i z_j} \cdot \left(\left[F(1-GF)^{-1}\colon \mathcal{E}\right]_B\right)_{z_B\to(I-}$$

In[∘]:=
```
{p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
 F = ℏ (f11 f12
        f12 f22), G = (g11 g12
                       g12 g22), 𝓔 = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Ev_p@⟨F, 𝓔 e^(B.G.B/2)⟩_B];
rhs = Series[Det[𝓘 - G.F]^(-1/2) e^(B.G.Inverse[𝓘-F.G].B/2), {ℏ, 0, p}]
   (Ev_p@⟨F.Inverse[𝓘 - G.F], 𝓔⟩_B /. Thread[B → Inverse[𝓘 - F.G].B]);
HL@FullSimplify[Normal[lhs - rhs] == 0]
```

Out[∘]= $\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$

Out[∘]= True

In[∘]:=
```
DeGauss@⟨F_, 𝓔_⟩_B_ := Module[{𝓘, Q, G, M, Δ},
  𝓘 = IdentityMatrix@Length@B;
  Q = Log[Normal[𝓔] /. ϵ → 0];
  G = Table[∂_{i,j} Q, {i, B}, {j, B}];
  M = Inverse[𝓘 - G.F];
  Δ = Simplify@Det@M;
  CF@⟨F.M, Δ^(1/2) 𝓔 e^(-B.G.B/2)⟩_B
 ]
```

In[∘]:=
```
{p = 2, B = {x, y}, F = ℏ (f11 f12; f12 f22),
  G = (g11 g12; g12 g22), ℰ = 1 + ∈ GenericPolynomial[{d, 0, 2}, B, c]};
lhs = ⟨F, ℰ 𝕖^(B.G.B/2)⟩_B
rhs = DeGauss@⟨F, ℰ 𝕖^(B.G.B/2)⟩_B
HL[Ev_p@lhs == Ev_p@rhs]
```

Out[∘]= $\left\langle \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \mathbb{e}^{\frac{1}{2}(x(x g_{11}+y g_{12})+y(x g_{12}+y g_{22}))}\left(1+\in\left(c_{0,0}+y\,c_{0,1}+y^2 c_{0,2}+x\,c_{1,0}+x\,y\,c_{1,1}+x^2 c_{2,0}\right)\right)\right\rangle_{\{x,y\}}$

Out[∘]= $$\Bigg\langle\Bigg\{\Bigg\{\frac{\hbar\left(\hbar f_{12}^2 g_{22}+f_{11}(1-\hbar f_{22} g_{22})\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)},$$
$$\frac{\hbar\left(f_{12}-\hbar f_{12}^2 g_{12}+\hbar f_{11} f_{22} g_{12}\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)}\Bigg\},$$
$$\Bigg\{\frac{\hbar\left(f_{12}-\hbar f_{12}^2 g_{12}+\hbar f_{11} f_{22} g_{12}\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)},$$
$$\frac{\hbar\left(\hbar f_{12}^2 g_{11}+f_{22}(1-\hbar f_{11} g_{11})\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)}\Bigg\}\Bigg\},$$
$$\sqrt{\frac{1}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)}}$$
$$\left(1+\in c_{0,0}+y\in c_{0,1}+y^2\in c_{0,2}+x\in c_{1,0}+x\,y\in c_{1,1}+x^2\in c_{2,0}\right)\Bigg\rangle_{\{x,y\}}$$

Out[∘]= True

Implementing / Testing Lemma 2

Lemma 2. $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B} y_i z_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+F y_B}\right\rangle_B.$

