

Pensieve Header: First consistent sighting of the MVA in β -calculus.

```
<< KnotTheory`
GD[K_] := GD @@ (
  PD[K] /. X[i_, j_, k_, l_] => If[PositiveQ[X[i, j, k, l]],
    Ar[l, i, +1], Ar[j, i, -1]
  ]
)
```

Loading KnotTheory` version of August 22, 2010, 13:36:57.55.

Read more at <http://katlas.org/wiki/KnotTheory>.

```
 $\beta$ Simplify = Factor;
SetAttributes[ $\beta$ Collect, Listable];
 $\beta$ Collect[B[ $\omega$ _,  $\mu$ _]] := B[
   $\beta$ Simplify[ $\omega$ ],
  Collect[ $\mu$ , _h, Collect[#, _t,  $\beta$ Simplify] &]
];
(* "L" for "Labels" *)
hL[ $\beta$ _] := Union[Cases[ $\beta$ , h[s_] => s, Infinity]];
tL[ $\beta$ _] := Union[Cases[ $\beta$ , t[s_] | T_s_ => s, Infinity]];
dL[ $\beta$ _] := Union[hL[ $\beta$ ], tL[ $\beta$ ]];
 $\beta$ Form[B[ $\omega$ _,  $\mu$ _]] := Module[
  {tails, heads, mat},
  tails = tL[B[ $\omega$ ,  $\mu$ ]]; heads = hL[B[ $\omega$ ,  $\mu$ ]];
  mat = Outer[ $\beta$ Simplify[Coefficient[ $\mu$ , h[#1] t[#2]]] &, heads, tails];
  PrependTo[mat, t /@ tails];
  mat = Prepend[Transpose[mat], Prepend[h /@ heads,  $\omega$ ]];
  MatrixForm[mat]
];
 $\beta$ Form[else_] := else /. { $\beta$ _B =>  $\beta$ Form[ $\beta$ ],  $\beta$ _Bu =>  $\beta$ Form[ $\beta$ ]};
```

```

tm[x_, y_, z_][β_] := β /. {t[x] → t[z], t[y] → t[z], Tx → Tz, Ty → Tz};
hm[x_, y_, z_][B[ω_, μ_]] := Module[
  {γx = D[μ, h[x]], γy = D[μ, h[y]], M = μ /. h[x] | h[y] → 0},
  B[ω, M + h[z] (γx + γy + (γx /. t[i_] → 1) γy)] // βCollect
];
thswap[y_, x_][B[ω_, μ_]] := Module[
  {α, β, γ, δ, ε},
  α = Coefficient[μ, h[x] t[y]];
  β = D[μ, t[y]] /. h[x] → 0;
  γ = D[μ, h[x]] /. t[y] → 0;
  δ = μ /. h[x] | t[y] → 0;
  ε = 1 + α;
  B[ω * ε, Plus[
    α (1 + (γ /. t[i_] → 1) / ε) h[x] t[y],
    β (1 + (γ /. t[i_] → 1) / ε) t[y],
    γ / ε h[x],
    δ - 1 / ε γ * β
  ]] // βCollect
];
dm[x_, y_, z_][β_] := β // thswap[x, y] // hm[x, y, z] // tm[x, y, z];
B /: B[ω1_, μ1_] B[ω2_, μ2_] := B[ω1 * ω2, μ1 + μ2];
R[x_, y_] := B[1, (Tx - 1) * t[x] h[y]];
Ri[x_, y_] := B[1, (1 / Tx - 1) * t[x] h[y]];
βZ[L_Link] := Module[
  {skel, β, s, k},
  skel = Skeleton[L];
  β = Times @@ GD[L] /. {Ar[x_, y_, +1] → R[x, y], Ar[x_, y_, -1] → Ri[x, y]};
  Do[
    Do[
      β = β // dm[skel[[s, 1]], skel[[s, k]], skel[[s, 1]]],
      {k, 2, Length[skel[[s]]]}
    ],
    {s, Length[skel]}
  ];
  β
]

(MVA = MultivariableAlexander)[L = Link["L6a5"]][T] /. T[i_] → Ti

$$\frac{-T_1 - T_2 + T_1 T_2 - T_3 + T_1 T_3 + T_2 T_3}{\sqrt{T_1} \sqrt{T_2} \sqrt{T_3}}$$


