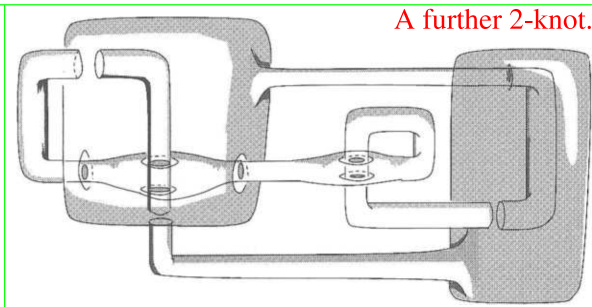
Knots in Three and Four Dimensions

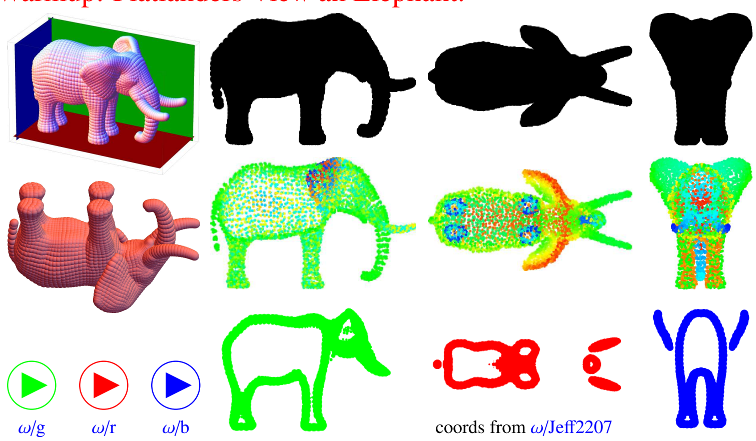
Abstract. Much as we can understand 3-dimensional objects by staring at their pictures and x-ray images and slices in 2-dimensions, so can we understand 4-dimensional objects by staring at their pictures and x-ray images and slices in 3-dimensions, capitalizing on the fact that we understand 3-dimensions pretty well. So we will spend some time staring at and understanding various 2-dimensional views of a 3-dimensional elephant, and then even more simply, various 2-dimensional views of some 3-dimensional knots. This achieved, we'll take the leap and visualize some 4-dimensional knots by their various traces in 3-dimensional space, and if we'll still have time, we'll prove that these knots are really knotted.

A further 2-knot.



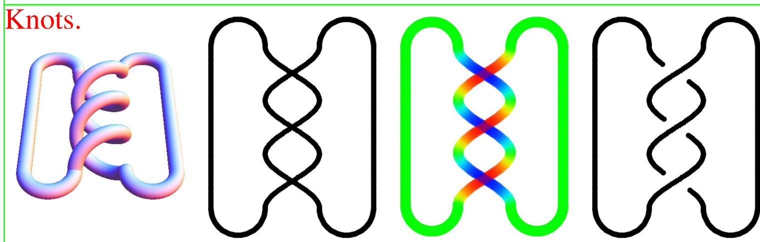
ω/CS

Warmup: Flatlanders View an Elephant.

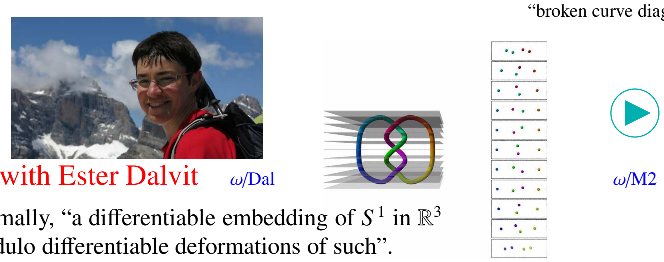


ω/g ω/r ω/b coords from ω/Jeff2207

Knots.



"broken curve diagram"



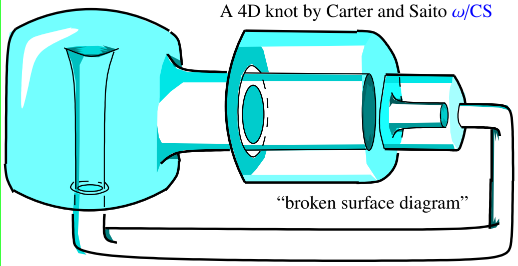
with Ester Dalvit ω/Dal ω/M2

Formally, "a differentiable embedding of S^1 in $\mathbb{R}^3$ modulo differentiable deformations of such".