In[∘]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], 𝓘 = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
  F = ℏ Table[f_{{10,1}.Sort[{i,j}]}, {i, n}, {j, n}], ℰ = 1 + ∈ GenericPolynomial[{d, 0, 3}, B, c]}
Timing[lhs = Ev_p@⟨F, ℰ 𝕖^(Y.B)⟩_B]
Timing[rhs = Ev_p@⟨F, 𝕖^(Y.F.Y/2) ℰ /. Thread[B → B + F.Y]⟩_B]
HL@Expand[lhs == rhs]
```

Out[∘]= $\{2, 2, \{z_1, z_2\}, \{\{1, 0\}, \{0, 1\}\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$
$1+\in\left(c_{0,0}+z_2 c_{0,1}+z_2^2 c_{0,2}+z_2^3 c_{0,3}+z_1 c_{1,0}+z_1 z_2 c_{1,1}+z_1 z_2^2 c_{1,2}+z_1^2 c_{2,0}+z_1^2 z_2 c_{2,1}+z_1^3 c_{3,0}\right)\}$

Out[◦]= $\{0.015625,$

$$(1+\in c_{0,0}) + \left(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \in f_{22} y_2^2 c_{0,0} + \in f_{12} y_1 c_{0,1} + \in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0}\right) \hbar +$$

$$\left(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 + \right.$$

$$\frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} +$$

$$\frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1} +$$

$$\frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 y_2 c_{0,2} +$$

$$\frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 c_{0,3} + 3 \in f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \in f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0} +$$

$$\in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1} + 2 \in f_{12}^2 y_1 y_2 c_{1,1} +$$

$$\in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1} + 2 \in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3 \in f_{12} f_{22} y_2 c_{1,2} +$$

$$\frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 c_{2,1} +$$

$$\left. 2 \in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3 \in f_{11}^2 y_1 c_{3,0} + 3 \in f_{11} f_{12} y_2 c_{3,0}\right) \hbar^2 + O[\hbar]^3 \}$$

Out[∘]= $\{0.015625,$

$(1+\in c_{0,0})+\left(\frac{1}{2}f_{11}y_1^2+f_{12}y_1y_2+\frac{1}{2}f_{22}y_2^2+\frac{1}{2}\in f_{11}y_1^2c_{0,0}+\in f_{12}y_1y_2c_{0,0}+\frac{1}{2}\in f_{22}y_2^2c_{0,0}+\right.$
$\left.\in f_{12}y_1c_{0,1}+\in f_{22}y_2c_{0,1}+\in f_{22}c_{0,2}+\in f_{11}y_1c_{1,0}+\in f_{12}y_2c_{1,0}+\in f_{12}c_{1,1}+\in f_{11}c_{2,0}\right)\hbar+$
$\left(\frac{1}{8}f_{11}^2y_1^4+\frac{1}{2}f_{11}f_{12}y_1^3y_2+\frac{1}{2}f_{12}^2y_1^2y_2^2+\frac{1}{4}f_{11}f_{22}y_1^2y_2^2+\frac{1}{2}f_{12}f_{22}y_1y_2^3+\frac{1}{8}f_{22}^2y_2^4+\right.$
$\frac{1}{8}\in f_{11}^2y_1^4c_{0,0}+\frac{1}{2}\in f_{11}f_{12}y_1^3y_2c_{0,0}+\frac{1}{2}\in f_{12}^2y_1^2y_2^2c_{0,0}+\frac{1}{4}\in f_{11}f_{22}y_1^2y_2^2c_{0,0}+$
$\frac{1}{2}\in f_{12}f_{22}y_1y_2^3c_{0,0}+\frac{1}{8}\in f_{22}^2y_2^4c_{0,0}+\frac{1}{2}\in f_{11}f_{12}y_1^3c_{0,1}+\in f_{12}^2y_1^2y_2c_{0,1}+\frac{1}{2}\in f_{11}f_{22}y_1^2y_2c_{0,1}+$
$\frac{3}{2}\in f_{12}f_{22}y_1y_2^2c_{0,1}+\frac{1}{2}\in f_{22}^2y_2^3c_{0,1}+\in f_{12}^2y_1^2c_{0,2}+\frac{1}{2}\in f_{11}f_{22}y_1^2c_{0,2}+3\in f_{12}f_{22}y_1y_2c_{0,2}+$
$\frac{3}{2}\in f_{22}^2y_2^2c_{0,2}+3\in f_{12}f_{22}y_1c_{0,3}+3\in f_{22}^2y_2c_{0,3}+\frac{1}{2}\in f_{11}^2y_1^3c_{1,0}+\frac{3}{2}\in f_{11}f_{12}y_1^2y_2c_{1,0}+$
$\in f_{12}^2y_1y_2^2c_{1,0}+\frac{1}{2}\in f_{11}f_{22}y_1y_2^2c_{1,0}+\frac{1}{2}\in f_{12}f_{22}y_2^3c_{1,0}+\frac{3}{2}\in f_{11}f_{12}y_1^2c_{1,1}+2\in f_{12}^2y_1y_2c_{1,1}+$
$\in f_{11}f_{22}y_1y_2c_{1,1}+\frac{3}{2}\in f_{12}f_{22}y_2^2c_{1,1}+2\in f_{12}^2y_1c_{1,2}+\in f_{11}f_{22}y_1c_{1,2}+3\in f_{12}f_{22}y_2c_{1,2}+$
$\frac{3}{2}\in f_{11}^2y_1^2c_{2,0}+3\in f_{11}f_{12}y_1y_2c_{2,0}+\in f_{12}^2y_2^2c_{2,0}+\frac{1}{2}\in f_{11}f_{22}y_2^2c_{2,0}+3\in f_{11}f_{12}y_1c_{2,1}+$
$\left.2\in f_{12}^2y_2c_{2,1}+\in f_{11}f_{22}y_2c_{2,1}+3\in f_{11}^2y_1c_{3,0}+3\in f_{11}f_{12}y_2c_{3,0}\right)\hbar^2+O[\hbar]^3\}$

Out[∘]= True

$$\left[F:\ \mathcal{E}\,e^{\sum_{i\in B}y_iz_i}\right]_B=e^{\frac{1}{2}\sum_{i,j\in B}F_{ij}y_iy_j+\sum_{i\in B}y_iz_i}\left[F:\ \mathcal{E}\big|_{z_B\to z_B+Fy_B}\right]_B$$

In[∘]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], I = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
 F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = 1 + ∈ GenericPolynomial[{d, 0, 3}, B, c]}
Timing[lhs = Ev_p@|F, ℰ e^(Y.B)|_B] // Short
Timing[rhs = e^(Y.B) Ev_p@|F, e^(Y.F.Y/2) ℰ /. Thread[B → B + F.Y]|_B];
HL@Expand[lhs == rhs]
```