```

$(\beta_0 = \beta_Z[L]) \text{ // } \beta\text{Form}$

$$\begin{pmatrix} \frac{(-1+T_1+T_5)(-1+T_1+T_9)(-1+T_5+T_9)}{T_1^2 T_5^2 T_9^2} & h[1] & h[5] \\ t[1] & -\frac{(-1+T_1)(1-T_1-T_5-T_9+T_5 T_9+T_1 T_5 T_9)}{T_5(-1+T_1+T_5) T_9(-1+T_1+T_9)} & -\frac{(-1+T_1) T_1}{(-1+T_1+T_5)(-1+T_1+T_9)} \\ t[5] & -\frac{(-1+T_5)(-T_1-T_5+T_1 T_5+T_1 T_9+T_5 T_9)}{(-1+T_1+T_5)(-1+T_1+T_9)(-1+T_5+T_9)} & -\frac{(-1+T_5)(-1+2 T_1-T_1^2+T_5-T_1 T_5+2 T_9-2 T_1 T_9-T_5 T_9-T_1}{T_1(-1+T_1+T_5) T_9(-1+T_1+T_9)} \\ t[9] & -\frac{(-1+T_9)(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)}{(-1+T_1+T_5)(-1+T_1+T_9)(-1+T_5+T_9)} & -\frac{(-1+T_9)(-T_5+T_1 T_5-T_9+T_1 T_9)}{(-1+T_1+T_5)(-1+T_1+T_9)(-1} \end{pmatrix}$$

$(\beta_1 = \beta\text{Collect}[\beta_0 /. B[\omega_, \mu_] \Rightarrow B[\omega, \omega * \mu]]) \text{ // } \beta\text{Form}$

$$\begin{pmatrix} \frac{(-1+T_1+T_5)(-1+T_1+T_9)(-1+T_5+T_9)}{T_1^2 T_5^2 T_9^2} & h[1] & h[5] \\ t[1] & -\frac{(-1+T_1)(-1+T_5+T_9)(1-T_1-T_5-T_9+T_5 T_9+T_1 T_5 T_9)}{T_1^2 T_5^2 T_9^2} & -\frac{(-1+T_1)(-}{T_1 T_5^2} \\ t[5] & -\frac{(-1+T_5)(-T_1-T_5+T_1 T_5+T_1 T_9+T_5 T_9)}{T_1^2 T_5^2 T_9^2} & -\frac{(-1+T_5)(-1+2 T_1-T_1^2+T_5-T_1 T_5+2 T_9-2 T_1 T_9-T_5 T_9-T_1}{T_1^3 T_5^2} \\ t[9] & -\frac{(-1+T_9)(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)}{T_1^2 T_5^2 T_9^2} & -\frac{(-1+T_9)(-T_5+T_1 T_5-T_9+T_1 T_9)}{T_1^2 T_5^2} \end{pmatrix}$$

$\text{Collect}[\beta_0[[2]] /. t[i_] \Rightarrow 1, _h, \text{Simplify}[1+\#] \&]$

$$\frac{h[9]}{T_1 T_5} + \frac{h[5]}{T_1 T_9} + \frac{h[1]}{T_5 T_9}$$

$\beta\text{Collect}[\text{Bu}[\omega_, \lambda_, \mu_]] := \text{Bu}[\$

$\beta\text{Simplify}[\omega],$
 $\text{Collect}[\lambda, _h, \beta\text{Simplify}],$
 $\text{Collect}[\mu, _h, \text{Collect}[\#, _t, \beta\text{Simplify}]] \&]$
 $];$