2-Knots / 4D Knots.

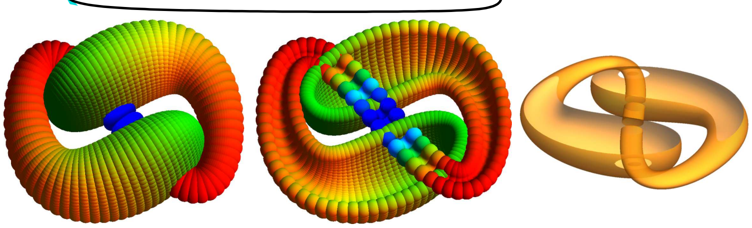
Formally, "a differentiable embedding of S^2 in $\mathbb{R}^4$ modulo differentiable deformations of such".

A 4D knot by Carter and Saito ω/CS

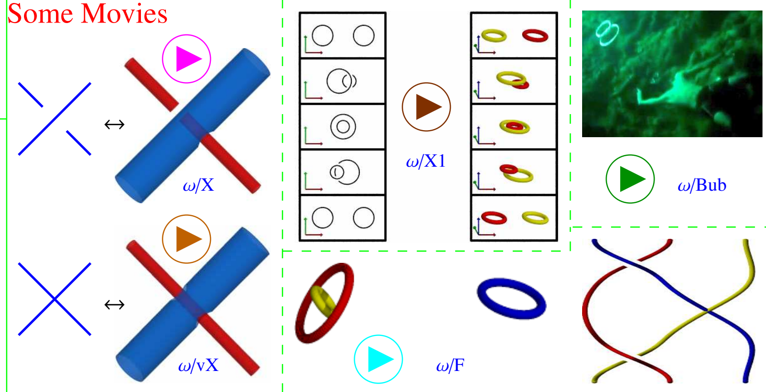


"broken surface diagram"

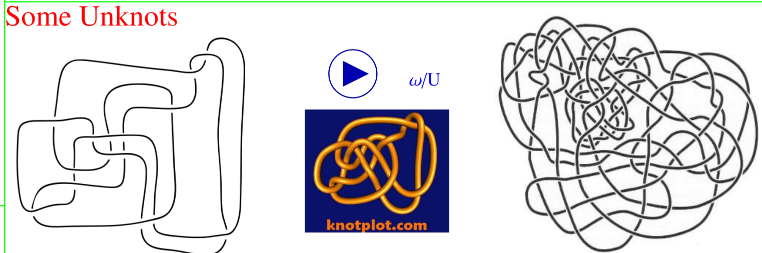
Carter, Banach, Saito



Some Movies



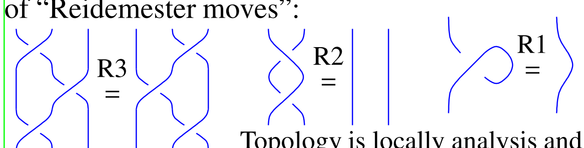
Some Unknots



Thistlethwaite's unknot Scharein's relaxation Haken's unknot

Reidemeister's Theorem.

(a) Every knot has a "broken curve diagram", made only of curves and "crossings" like ⨉. (b) Two knot diagrams represent the same 3D knot iff they differ by a sequence of "Reidemester moves":

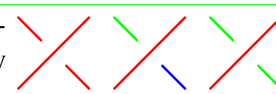


Kurt Reidemeister

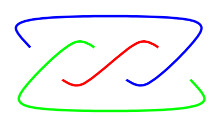
Topology is locally analysis and globally algebra

3-Colourings.

Colour the arcs of a broken arc diagram in RGB so that every crossing is either mono-chromatic or tri-chromatic. Let $\lambda(K)$ be the number of such 3-colourings that K has.