Out[∘]= $\{2, 2, \{z_1, z_2\}, \{\{1, 0\}, \{0, 1\}\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$
$1+\in(c_{0,0}+z_2c_{0,1}+z_2^2c_{0,2}+z_2^3c_{0,3}+z_1c_{1,0}+z_1z_2c_{1,1}+z_1z_2^2c_{1,2}+z_1^2c_{2,0}+z_1^2z_2c_{2,1}+z_1^3c_{3,0})\}$

Out[∘]//Short= {0.015625, <<1>> }

Out[∘]= True

In[•]:=
```
DeLin@⟨F_, ℰ_⟩_B_ := Module[{L, Y},
  L = PowerExpand@Log[Normal[ℰ] /. ∈ → 0];
  Y = Table[∂_i L, {i, B}];
  CF@⟨F, e^(Y.F.Y/2) (e^(-B.Y) ℰ /. Thread[B → B + F.Y])⟩_B
 ]
```

In[•]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], Y = Table[y_i, {i, n}],
 F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = 1 + ∈ GenericPolynomial[{d, 0, 3}, B, c]}
Timing[lhs = Ev_p@⟨F, ℰ e^(Y.B)⟩_B]
Timing[rhs = Ev_p@DeLin@⟨F, ℰ e^(Y.B)⟩_B]
HL@Simplify[lhs == rhs]
```

Out[•]=
$$\{2, 2, \{z_1, z_2\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, 1 + \in (c_{0,0} + z_2 c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0})\}$$

Out[•]=
$$\Big\{0.03125, (1 + \in c_{0,0}) + \Big(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \in f_{22} y_2^2 c_{0,0} + \in f_{12} y_1 c_{0,1} + \in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0}\Big) \hbar +$$
$$\Big(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 +$$
$$\frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} +$$
$$\frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1} +$$
$$\frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 y_2 c_{0,2} +$$
$$\frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 c_{0,3} + 3 \in f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \in f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0} +$$
$$\in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1} + 2 \in f_{12}^2 y_1 y_2 c_{1,1} +$$
$$\in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1} + 2 \in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3 \in f_{12} f_{22} y_2 c_{1,2} +$$
$$\frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 c_{2,1} +$$
$$2 \in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3 \in f_{11}^2 y_1 c_{3,0} + 3 \in f_{11} f_{12} y_2 c_{3,0}\Big) \hbar^2 + O[\hbar]^3\Big\}$$

Out[∘]= $\{0., (1+\epsilon\, c_{0,0}) + \left(\frac{1}{2} f_{11}\, y_1^2 + f_{12}\, y_1\, y_2 + \frac{1}{2} f_{22}\, y_2^2 + \frac{1}{2} \epsilon\, f_{11}\, y_1^2\, c_{0,0} + \epsilon\, f_{12}\, y_1\, y_2\, c_{0,0} + \frac{1}{2} \epsilon\, f_{22}\, y_2^2\, c_{0,0} + \epsilon\, f_{12}\, y_1\, c_{0,1} + \epsilon\, f_{22}\, y_2\, c_{0,1} + \epsilon\, f_{22}\, c_{0,2} + \epsilon\, f_{11}\, y_1\, c_{1,0} + \epsilon\, f_{12}\, y_2\, c_{1,0} + \epsilon\, f_{12}\, c_{1,1} + \epsilon\, f_{11}\, c_{2,0}\right) \hbar +$