$\beta\text{Form}[\text{Bu}[\omega_, \lambda_, \mu_]] := \text{Module}[\$

$\{\text{tails}, \text{heads}, \text{mat}\},$
 $\text{tails} = \text{tL}[\text{B}[\omega, \lambda, \mu]];$
 $\text{heads} = \text{hL}[\text{B}[\omega, \lambda, \mu]];$
 $\text{mat} = \text{Outer}[\beta\text{Simplify}[\text{Coefficient}[\mu, h[\#1] t[\#2]]] \&, \text{heads}, \text{tails}];$
 $\text{PrependTo}[\text{mat}, t /@ \text{tails}];$
 $\text{mat} = \text{Prepend}[\text{Transpose}[\text{mat}], \text{Prepend}[\lambda, \omega]];$
 $\text{MatrixForm}[\text{mat}]$
 $];$

$\text{Bu}[n_Integer, \beta_B] := \text{Bu}[h /@ \text{Range}[n], \beta];$

$\text{Bu}[\eta s_List, B[\omega_, \mu_]] := \text{Module}[\{\lambda,$

$\lambda = (1 + \text{Coefficient}[\mu, \#] /. t[i_] \rightarrow 1) \& /@ \eta s;$
 $\text{Bu}[\omega,$
 $\text{Thread}[\eta s \rightarrow \lambda],$
 $-\mu + (\eta s /. h[j_] \Rightarrow t[j] h[j]) . \lambda$
 $] \text{ // } \beta\text{Collect}$
 $];$

$B[\text{Bu}[\omega_, \lambda_, \mu_]] := 0;$

```
{β0, β1 = Bu[h /@ {1, 5, 9}, β0], Last[β1] /. t[i_] => 1 // Simplify} // βForm
```

$$\left\{ \begin{array}{l} \frac{(-1+T_1+T_5)(-1+T_1+T_9)(-1+T_5+T_9)}{T_1^2 T_5^2 T_9^2} h[1] \\ t[1] \\ t[5] \\ t[9] \end{array} \right\} \begin{array}{l} h[5] \\ - \frac{(-1+T_1)(1-T_1-T_5-T_9+T_5 T_9+T_1 T_5 T_9)}{T_5(-1+T_1+T_5) T_9(-1+T_1+T_9)} - \frac{(-1+T_1) T_1}{(-1+T_1+T_5)(-1+T_1+T_9)} \\ - \frac{(-1+T_5)(-T_1-T_5+T_1 T_5+T_1 T_9+T_5 T_9)}{(-1+T_1+T_5)(-1+T_1+T_9)(-1+T_5+T_9)} - \frac{(-1+T_5)(-1+2 T_1-T_1^2+T_5-T_1 T_5+2 T_9-2 T_1 T_9-T_5 T_9-T_1 T_5 T_9)}{T_1(-1+T_1+T_5) T_9(-1+T_1+T_9)} \\ - \frac{(-1+T_9)(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)}{(-1+T_1+T_5)(-1+T_1+T_9)(-1+T_5+T_9)} - \frac{(-1+T_9)(-T_5+T_1 T_5-T_9+T_1 T_5 T_9)}{(-1+T_1+T_5)(-1+T_1+T_9)(-1+T_5+T_9)} \end{array}$$

```
Bu[2, R[1, 2]]
```

```
Bu[1, {h[1] → 1, h[2] → T1}, h[1] t[1] + h[2] ((1 - T1) t[1] + T1 t[2])]
```

```
Bu[2, R[1, 2]] // βForm
```

$$\left(\begin{array}{ccc} 1 & h[1] \rightarrow 1 & h[2] \rightarrow T_1 \\ t[1] & 1 & 1 - T_1 \\ t[2] & 0 & T_1 \end{array} \right)$$