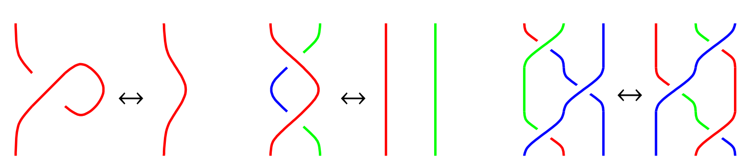
good good bad

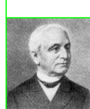


Example. $\lambda(\bigcirc) = 3$ while $\lambda(\ ) = 9$; so $\bigcirc \neq$.

Riddle. Is $\lambda(K)$ always a power of 3?

Proof sketch. It is enough to show that for each Reidemeister move, there is an end-colours-preserving bijection between the colourings of the two sides. E.g.:





"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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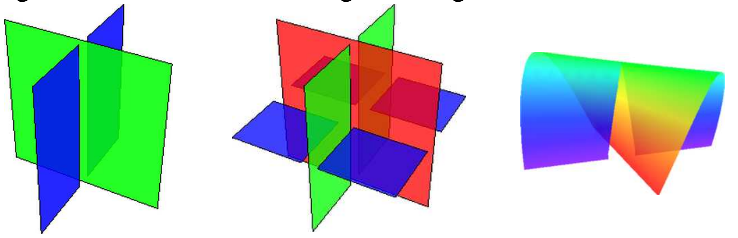
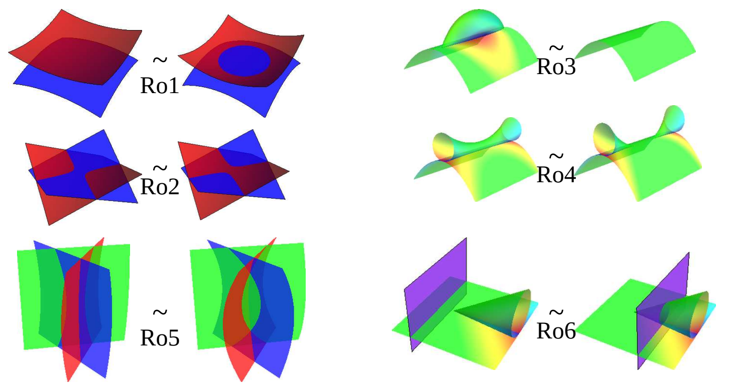
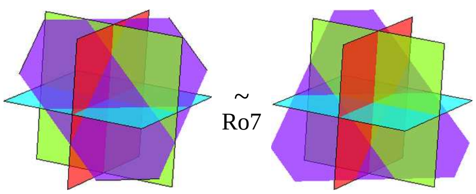
Knots in Three and Four Dimensions, 2

Theorem. Every 2-knot can be represented by a "broken surface diagram" made of the following basic ingredients,



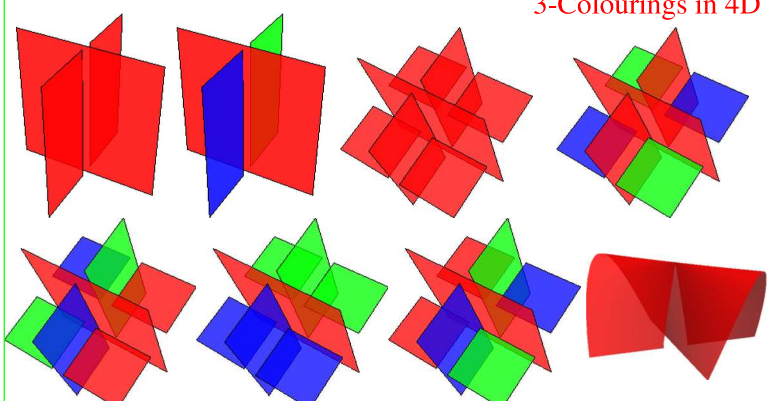
...and any two representations of the same knot differ by a sequence of the following "Roseman moves":