$\left(\frac{1}{8} f_{11}^2\, y_1^4 + \frac{1}{2} f_{11}\, f_{12}\, y_1^3\, y_2 + \frac{1}{2} f_{12}^2\, y_1^2\, y_2^2 + \frac{1}{4} f_{11}\, f_{22}\, y_1^2\, y_2^2 + \frac{1}{2} f_{12}\, f_{22}\, y_1\, y_2^3 + \frac{1}{8} f_{22}^2\, y_2^4 + \right.$
$\frac{1}{8} \epsilon\, f_{11}^2\, y_1^4\, c_{0,0} + \frac{1}{2} \epsilon\, f_{11}\, f_{12}\, y_1^3\, y_2\, c_{0,0} + \frac{1}{2} \epsilon\, f_{12}^2\, y_1^2\, y_2^2\, c_{0,0} + \frac{1}{4} \epsilon\, f_{11}\, f_{22}\, y_1^2\, y_2^2\, c_{0,0} +$
$\frac{1}{2} \epsilon\, f_{12}\, f_{22}\, y_1\, y_2^3\, c_{0,0} + \frac{1}{8} \epsilon\, f_{22}^2\, y_2^4\, c_{0,0} + \frac{1}{2} \epsilon\, f_{11}\, f_{12}\, y_1^3\, c_{0,1} + \epsilon\, f_{12}^2\, y_1^2\, y_2\, c_{0,1} + \frac{1}{2} \epsilon\, f_{11}\, f_{22}\, y_1^2\, y_2\, c_{0,1} +$
$\frac{3}{2} \epsilon\, f_{12}\, f_{22}\, y_1\, y_2^2\, c_{0,1} + \frac{1}{2} \epsilon\, f_{22}^2\, y_2^3\, c_{0,1} + \epsilon\, f_{12}^2\, y_1^2\, c_{0,2} + \frac{1}{2} \epsilon\, f_{11}\, f_{22}\, y_1^2\, c_{0,2} + 3\, \epsilon\, f_{12}\, f_{22}\, y_1\, y_2\, c_{0,2} +$
$\frac{3}{2} \epsilon\, f_{22}^2\, y_2^2\, c_{0,2} + 3\, \epsilon\, f_{12}\, f_{22}\, y_1\, c_{0,3} + 3\, \epsilon\, f_{22}^2\, y_2\, c_{0,3} + \frac{1}{2} \epsilon\, f_{11}^2\, y_1^3\, c_{1,0} + \frac{3}{2} \epsilon\, f_{11}\, f_{12}\, y_1^2\, y_2\, c_{1,0} +$
$\epsilon\, f_{12}^2\, y_1\, y_2^2\, c_{1,0} + \frac{1}{2} \epsilon\, f_{11}\, f_{22}\, y_1\, y_2^2\, c_{1,0} + \frac{1}{2} \epsilon\, f_{12}\, f_{22}\, y_2^3\, c_{1,0} + \frac{3}{2} \epsilon\, f_{11}\, f_{12}\, y_1^2\, c_{1,1} + 2\, \epsilon\, f_{12}^2\, y_1\, y_2\, c_{1,1} +$
$\epsilon\, f_{11}\, f_{22}\, y_1\, y_2\, c_{1,1} + \frac{3}{2} \epsilon\, f_{12}\, f_{22}\, y_2^2\, c_{1,1} + 2\, \epsilon\, f_{12}^2\, y_1\, c_{1,2} + \epsilon\, f_{11}\, f_{22}\, y_1\, c_{1,2} + 3\, \epsilon\, f_{12}\, f_{22}\, y_2\, c_{1,2} +$
$\frac{3}{2} \epsilon\, f_{11}^2\, y_1^2\, c_{2,0} + 3\, \epsilon\, f_{11}\, f_{12}\, y_1\, y_2\, c_{2,0} + \epsilon\, f_{12}^2\, y_2^2\, c_{2,0} + \frac{1}{2} \epsilon\, f_{11}\, f_{22}\, y_2^2\, c_{2,0} + 3\, \epsilon\, f_{11}\, f_{12}\, y_1\, c_{2,1} +$
$\left. 2\, \epsilon\, f_{12}^2\, y_2\, c_{2,1} + \epsilon\, f_{11}\, f_{22}\, y_2\, c_{2,1} + 3\, \epsilon\, f_{11}^2\, y_1\, c_{3,0} + 3\, \epsilon\, f_{11}\, f_{12}\, y_2\, c_{3,0}\right) \hbar^2 + O[\hbar]^3\}$

Out[∘]= True

Testing Lemma 3

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

In[∘]:=
```
{n = 2, p = 2, B = Table[b_i, {i, n}],
 F = ℏ Table[f_{{10,1}.Sort[{i,j}]}, {i, n}, {j, n}], P = GenericPolynomial[{d, 0, 2}, B, c]}
```

Out[∘]= $\{2, 2, \{b_1, b_2\}, \{\{\hbar\, f_{11}, \hbar\, f_{12}\}, \{\hbar\, f_{12}, \hbar\, f_{22}\}\}, c_{0,0} + b_2\, c_{0,1} + b_2^2\, c_{0,2} + b_1\, c_{1,0} + b_1\, b_2\, c_{1,1} + b_1^2\, c_{2,0}\}$