$$\left(\frac{2 - 3 T_5 + 2 T_5^2 - T_5^3 + T_1 T_5^3}{T_5^2 (1 + T_1 - T_5 - 2 T_1 T_5 + 2 T_1 T_5^2)} - 1 \right) * \left(\frac{1 + T_1 - T_5 - 2 T_1 T_5 + 2 T_1 T_5^2}{T_1 T_5^3} \right) // \text{Simplify}$$

$$- \frac{(-1 + T_5) (2 - T_5 - T_1 T_5^2 + 2 T_1 T_5^3)}{T_1 T_5^5}$$

```
βMVA[Bu[ω_, λ_, μ_]] := Module[
```

```
{lbls},
```

```
lbls = Rest[First /@ λ];
```

```
ω * Det[Outer[
```

```
Coefficient[μ - lbls.(lbls /. h[i_] => t[i]), #1 * #2] &,
```

```
lbls, lbls /. h[i_] => t[i]
```

```
]] / (1 - λ[[1, 1]] /. h[i_] → Ti) // Factor
```



```
βMVA[L_Link] := βMVA[Bu[h /@ (First /@ Skeleton[L]), βZ[L]]]
```

```
{βMVA[Bu[{h[1], h[5], h[9]}, β0]], βMVA[L]}
```

$$\left\{ - \frac{-T_1 - T_5 + T_1 T_5 - T_9 + T_1 T_9 + T_5 T_9}{T_1^2 T_5^2 T_9^2}, - \frac{-T_1 - T_5 + T_1 T_5 - T_9 + T_1 T_9 + T_5 T_9}{T_1^2 T_5^2 T_9^2} \right\}$$

```
{ω, λ, μ} = List @@ Bu[{h[1], h[5]}, β0]
```

$$\left\{ \frac{1 + T_1 - T_5 - 2 T_1 T_5 + 2 T_1 T_5^2}{T_1 T_5^3}, \left\{ h[1] \rightarrow \frac{1}{T_5^2}, h[5] \rightarrow \frac{1}{T_1^2 T_5^5} \right\}, \right.$$

$$h[1] \left(\frac{(2 - 3 T_5 + 2 T_5^2 - T_5^3 + T_1 T_5^3) t[1]}{T_5^2 (1 + T_1 - T_5 - 2 T_1 T_5 + 2 T_1 T_5^2)} + \frac{(-1 + T_5) (2 - T_5 - T_1 T_5^2 + 2 T_1 T_5^3) t[5]}{T_5^2 (1 + T_1 - T_5 - 2 T_1 T_5 + 2 T_1 T_5^2)} \right) +$$

$$h[5] \left(\frac{(-1 + T_1) (2 - T_5 - T_1 T_5^2 + 2 T_1 T_5^3) t[1]}{T_1 T_5^3 (1 + T_1 - T_5 - 2 T_1 T_5 + 2 T_1 T_5^2)} + \right.$$

$$\left. \left((2 - 2 T_1 - T_5 + T_1 T_5 - T_1 T_5^2 + T_1^2 T_5^2 + 3 T_1 T_5^3 - T_1^2 T_5^3 - T_1 T_5^4 - 2 T_1^2 T_5^4 + 2 T_1^2 T_5^5) t[5] \right) / \right.$$