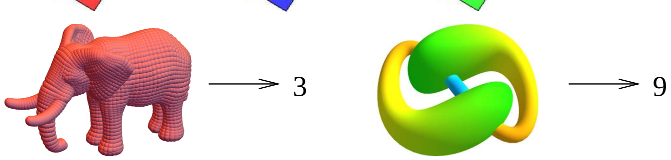




D. Roseman

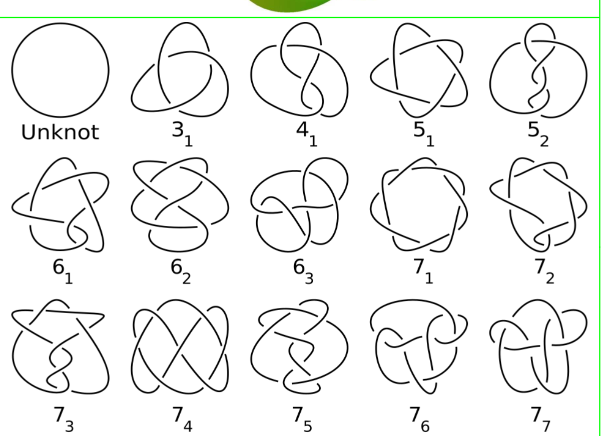
3-Colourings in 4D





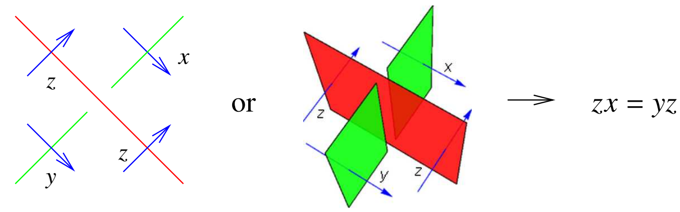
A Knot Table

There are many more!



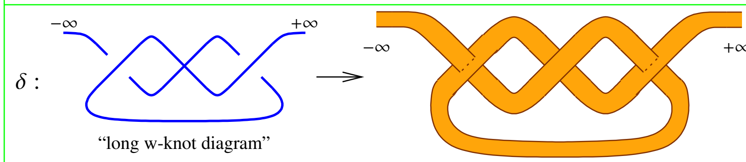
ω/KT

A Stronger Invariant. There is an assigment of groups to knots / 2-knots as follows. Put an arrow "under" every un-broken curve / surface in a broken curve / surface diagram and label it with the name of a group generator. Then mod out by relations as below.



Facts. The resulting "Fundamental group" $\pi_1(K)$ of a knot / 2-knot K is a very strong but not very computable invariant of K. Though it has computable projections; e.g., for any finite G, count the homomorphisms from $\pi_1(K)$ to G.

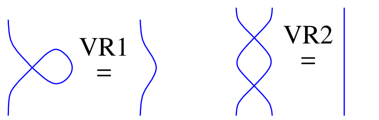
Exercise. Show that $|\mathrm{Hom}(\pi_1(K) \to S_3)| = \lambda(K) + 3$.

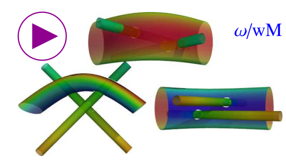


$\longrightarrow$ "simple long knotted 2D tube in 4D"

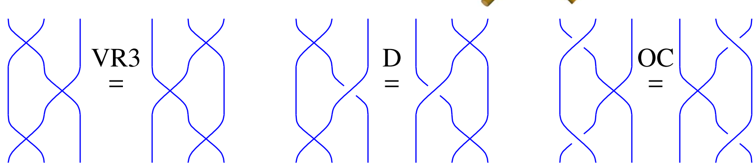
Satoh's Conjecture. (Satoh, *Virtual Knot Presentations of Ribbon Torus-Knots,* J. Knot Theory and its Ramifications **9** (2000) 531–542). Two long w-knot diagrams represent via the map δ the same simple long 2D knotted tube in 4D iff they differ by a sequence of R-moves as above and the "w-moves" VR1–VR3, D and OC listed below:

Shin Satoh





ω/wM



Some knot theory books.

- Colin C. Adams, *The Knot Book, an Elementary Introduction to the Mathematical Theory of Knots,* American Mathematical Society, 2004.
- Meike Akveld and Andrew Jobbings, *Knots Unravelled, from Strings to Mathematics,* Arbelos 2011.
- J. Scott Carter and Masahico Saito, *Knotted Surfaces and Their Diagrams,* American Mathematical Society, 1997.
- Peter Cromwell, *Knots and Links,* Cambridge University Press, 2004.
- W.B. Raymond Lickorish, *An Introduction to Knot Theory,* Springer 1997.