Pensieve Header: bb-calculus, revision 1.

```
bbSimplify = Simplify;
SetAttributes[bbCollect, Listable];
bbCollect[B[ω_, σ_, μ_]] := B[
   bbSimplify[ω], σ,
   Collect[μ, _h, Collect[#, _t, bbSimplify] &]
  ];
hL[b_] := Union[Cases[b, h[s_] :> s, Infinity]];
tL[b_] := Union[Cases[b, t[s_] | T_s_ :> s, Infinity]];
dL[b_] := Union[hL[b], tL[b]];
σ_ ⊢ h_ := (∂_h σ /. 0 → 1);
bbForm[B[ω_, σ_, μ_]] := Module[
   {tails, heads, mat},
   tails = tL[B[ω, σ, μ]]; heads = hL[B[ω, σ, μ]];
   mat = Outer[bbSimplify[∂_{h[#1],t[#2]} μ] &, heads, tails];
   PrependTo[mat, t /@ tails];
   mat = Join[
     {Prepend[h /@ heads, ω]},
     Transpose[mat],
     {Prepend[(σ ⊢ h[#]) & /@ heads, "1+Σ/ω"]}
    ];
   MatrixForm[mat]
  ];
bbForm[else_] := else /. b_B :> bbForm[b];
Format[b_B, StandardForm] := bbForm[b];
B /: B[ω1_, σ1_, μ1_] == B[ω2_, σ2_, μ2_] := Module[
   {heads, tails},
   tails = tL[{B[ω1, σ1, μ1], B[ω2, σ2, μ2]}];
   heads = hL[{B[ω1, σ1, μ1], B[ω2, σ2, μ2]}];
   (ω1 == ω2) && (σ1 == σ2) && (
     And @@ Flatten[Outer[
         (Coefficient[μ1, t[#1] h[#2]] == Coefficient[μ2, t[#1] h[#2]]) &,
         tails, heads
        ]]
    )
  ];

tm[x_, y_, z_][b_] := b /. {t[x] → t[z], t[y] → t[z], T_x → T_z, T_y → T_z};
hm[x_, y_, z_][B[ω_, σ_, μ_]] := B[ω,
    h[z] (σ ⊢ h[x]) (σ ⊢ h[y]) + (σ /. h[x] | h[y] → 0),
    h[z] (D[μ, h[x]] + (σ ⊢ h[x]) ∂_{h[y]} μ) + (μ /. h[x] | h[y] → 0)
   ] // bbCollect;
swapth[y_, x_][B[ω_, σ_, μ_]] := Module[
   {α, β, γ, δ},
   (α β; γ δ) = (Coefficient[μ, t[y] h[x]]   D[μ, t[y]] /. h[x] → 0;
                 D[μ, h[x]] /. t[y] → 0     μ /. h[x] | t[y] → 0);
```

```
  B[ω + α, σ, {(σ ⊢ h[x]) t[y], 1}.(α  β; γ  ((ω + α) δ - γ * β) / ω).{h[x], 1}] // bbCollect
 ];
dm[x_, y_, z_][b_] := b // swapth[x, y] // hm[x, y, z] // tm[x, y, z];
B /: B[ω1_, σ1_, μ1_] B[ω2_, σ2_, μ2_] := B[ω1 * ω2, σ1 + σ2, ω2 μ1 + ω1 μ2];

Unprotect[NonCommutativeMultiply];
b1_B ** b2_B := Module[
  {ρ, σ, labels},
  ρ = b1 * (b2 /. {h[s_] :> h[σ[s]], t[s_] :> t[σ[s]], T_s_ :> T_σ[s]});
  labels = dL[{b1, b2}];
  Do[ρ = ρ // dm[s, σ[s], s], {s, labels}];
  ρ
 ];

Rp[x_, y_] := B[1, h[x] + T_x h[y], (T_x - 1) * t[x] h[y]];
Rm[x_, y_] := B[1, h[x] + h[y] / T_x, (1 / T_x - 1) * t[x] h[y]];
```

```
{Rp[1, 2], Rm[1, 2]}
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & 0 & -1+T_1 \\ 1+\Sigma/\omega & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & 0 & -1+\frac{1}{T_1} \\ 1+\Sigma/\omega & 1 & \frac{1}{T_1} \end{pmatrix}\right\}$$

```
{Rp[1, 2] ** Rp[1, 3] ** Rp[2, 3], Rp[2, 3] ** Rp[1, 3] ** Rp[1, 2]}
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & 0 & -1+T_1 & -1+T_1 \\ t[2] & 0 & 0 & T_1(-1+T_2) \\ 1+\Sigma/\omega & 1 & T_1 & T_1 T_2 \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & 0 & -1+T_1 & -1+T_1 \\ t[2] & 0 & 0 & T_1(-1+T_2) \\ 1+\Sigma/\omega & 1 & T_1 & T_1 T_2 \end{pmatrix}\right\}$$

```
<< KnotTheory`
{b = Times @@ (PD[Knot[8, 17]] /.
     X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]], Rp[l, i], Rm[j, i]]);
 Do[b = dm[1, k, 1][b], {k, 2, 16}]; b,
 Alexander[Knot[8, 17]][T_1] // bbSimplify
}
```

Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.