$$\left. \left(T_1 T_5^3 (1 + T_1 - T_5 - 2 T_1 T_5 + 2 T_1 T_5^2) \right) \right\}$$

```
lbls = Rest[First /@ λ]
```

```
{h[5]}
```

```
Factor[(MVA[#][T] /. T[i_] => TSkeleton#[[i,1]]) / βMVA[#]] & /@ AllLinks[10]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. >>

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General::stop : Further output of Power::infy will be suppressed during this calculation. >>

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. >>

General::stop : Further output of Infinity::indet will be suppressed during this calculation. >>

$$\left\{ -\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} \sqrt{T_5}, -\sqrt{T_1} \sqrt{T_5}, -\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} \sqrt{T_5}, \right.$$

$$-\sqrt{T_1} \sqrt{T_5}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -T_1^{3/2} T_5^{7/2}, -T_1^{3/2} \sqrt{T_5}, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1} \sqrt{T_5}, -T_1^{3/2} T_5^{5/2},$$

$$-T_1^{3/2} T_5^{5/2}, -\frac{\sqrt{T_1}}{T_5^{7/2}}, -\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} \sqrt{T_5}, -\sqrt{T_1} \sqrt{T_5}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -\sqrt{T_1} T_5^{5/2},$$

$$-\sqrt{T_1} T_5^{5/2}, -T_1^{3/2} T_5^{7/2}, -T_1^{3/2} T_5^{3/2}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -\sqrt{T_1} \sqrt{T_5}, -\sqrt{T_1} \sqrt{T_5}, -\frac{T_1^{3/2}}{T_5^{3/2}}, -\sqrt{T_1} \sqrt{T_5},$$

$$-\frac{\sqrt{T_1}}{T_5^{3/2}}, -T_1^{3/2} \sqrt{T_5}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} T_5^{5/2}, -T_1^{3/2} \sqrt{T_5}, -T_1^{3/2} T_5^{11/2}, -\sqrt{T_1} \sqrt{T_5},$$

$$-T_1^{3/2} \sqrt{T_5}, -\sqrt{T_1} \sqrt{T_5}, -T_1^{3/2} T_5^{5/2}, -\frac{T_1^{3/2}}{T_5^{3/2}}, -\frac{T_1^{3/2}}{T_5^{3/2}}, -\frac{T_1^{3/2}}{T_5^{3/2}}, -\frac{T_1^{3/2}}{T_5^{3/2}}, -\frac{T_1^{3/2}}{T_5^{3/2}}, -\frac{T_1^{3/2}}{T_5^{3/2}},$$

$$\begin{aligned}
& -T_1^{3/2} T_5^{9/2}, -T_1^{3/2} T_5^{9/2}, -T_1, -T_1, -T_1, -T_1 T_7^2, -T_1, -T_1, -T_1 T_7^2, -T_1^2 T_7^2, -T_1, -1, -\frac{1}{T_7^2}, \\
& -\frac{T_1}{T_7^2}, -T_1 T_7^2, -\frac{T_1}{T_7^2}, -T_1 T_7^2, -T_1 T_7^2, -T_1^2 T_7^2, -T_1, -T_7, -T_7, -1, -T_1^2 T_7^4, -T_1^2 T_7^4, -T_1^2 T_7^4, \\