In[∘]:= **Z = PowerExpand@Expand@Log$\left[\mathrm{Ev}_p @ \left|\lambda\, F,\ e^P\right|_B\right]$**

Out[∘]=
$$
\begin{aligned}
&\left(c_{0,0}+b_2\, c_{0,1}+b_2^2\, c_{0,2}+b_1\, c_{1,0}+b_1\, b_2\, c_{1,1}+b_1^2\, c_{2,0}\right)+\\
&\Big(\frac{1}{2}\lambda\, f_{22}\, c_{0,1}^2+\lambda\, f_{22}\, c_{0,2}+2\lambda\, b_2\, f_{22}\, c_{0,1}\, c_{0,2}+2\lambda\, b_2^2\, f_{22}\, c_{0,2}^2+\lambda\, f_{12}\, c_{0,1}\, c_{1,0}+2\lambda\, b_2\, f_{12}\, c_{0,2}\, c_{1,0}+\\
&\quad\frac{1}{2}\lambda\, f_{11}\, c_{1,0}^2+\lambda\, f_{12}\, c_{1,1}+\lambda\, b_2\, f_{12}\, c_{0,1}\, c_{1,1}+\lambda\, b_1\, f_{22}\, c_{0,1}\, c_{1,1}+2\lambda\, b_2^2\, f_{12}\, c_{0,2}\, c_{1,1}+\\
&\quad 2\lambda\, b_1\, b_2\, f_{22}\, c_{0,2}\, c_{1,1}+\lambda\, b_2\, f_{11}\, c_{1,0}\, c_{1,1}+\lambda\, b_1\, f_{12}\, c_{1,0}\, c_{1,1}+\frac{1}{2}\lambda\, b_2^2\, f_{11}\, c_{1,1}^2+\\
&\quad\lambda\, b_1\, b_2\, f_{12}\, c_{1,1}^2+\frac{1}{2}\lambda\, b_1^2\, f_{22}\, c_{1,1}^2+\lambda\, f_{11}\, c_{2,0}+2\lambda\, b_1\, f_{12}\, c_{0,1}\, c_{2,0}+4\lambda\, b_1\, b_2\, f_{12}\, c_{0,2}\, c_{2,0}+\\
&\quad 2\lambda\, b_1\, f_{11}\, c_{1,0}\, c_{2,0}+2\lambda\, b_1\, b_2\, f_{11}\, c_{1,1}\, c_{2,0}+2\lambda\, b_1^2\, f_{12}\, c_{1,1}\, c_{2,0}+2\lambda\, b_1^2\, f_{11}\, c_{2,0}^2\Big)\,\hbar+\\
&\Big(\lambda^2 f_{22}^2\, c_{0,1}^2\, c_{0,2}+\lambda^2 f_{22}^2\, c_{0,2}^2+4\lambda^2 b_2\, f_{22}^2\, c_{0,1}\, c_{0,2}^2+4\lambda^2 b_2^2\, f_{22}^2\, c_{0,2}^3+2\lambda^2 f_{12}\, f_{22}\, c_{0,1}\, c_{0,2}\, c_{1,0}+\\
&\quad 4\lambda^2 b_2\, f_{12}\, f_{22}\, c_{0,2}^2\, c_{1,0}+\lambda^2 f_{12}^2\, c_{0,2}\, c_{1,0}^2+\lambda^2 f_{12}\, f_{22}\, c_{0,1}^2\, c_{1,1}+2\lambda^2 f_{12}\, f_{22}\, c_{0,2}\, c_{1,1}+\\
&\quad 6\lambda^2 b_2\, f_{12}\, f_{22}\, c_{0,1}\, c_{0,2}\, c_{1,1}+2\lambda^2 b_1\, f_{22}^2\, c_{0,1}\, c_{0,2}\, c_{1,1}+8\lambda^2 b_2^2\, f_{12}\, f_{22}\, c_{0,2}^2\, c_{1,1}+4\lambda^2 b_1\, b_2\, f_{22}^2\, c_{0,2}^2\, c_{1,1}+\\
&\quad\lambda^2 f_{12}^2\, c_{0,1}\, c_{1,0}\, c_{1,1}+\lambda^2 f_{11}\, f_{22}\, c_{0,1}\, c_{1,0}\, c_{1,1}+4\lambda^2 b_2\, f_{12}^2\, c_{0,2}\, c_{1,0}\, c_{1,1}+2\lambda^2 b_2\, f_{11}\, f_{22}\, c_{0,2}\, c_{1,0}\, c_{1,1}+\\
&\quad 2\lambda^2 b_1\, f_{12}\, f_{22}\, c_{0,2}\, c_{1,0}\, c_{1,1}+\lambda^2 f_{11}\, f_{12}\, c_{1,0}^2\, c_{1,1}+\frac{1}{2}\lambda^2 f_{12}^2\, c_{1,1}^2+\frac{1}{2}\lambda^2 f_{11}\, f_{22}\, c_{1,1}^2+\lambda^2 b_2\, f_{12}^2\, c_{0,1}\, c_{1,1}^2+\\
&\quad\lambda^2 b_2\, f_{11}\, f_{22}\, c_{0,1}\, c_{1,1}^2+2\lambda^2 b_1\, f_{12}\, f_{22}\, c_{0,1}\, c_{1,1}^2+3\lambda^2 b_2^2\, f_{12}^2\, c_{0,2}\, c_{1,1}^2+2\lambda^2 b_2^2\, f_{11}\, f_{22}\, c_{0,2}\, c_{1,1}^2+\\