KnotTheory::loading : Loading precomputed data in PD4Knots`.

$$\left\{\begin{pmatrix} -8-\frac{1}{T_1^2}+\frac{4}{T_1}+11\,T_1-8\,T_1^2+4\,T_1^3-T_1^4 & h[1] \\ t[1] & 0 \\ 1+\Sigma/\omega & 1 \end{pmatrix}, 11-\frac{1}{T_1^3}+\frac{4}{T_1^2}-\frac{8}{T_1}-8\,T_1+4\,T_1^2-T_1^3\right\}$$

```
{n = 4;
  b = B[ω, Sum[σ_j h[j], {j, n}], Sum[α_{10 i+j} t[i] h[j], {i, n}, {j, n}]],
  b // dm[1, 2, 1]
 } // ColumnForm
```

ω	h[1]	h[2]	h[3]	h[4]
t[1]	α_{11}	α_{12}	α_{13}	α_{14}
t[2]	α_{21}	α_{22}	α_{23}	α_{24}
t[3]	α_{31}	α_{32}	α_{33}	α_{34}
t[4]	α_{41}	α_{42}	α_{43}	α_{44}
$1+\Sigma/\omega$	σ_1	σ_2	σ_3	σ_4

$\omega+\alpha_{12}$	h[1]	h[3]	h[4]
t[1]	$\frac{(\omega+\alpha_{12})\,\alpha_{21}+\alpha_{22}\,(-\alpha_{11}+\omega\,\sigma_1)}{\omega}+(\alpha_{11}+\alpha_{12}\,\sigma_1)\,\sigma_2$	$\frac{(\omega+\alpha_{12})\,\alpha_{23}+\alpha_{13}\,(-\alpha_{22}+\omega\,\sigma_2)}{\omega}$	$\frac{(\omega+\alpha_{12})\,\alpha_{24}+\alpha_{14}\,(-\alpha_{22}+\omega\,\sigma_2)}{\omega}$
t[3]	$\frac{(\omega+\alpha_{12})\,\alpha_{31}+\alpha_{32}\,(-\alpha_{11}+\omega\,\sigma_1)}{\omega}$	$\frac{-\alpha_{13}\,\alpha_{32}+(\omega+\alpha_{12})\,\alpha_{33}}{\omega}$	$\frac{-\alpha_{14}\,\alpha_{32}+(\omega+\alpha_{12})\,\alpha_{34}}{\omega}$
t[4]	$\frac{(\omega+\alpha_{12})\,\alpha_{41}+\alpha_{42}\,(-\alpha_{11}+\omega\,\sigma_1)}{\omega}$	$\frac{-\alpha_{13}\,\alpha_{42}+(\omega+\alpha_{12})\,\alpha_{43}}{\omega}$	$\frac{-\alpha_{14}\,\alpha_{42}+(\omega+\alpha_{12})\,\alpha_{44}}{\omega}$
$1+\Sigma/\omega$	$\sigma_1\,\sigma_2$	σ_3	σ_4

```
{n = 4;
  b = B[ω, Sum[σ_j h[j], {j, n}], Sum[α_{10 i+j} t[i] h[j], {i, n}, {j, n}]],
  t1 = b // dm[1, 2, 1] // dm[1, 3, 1],
  t2 = b // dm[2, 3, 2] // dm[1, 2, 1],
  t1 == t2 // Simplify
 } // ColumnForm
```

ω	h[1]	h[2]	h[3]	h[4]
t[1]	α_{11}	α_{12}	α_{13}	α_{14}
t[2]	α_{21}	α_{22}	α_{23}	α_{24}
t[3]	α_{31}	α_{32}	α_{33}	α_{34}
t[4]	α_{41}	α_{42}	α_{43}	α_{44}
$1+\Sigma/\omega$	σ_1	σ_2	σ_3	σ_4

$\omega+\alpha_{23}+\frac{\alpha_{12}\,(\omega+\alpha_{23})}{\omega}+\alpha_{13}\,\left(-\frac{\alpha_{22}}{\omega}+\sigma_2\right)$	
t[1]	$\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\omega\,\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+\omega^2\,\alpha_{32}\,\sigma_1+\omega\,\alpha_{23}\,\alpha_{32}\,\sigma_1-\omega\,\alpha_{22}\,\alpha_{33}\,\sigma_1+\omega^2\,\alpha_{33}\,\sigma_1\,\sigma_2+\alpha_{31}\,(\omega\,(\omega+\alpha$
t[4]	$\alpha_{13}\,\alpha_{21}\,\alpha_{42}-\omega\,\alpha_{21}\,\alpha_{43}-\alpha_{12}\,\alpha_{21}\,\alpha_{43}+\omega^2\,\alpha_{42}\,\sigma_1+\omega\,\alpha_2$
$1+\Sigma/\omega$	

$\omega+\alpha_{23}+\frac{\alpha_{12}\,(\omega+\alpha_{23})}{\omega}+\alpha_{13}\,\left(-\frac{\alpha_{22}}{\omega}+\sigma_2\right)$	
t[1]	$\alpha_{31}\,(\omega\,(\omega+\alpha_{23})+\alpha_{12}\,(\omega+\alpha_{23})+\alpha_{13}\,(-\alpha_{22}+\omega\,\sigma_2))+\omega\,\sigma_2\,(\alpha_{11}\,(\omega+\alpha_{23})+\alpha_{12}\,(\omega+\alpha_{23})\,\sigma_1-\alpha_{13}\,(\alpha_{21}+\sigma$
t[4]	$\alpha_{21}\,(\alpha_{13}\,\alpha_{42}-(\omega+\alpha_{12})\,\alpha_{43})+\alpha_{41}\,($
$1+\Sigma/\omega$	

True