& -T_1^2 T_7^4, -T_7^3, -T_1^2 T_7^4, -T_1^2, -T_1 T_7^2, -T_1 T_7, -T_1^2 T_7^2, -\frac{T_1}{T_7}, -T_1^2 T_7^2, -T_7, -T_1^2, -1, -T_1^2 T_7^4, \\
& -T_1^{3/2} T_9^{3/2}, -\frac{\sqrt{T_1}}{T_9^{3/2}}, -\sqrt{T_1} \sqrt{T_9}, -\frac{\sqrt{T_1}}{T_9^{3/2}}, -\frac{T_1^{3/2}}{\sqrt{T_9}}, -T_1^{3/2} T_9^{3/2}, -T_1^{5/2} T_9^{7/2}, -\sqrt{T_1} T_9^{5/2}, \\
& -T_1^{5/2} T_9^{7/2}, -T_1^{5/2} T_9^{3/2}, -T_1^{5/2} T_9^{7/2}, -\frac{T_1^{3/2}}{\sqrt{T_9}}, -T_1^{5/2} T_9^{7/2}, -T_1^{5/2} T_9^{7/2}, -T_1^{5/2} T_9^{7/2}, -\frac{\sqrt{T_1}}{\sqrt{T_9}}, \\
& -\frac{T_9^{3/2}}{\sqrt{T_1}}, -T_1^{5/2} T_9^{3/2}, -T_1^{3/2} T_9^{3/2}, -T_1^{5/2} T_9^{3/2}, -T_1^{7/2} T_{11}^{7/2}, -\frac{T_1^{5/2}}{T_{11}^{3/2}}, -\frac{1}{\sqrt{T_1} \sqrt{T_{11}}}, -T_1^{3/2} T_{11}^{3/2}, \\
& -\frac{T_{11}^{3/2}}{\sqrt{T_1}}, -\frac{1}{\sqrt{T_1} \sqrt{T_{11}}}, -T_1^3 T_{11}^3, -T_1^3 T_{11}^3, -T_1^3 T_{11}^3, -T_1^3 T_{11}^3, -T_1^3 T_{11}^3, -T_1^3 T_{11}^3, -T_1^3 T_{11}^3, \\
& -T_1^3 T_{11}^3, -\frac{T_1^{3/2} \sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{3/2} T_9^{3/2}, -T_1^{3/2} \sqrt{T_5} T_9^{3/2}, -\frac{T_1^{3/2} T_5^{3/2}}{\sqrt{T_9}}, -\frac{T_1^{3/2} T_5^{3/2}}{\sqrt{T_9}}, -\frac{\sqrt{T_1}}{T_{11}}, \\
& -\frac{T_1^{3/2} T_5}{T_{11}}, -T_1^{3/2} T_5 T_{11}, -T_1^{3/2} T_5^2 T_{11}^3, -\frac{\sqrt{T_1}}{T_{11}^3}, -T_1^{3/2} T_5^2 T_{11}, -\frac{\sqrt{T_1}}{T_{11}^3}, -\sqrt{T_1} T_5^2, -\frac{\sqrt{T_1} T_5}{T_{11}^2}, \\
& -\sqrt{T_1} T_5, -\frac{\sqrt{T_1}}{\sqrt{T_5} \sqrt{T_{13}}}, -\frac{\sqrt{T_1}}{\sqrt{T_5} \sqrt{T_{13}}}, -\frac{T_1^{3/2} \sqrt{T_5}}{\sqrt{T_{13}}}, -\sqrt{T_1} \sqrt{T_5} \sqrt{T_{13}}, -\sqrt{T_1} \sqrt{T_5} \sqrt{T_{13}}, \\
& -T_1^{3/2} T_5^{5/2} T_{13}^{5/2}, -T_1^{3/2} T_5^{5/2} T_{13}^{5/2}, -\frac{\sqrt{T_1}}{\sqrt{T_5} T_{13}^{5/2}}, -\frac{\sqrt{T_1}}{\sqrt{T_5} T_{13}^{5/2}}, -\frac{\sqrt{T_1}}{\sqrt{T_5} T_{13}^{5/2}}, -\frac{\sqrt{T_1} \sqrt{T_5}}{T_{13}^{3/2}}, \\