&\quad 6\lambda^2 b_1\, b_2\, f_{12}\, f_{22}\, c_{0,2}\, c_{1,1}^2+\lambda^2 b_1^2\, f_{22}^2\, c_{0,2}\, c_{1,1}^2+2\lambda^2 b_2\, f_{11}\, f_{12}\, c_{1,0}\, c_{1,1}^2+\lambda^2 b_1\, f_{12}^2\, c_{1,0}\, c_{1,1}^2+\\
&\quad\lambda^2 b_1\, f_{11}\, f_{22}\, c_{1,0}\, c_{1,1}^2+\lambda^2 b_2^2\, f_{11}\, f_{12}\, c_{1,1}^3+\lambda^2 b_1\, b_2\, f_{12}^2\, c_{1,1}^3+\lambda^2 b_1\, b_2\, f_{11}\, f_{22}\, c_{1,1}^3+\lambda^2 b_1^2\, f_{12}\, f_{22}\, c_{1,1}^3+\\
&\quad\lambda^2 f_{12}^2\, c_{0,1}^2\, c_{2,0}+2\lambda^2 f_{12}^2\, c_{0,2}\, c_{2,0}+4\lambda^2 b_2\, f_{12}^2\, c_{0,1}\, c_{0,2}\, c_{2,0}+4\lambda^2 b_1\, f_{12}\, f_{22}\, c_{0,1}\, c_{0,2}\, c_{2,0}+\\
&\quad 4\lambda^2 b_2^2\, f_{12}^2\, c_{0,2}^2\, c_{2,0}+8\lambda^2 b_1\, b_2\, f_{12}\, f_{22}\, c_{0,2}^2\, c_{2,0}+2\lambda^2 f_{11}\, f_{12}\, c_{0,1}\, c_{1,0}\, c_{2,0}+\\
&\quad 4\lambda^2 b_2\, f_{11}\, f_{12}\, c_{0,2}\, c_{1,0}\, c_{2,0}+4\lambda^2 b_1\, f_{12}^2\, c_{0,2}\, c_{1,0}\, c_{2,0}+\lambda^2 f_{11}^2\, c_{1,0}^2\, c_{2,0}+2\lambda^2 f_{11}\, f_{12}\, c_{1,1}\, c_{2,0}+\\
&\quad 2\lambda^2 b_2\, f_{11}\, f_{12}\, c_{0,1}\, c_{1,1}\, c_{2,0}+4\lambda^2 b_1\, f_{12}^2\, c_{0,1}\, c_{1,1}\, c_{2,0}+2\lambda^2 b_1\, f_{11}\, f_{22}\, c_{0,1}\, c_{1,1}\, c_{2,0}+\\
&\quad 4\lambda^2 b_2^2\, f_{11}\, f_{12}\, c_{0,2}\, c_{1,1}\, c_{2,0}+12\lambda^2 b_1\, b_2\, f_{12}^2\, c_{0,2}\, c_{1,1}\, c_{2,0}+4\lambda^2 b_1\, b_2\, f_{11}\, f_{22}\, c_{0,2}\, c_{1,1}\, c_{2,0}+\\
&\quad 4\lambda^2 b_1^2\, f_{12}\, f_{22}\, c_{0,2}\, c_{1,1}\, c_{2,0}+2\lambda^2 b_2\, f_{11}^2\, c_{1,0}\, c_{1,1}\, c_{2,0}+6\lambda^2 b_1\, f_{11}\, f_{12}\, c_{1,0}\, c_{1,1}\, c_{2,0}+\\
&\quad\lambda^2 b_2^2\, f_{11}^2\, c_{1,1}^2\, c_{2,0}+6\lambda^2 b_1\, b_2\, f_{11}\, f_{12}\, c_{1,1}^2\, c_{2,0}+3\lambda^2 b_1^2\, f_{12}^2\, c_{1,1}^2\, c_{2,0}+2\lambda^2 b_1^2\, f_{11}\, f_{22}\, c_{1,1}^2\, c_{2,0}+\\
&\quad\lambda^2 f_{11}^2\, c_{2,0}^2+4\lambda^2 b_1\, f_{11}\, f_{12}\, c_{0,1}\, c_{2,0}^2+8\lambda^2 b_1\, b_2\, f_{11}\, f_{12}\, c_{0,2}\, c_{2,0}^2+4\lambda^2 b_1^2\, f_{12}^2\, c_{0,2}\, c_{2,0}^2+\\
&\quad 4\lambda^2 b_1\, f_{11}^2\, c_{1,0}\, c_{2,0}^2+4\lambda^2 b_1\, b_2\, f_{11}^2\, c_{1,1}\, c_{2,0}^2+8\lambda^2 b_1^2\, f_{11}\, f_{12}\, c_{1,1}\, c_{2,0}^2+4\lambda^2 b_1^2\, f_{11}^2\, c_{2,0}^3\Big)\,\hbar^2+O[\hbar]^3
\end{aligned}
$$

In[∘]:= **Z /. λ → 0**

Out[∘]= $\left(c_{0,0}+b_2\, c_{0,1}+b_2^2\, c_{0,2}+b_1\, c_{1,0}+b_1\, b_2\, c_{1,1}+b_1^2\, c_{2,0}\right)+O[\hbar]^3$

In[∘]:= **(Z /. λ → 0) - P**

Out[∘]= $O[\hbar]^3$

In[∘]:= **lhs = ∂$_\lambda$Z**

Out[∘]=

$$\begin{aligned}
&\Big(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + 2 b_2^2 f_{22} c_{0,2}^2 + f_{12} c_{0,1} c_{1,0} + 2 b_2 f_{12} c_{0,2} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + \\
