Gillet@FI: What is infinity factorial (and why might we care)?

November-25-08
4:05 PM

<u>claim</u> $1 \cdot 2 \cdot 3 \cdot 4 \cdots = \sqrt{2\pi}$

$1 + 1 + \cdots + 1 = -\frac{1}{2}$

<u>General Q</u> How and why do we assign values to divergent sums & products.

$\log(\infty!) = \log 1 + \log 2 + \cdots$

$\sum r^n = \frac{1}{1-r}$ so at $r = 1$

$1 - 1 + 1 - 1 \cdots = \frac{1}{1-r} = \frac{1}{2}$

<u>General Question</u> Give a reasonable definition for $\sum a_i$ s.t. 1. If $\sum a_i$ converges in the usual sense, we agree w/ the usual answer.

2. Linearity.
3. Good behaviour under First & Rest.

Connection between $\infty!$ & Wallis' formula for π.

Cavalieri, Wallis could integrate $\int_0^1 x^n\,dx$.

How about $\int_0^1 \sqrt{1 - x^2}\,dx$ (should be $\frac{\pi}{4}$)

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots$$

$= W = \frac{\pi}{2}$

Exercise If $\infty! = \sqrt{2\pi}$, deduce Wallis' formula

Euler For integer k, computed

$$1+2^k+3^k+\ldots$$

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$ The Riemann ζ-function.

Converges for $s>1$ (really, $Re(s)>1$)

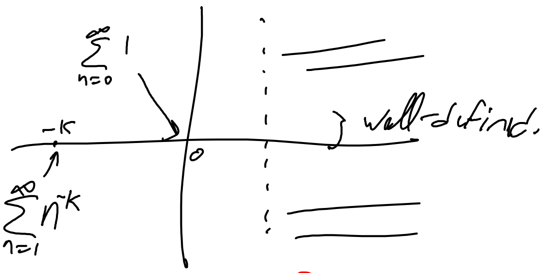
E.g. $\zeta(2) = \frac{\pi^2}{6} \Longleftrightarrow$ The probability that two non-negative numbers are co-prime is $\cancel{6}/\pi^2$

Abel summation: If a_i is a seq s.t. $\sum a_n x^n$ converges for $|x|<1$, and $\lim\limits_{x \nearrow 1} \sum a_n x^n = s$, we say that $\sum a_i = s$ & (a_i) is Abel-summable.

Cesaro summation $S_n = a_0 + \ldots + a_n$

If $\frac{S_0+\ldots+S_n}{n+1}$ converges, we say that (a_i) is Cesaro summable.

Back to ζ:



Euler: $\sum n^k = -\frac{B_{k+1}}{k+1}$

where

$$\frac{x}{e^x-1} = \sum_{n=1}^{\infty} B_n \frac{x^n}{n!}$$

Lie interpretation?

$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} \ldots$ converges for $s>1$

$L(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots$ converges for $s>0$

$L(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots$ converges for $s > 0$

$\zeta(s) - L(s) = 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \ldots\right) = 2\frac{1}{2^s}\left(1 + \frac{1}{2^s} + \ldots\right) = 2^{1-s}\zeta(s)$

$$\Rightarrow \zeta(s) = \underbrace{\frac{1}{1-2^{1-s}}}_{\to\infty \text{ as } s\to 1} \underbrace{L(s)}_{\text{finite for } s>0}$$

$\Rightarrow$ we've "isolated" the singularity

$1 + 1 + \ldots = \zeta(0)$

$\zeta'(s) = \sum_{n=1}^{\infty} \frac{-\log n}{n^s}$ so formally, $\zeta'(0) = \sum_{n=1}^{\infty} -\log n$

$= -\log \infty!$

<u>Claim</u> $\zeta(0) = -\frac{1}{2}$ $\zeta'(0) = -\log\sqrt{2\pi}$

$\zeta(0) = \frac{1}{1-2^1} L(0) = -(1 - 1 + 1 - 1 \ldots) = -\frac{1}{2}$

<u>Euler Transformation</u> $S = a_0 - a_1 + a_2 - \ldots$

$$= \frac{1}{2}a_0 + \frac{1}{2}\underbrace{\left((a_0 - a_1) - (a_1 - a_2) + (a_2 - a_3) - \ldots\right)}_{\text{may have better convergence properties.}}$$

$\Rightarrow L(s) = \frac{1}{2}\left(1 + \left(1 - \frac{1}{2^s}\right) - \left(\frac{1}{2^s} - \frac{1}{3^s}\right) + \ldots\right)$

converges for $s > -1$!

$\Rightarrow L(0) = \frac{1}{2}(1 + 0 - 0 + 0 \ldots) = \frac{1}{2}$

and $(n^{-s})'|_{s=0} = -\log n$

$L'(0) = \frac{1}{2}\left(+\log 2 - (-\log 2 + \log 3) + (-\log 3 + \log 4) - \ldots\right)$

$= \frac{1}{2}\left[\log 2 + \log\frac{2}{3} + \log\frac{4}{3} + \ldots\right]$

$$= \frac{1}{2}\left[\log 2 + \log\frac{2}{3} + \log\frac{4}{3} + \dots\right]$$

$$= \frac{1}{2}\log(\text{Wallis' product}) = \frac{1}{2}\log \pi/2$$

$$\Longrightarrow \zeta'(0) = -\log\sqrt{2\pi}$$

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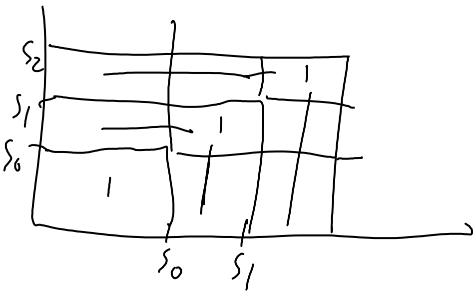
pf of Wallis' formula by Wästlund:

$$s_n = \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots$$ but

$$W_n = \frac{2n}{(s_n)^2}$$



$$\Longrightarrow \frac{n}{\pi/2} \le s_n^2 \le \frac{n+1}{\pi/2}$$