& -\frac{\sqrt{T_1} \sqrt{T_5}}{T_{13}^{3/2}}, -\frac{\sqrt{T_1} \sqrt{T_5}}{T_{15}^{3/2}}, -\frac{\sqrt{T_1}}{T_5^{3/2} T_{15}^{3/2}}, -\frac{\sqrt{T_1} \sqrt{T_{15}}}{\sqrt{T_5}}, -\frac{T_1^{3/2}}{T_{15}^2}, -T_1^{3/2} T_5^3 T_{15}^2, -T_1^{3/2} T_5^{3/2} T_{17}^{3/2}, \\
& -\frac{T_1^{3/2} T_5^{3/2}}{T_{17}^{3/2}}, -\frac{T_7}{\sqrt{T_{13}}}, -T_1 \sqrt{T_{13}}, -T_1 \sqrt{T_{13}}, -T_1^2 T_7^2 T_{13}^{5/2}, -\frac{1}{T_{13}^{5/2}}, -\frac{1}{T_{13}^{5/2}}, -\frac{T_1}{T_{13}^{3/2}}, -\frac{T_1 T_7}{\sqrt{T_{13}}}, \\
& -T_1^2 T_7^2 T_{13}^{5/2}, -T_1^{3/2} T_5^{3/2} T_9^{3/2} \sqrt{T_{13}}, -T_1^{3/2} T_5^{3/2} T_9^2 T_{15}^2, -\frac{T_1^{3/2} \sqrt{T_5}}{T_{15}^2}, -\frac{T_1^{3/2} T_5^{3/2} \sqrt{T_9}}{T_{17}^{3/2}}, \\
& -\sqrt{T_1} T_5 \sqrt{T_{11}} T_{15}, -\sqrt{T_1} \sqrt{T_{11}} T_{15}, -\frac{T_1^{3/2} T_5^2 T_{11}}{\sqrt{T_{17}}}, -\frac{\sqrt{T_1} \sqrt{T_{17}}}{T_{11}}, -T_1^{3/2} T_5 T_{11} T_{17}^{3/2}, \\
& -T_1^{3/2} T_5^{3/2} T_9^{3/2} T_{13}^{3/2} T_{17}^{3/2}, -T_1^{3/2} T_5^{5/2}, -\sqrt{T_1} T_5^{3/2}, -\sqrt{T_1} T_5^{3/2}, -T_1^{3/2} \sqrt{T_5}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, \\
& -T_1^{3/2} \sqrt{T_5}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{5/2}, -\sqrt{T_1} T_5^{3/2}, -\sqrt{T_1} T_5^{3/2}, -T_1^{3/2} T_5^{5/2}, -\sqrt{T_1} T_5^{3/2},
\end{aligned}$$

$$\begin{aligned}
& -\frac{\sqrt{T_1}}{T_5^{3/2}}, -\sqrt{T_1} T_5^{7/2}, -T_1^{3/2} T_5^{5/2}, -\sqrt{T_1} T_5^{3/2}, -T_1^{3/2} T_5^{5/2}, -\sqrt{T_1} T_5^{3/2}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -\sqrt{T_1} T_5^{5/2}, \\
& -\frac{\sqrt{T_1}}{T_5^{5/2}}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{3/2}, -T_1^{3/2} T_5^{3/2}, -T_1^{3/2} T_5^{7/2}, -T_1^{3/2} T_5^{5/2}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{3/2}, -T_1^{3/2} T_5^{5/2}, \\
& \text{Indeterminate}, -\sqrt{T_1} T_5^{3/2}, -\frac{\sqrt{T_1}}{T_5^{5/2}}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, \text{Indeterminate}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{5/2}, \\
& -\frac{\sqrt{T_1}}{T_5^{5/2}}, -1, -T_1 T_7, -\frac{1}{T_7^3}, -\frac{T_1}{T_7^2}, -T_1 T_7^2, -T_1 T_7, -T_1^2 T_7^2, -T_1^2, -T_1 T_7, -T_1^2 T_7^2, -T_1 T_7^2, \\