&\quad f_{12} c_{1,1} + b_2 f_{12} c_{0,1} c_{1,1} + b_1 f_{22} c_{0,1} c_{1,1} + 2 b_2^2 f_{12} c_{0,2} c_{1,1} + 2 b_1 b_2 f_{22} c_{0,2} c_{1,1} + b_2 f_{11} c_{1,0} c_{1,1} + \\
&\quad b_1 f_{12} c_{1,0} c_{1,1} + \frac{1}{2} b_2^2 f_{11} c_{1,1}^2 + b_1 b_2 f_{12} c_{1,1}^2 + \frac{1}{2} b_1^2 f_{22} c_{1,1}^2 + f_{11} c_{2,0} + 2 b_1 f_{12} c_{0,1} c_{2,0} + \\
&\quad 4 b_1 b_2 f_{12} c_{0,2} c_{2,0} + 2 b_1 f_{11} c_{1,0} c_{2,0} + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2\Big) \hbar + \\
&\Big(2 \lambda f_{22}^2 c_{0,1}^2 c_{0,2} + 2 \lambda f_{22}^2 c_{0,2}^2 + 8 \lambda b_2 f_{22}^2 c_{0,1} c_{0,2}^2 + 8 \lambda b_2^2 f_{22}^2 c_{0,2}^3 + 4 \lambda f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + \\
&\quad 8 \lambda b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + 2 \lambda f_{12}^2 c_{0,2} c_{1,0}^2 + 2 \lambda f_{12} f_{22} c_{0,1}^2 c_{1,1} + 4 \lambda f_{12} f_{22} c_{0,2} c_{1,1} + \\
&\quad 12 \lambda b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 4 \lambda b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 16 \lambda b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} + \\
&\quad 8 \lambda b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} + 2 \lambda f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 8 \lambda b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} + \\
&\quad 4 \lambda b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} + 4 \lambda b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{12} c_{1,0}^2 c_{1,1} + \lambda f_{12}^2 c_{1,1}^2 + \\
&\quad \lambda f_{11} f_{22} c_{1,1}^2 + 2 \lambda b_2 f_{12}^2 c_{0,1} c_{1,1}^2 + 2 \lambda b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 4 \lambda b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 + \\
&\quad 6 \lambda b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 4 \lambda b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 + 12 \lambda b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + 2 \lambda b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 + \\
&\quad 4 \lambda b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{12}^2 c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + 2 \lambda b_2^2 f_{11} f_{12} c_{1,1}^3 + \\
&\quad 2 \lambda b_1 b_2 f_{12}^2 c_{1,1}^3 + 2 \lambda b_1 b_2 f_{11} f_{22} c_{1,1}^3 + 2 \lambda b_1^2 f_{12} f_{22} c_{1,1}^3 + 2 \lambda f_{12}^2 c_{0,1}^2 c_{2,0} + 4 \lambda f_{12}^2 c_{0,2} c_{2,0} + \\
&\quad 8 \lambda b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0} + 16 \lambda b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} + \\