& -T_1^2 T_7^2, -T_1 T_7, -T_1^2 T_7^2, -\frac{1}{\sqrt{T_1} T_9^{5/2}}, -T_1^{5/2} T_9^{5/2}, -\sqrt{T_1} T_9^{3/2}, -\sqrt{T_1} T_9^{3/2}, -\frac{1}{\sqrt{T_1} \sqrt{T_{11}}}, \\
& -\frac{T_{11}^{5/2}}{\sqrt{T_1}}, -\sqrt{T_1} \sqrt{T_{11}}, -\sqrt{T_1} T_{11}^{7/2}, -\frac{1}{\sqrt{T_1} T_{11}^{3/2}}, -T_1^{3/2} T_{11}^{3/2}, -T_1^{3/2} T_{11}^{5/2}, -\frac{\sqrt{T_1} T_5^{3/2}}{\sqrt{T_9}}, \\
& -\frac{\sqrt{T_1} T_5^{3/2}}{\sqrt{T_9}}, -\frac{T_1^{3/2} \sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^2 T_{11}^3, -\sqrt{T_1} T_5^2 T_{11}^2, -\sqrt{T_1} T_5^{3/2} T_{11}^{5/2}, -T_1^{3/2} T_{11}, -\sqrt{T_1}, \\
& -\sqrt{T_1} T_5, -T_1^{3/2} T_5^{5/2} T_{13}^{5/2}, -\sqrt{T_1} T_5^{5/2} T_{13}^{3/2}, -\sqrt{T_1} T_5^2 T_{13}^2, -T_1^{3/2} T_5^{5/2} T_{13}^{5/2}, -\sqrt{T_1} T_5^{5/2} T_{13}^{3/2}, \\
& -\sqrt{T_1} T_5^2 T_{13}^2, -\frac{T_1^{3/2} T_5^{5/2}}{\sqrt{T_{13}}}, -\sqrt{T_1} T_5^{5/2} T_{13}^{3/2}, -\sqrt{T_1} T_5^{5/2} \sqrt{T_{13}}, -\sqrt{T_1} T_5^{5/2} T_{15}^{3/2}, -T_1^{3/2} T_5^3 T_{15}, \\
& -\frac{T_1^{3/2} T_5}{T_{15}}, -T_1^{3/2} T_5^2, -\frac{T_1^{3/2}}{T_{15}}, -\frac{T_1^{3/2} T_5^{3/2}}{\sqrt{T_{17}}}, -T_1^{3/2} T_5^{7/2} T_{17}^{3/2}, -\sqrt{T_1} \sqrt{T_5} \sqrt{T_{17}}, -T_1^{3/2} T_5^{3/2} \sqrt{T_{17}}, \\
& -\frac{T_1^2}{\sqrt{T_{13}}}, -T_1^2 T_7^2 T_{13}^{3/2}, -\frac{1}{T_{13}^{5/2}}, -T_1^2 \sqrt{T_{13}}, -\frac{\sqrt{T_1} \sqrt{T_5}}{T_9 T_{15}}, -\sqrt{T_1} \sqrt{T_5} T_9 T_{15}, -T_1^{3/2} T_5^{3/2}, \\
& -\sqrt{T_1} \sqrt{T_5} \sqrt{T_9} \sqrt{T_{17}}, -\frac{\sqrt{T_1} \sqrt{T_5}}{\sqrt{T_9} \sqrt{T_{17}}}, -\frac{\sqrt{T_1} \sqrt{T_5}}{T_9^{3/2} T_{17}^{3/2}}, -T_1^{3/2} T_5^{3/2} T_9^{5/2} \sqrt{T_{17}}, -\frac{T_1^{3/2} T_5^{3/2} T_9^{3/2}}{\sqrt{T_{17}}}, \\
& -T_1^{3/2} T_5 T_{11}^{3/2} T_{15}, -\sqrt{T_1} T_5 T_{11}^{3/2}, -T_1^{3/2} T_5 \sqrt{T_{11}} T_{15}, \text{Indeterminate}, -T_1^{3/2} T_5^2 \sqrt{T_{11}} T_{15}, \\
& -\sqrt{T_1} T_5^2 \sqrt{T_{11}}, -\frac{\sqrt{T_1} T_5}{\sqrt{T_{17}}}, -\frac{T_1^{3/2} T_5 T_{11}}{\sqrt{T_{17}}}, -\frac{\sqrt{T_1} \sqrt{T_5}}{\sqrt{T_9} \sqrt{T_{13}} \sqrt{T_{17}}}, -\frac{\sqrt{T_1} \sqrt{T_5} \sqrt{T_{13}} \sqrt{T_{17}}}{\sqrt{T_9}} \}
\end{aligned}$$