&\quad 4 \lambda f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + 8 \lambda b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + 2 \lambda f_{11}^2 c_{1,0}^2 c_{2,0} + \\
&\quad 4 \lambda f_{11} f_{12} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 4 \lambda b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} + \\
&\quad 8 \lambda b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 24 \lambda b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 8 \lambda b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} + \\
&\quad 8 \lambda b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 12 \lambda b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} + \\
&\quad 2 \lambda b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 12 \lambda b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 6 \lambda b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 4 \lambda b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} + \\
&\quad 2 \lambda f_{11}^2 c_{2,0}^2 + 8 \lambda b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 16 \lambda b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 8 \lambda b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 + \\
&\quad 8 \lambda b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 8 \lambda b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 16 \lambda b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 8 \lambda b_1^2 f_{11}^2 c_{2,0}^3\Big) \hbar^2 + O[\hbar]^3
\end{aligned}$$

In[∘]:= **Short[rhs = Expand@Sum[(∂$_{b1,b2}$(B.F.B)) (∂$_{b1,b2}$Z + (∂$_{b1}$Z) (∂$_{b2}$Z)) / 4, {b1, B}, {b2, B}]]**

Out[∘]//Short=

$$\Big(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + \ll 18 \gg + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2\Big) \hbar + (\ll 1 \gg) \ll 1 \gg + \ll 1 \gg + O[\hbar]^4$$

In[∘]:= **HL[Normal[lhs − rhs] == 0]**

Out[∘]= True

In[∘]:= **Z /. λ → 1 /. Alternatives @@ B → 0**

Out[∘]=

$$\begin{aligned}
&c_{0,0} + \Big(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + f_{12} c_{0,1} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + f_{11} c_{2,0}\Big) \hbar + \\
&\Big(f_{22}^2 c_{0,1}^2 c_{0,2} + f_{22}^2 c_{0,2}^2 + 2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + f_{12}^2 c_{0,2} c_{1,0}^2 + f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 f_{12} f_{22} c_{0,2} c_{1,1} + \\
&\quad f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2} f_{12}^2 c_{1,1}^2 + \frac{1}{2} f_{11} f_{22} c_{1,1}^2 + f_{12}^2 c_{0,1}^2 c_{2,0} + \\
&\quad 2 f_{12}^2 c_{0,2} c_{2,0} + 2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + f_{11}^2 c_{1,0}^2 c_{2,0} + 2 f_{11} f_{12} c_{1,1} c_{2,0} + f_{11}^2 c_{2,0}^2\Big) \hbar^2 + O[\hbar]^3
\end{aligned}$$