

# Chord Diagrams, Knots, and Lie Algebras

**Abstract.** This will be a service talk on ancient material — I will briefly describe how the exact same type of chord diagrams (and relations between them) occur in a natural way in both knot theory and in the theory of Lie algebras.

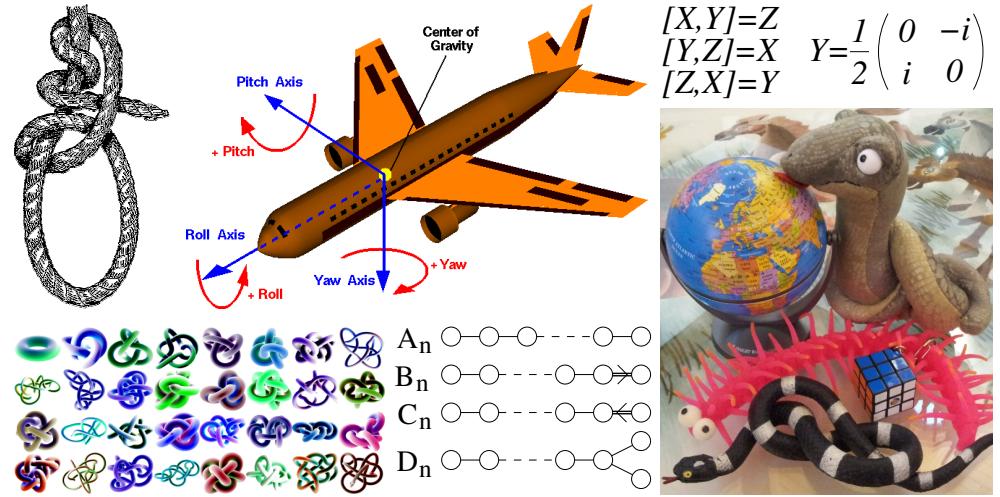
[Book] *Introduction to Vassiliev Knot Invariants*, by S. Chmutov, S. Duzhin, and J. Mostovoy, Cambridge University Press, Cambridge UK, 2012, xvi+504 pp., hardback, \$70.00, ISBN 978-1-10702-083-2.

Merely 30 **36** years ago, if you had asked even the best informed mathematician about the relationship between knots and Lie algebras, she would have laughed, for there isn't and there can't be. Knots are flexible; Lie algebras are rigid. Knots are irregular; Lie algebras are symmetric. The list of knots is a lengthy mess; the collection of Lie algebras is well-organized. Knots are useful for sailors, scouts, and hangmen; Lie algebras for navigators, engineers, and high energy physicists. Knots are blue collar; Lie algebras are white. They are as similar as worms and crystals: both well-studied, but hardly ever together.

Then in the 1980s came Jones, and Witten, and Reshetikhin and Turaev [Jo, Wi, RT] and showed that if you really are the best informed, and you know your quantum field theory and conformal field theory and quantum groups, then you know that the two disjoint fields are in fact intricately related. This “quantum” approach remains the most powerful way to get computable knot invariants out of (certain) Lie algebras (and representations thereof). Yet shortly later, in the late 80s and early 90s, an alternative perspective arose, that of “finite-type” or “Vassiliev-Goussarov” invariants [Va1, Va2, Go1, Go2, BL, Ko1, Ko2, BN1], which made the surprising relationship between knots and Lie algebras appear simple and almost inevitable.

The reviewed [Book] is about that alternative perspective, the one reasonable sounding but not entirely trivial theorem that is crucially needed within it (the “Fundamental Theorem” or the “Kontsevich integral”), and the

*While preparing for this talk I realized that I've done it before, much better, within a book review. So here's that review! It has been modified from its original version: it had been formatted to fit this page, parts were highlighted, and commentary had been added in green italics.*



A knot and a Lie algebra, a list of knots and a list of Lie algebras, and an unusual conference of the symmetric and the knotted.

many threads that begin with that perspective. Let me start with a brief summary of the mathematics, and even before, an even briefer summary.

*In briefest, a certain space  $\mathcal{A}$  of chord diagrams is the dual to the dual of the space of knots, and at the same time, it is dual to Lie algebras.*

The briefer summary is that in some combinatorial sense it is possible to “differentiate” knot invariants, and hence it makes sense to talk about “polynomials” on the space of knots — these are functions on the set of knots (namely, these are knot invariants) whose sufficiently high derivatives vanish. Such polynomials can be fairly conjectured to separate knots — elsewhere in math in lucky cases polynomials separate points, and in our case, specific computations are encouraging. Also, such polynomials are determined by their “coefficients”, and each of these, by the one-side-easy “Fundamental Theorem”, is a linear functional on some finite space of

graphs modulo relations. These same graphs turn out to parameterize formulas that make sense in a wide class of Lie algebras, and the said relations match exactly with the relations in the definition of a Lie algebra — antisymmetry and the Jacobi identity. Hence what is more or less dual to knots (invariants), is also, after passing to the coefficients, dual to certain graphs which are more or less dual to Lie algebras. QED, and on to the less brief summary<sup>1</sup>.

Let  $V$  be an arbitrary invariant of oriented knots in oriented space with values in (say)  $\mathbb{Q}$ . Extend  $V$  to be an invariant of 1-singular knots, knots that have a single singularity that locally looks like a double point  $\times$ , using the formula

$$(1) \quad V(\times) = V(\nearrow\searrow) - V(\nwarrow\nearrow).$$

Further extend  $V$  to the set  $\mathcal{K}^m$  of  $m$ -singular knots (knots with  $m$  such double points) by repeatedly using (1).

**Definition 1.** We say that  $V$  is of type  $m$  (or “Vassiliev of type  $m$ ”) if its extension  $V|_{\mathcal{K}^{m+1}}$  to  $(m+1)$ -singular knots vanishes identically. We say that  $V$  is of finite type (or “Vassiliev”) if it is of type  $m$  for some  $m$ .

Repeated differences are similar to repeated derivatives and hence it is fair to think of the definition of  $V|_{\mathcal{K}^m}$  as repeated differentiation. With this in mind, the above definition imitates the definition of polynomials of degree  $m$ . Hence finite type invariants can be thought of as “polynomials” on the space of knots<sup>2</sup>. It is known (see e.g. [Book]) that the class of finite type invariants is large and powerful. Yet the first question on finite type invariants remains unanswered:

**Problem 2.** Honest polynomials are dense in the space of functions. Are finite type invariants dense within the space of all knot invariants? Do they separate knots?

The top derivatives of a multi-variable polynomial form a system of constants that determine that polynomial up to polynomials of lower degree. Likewise the  $m$ th derivative<sup>3</sup>  $V^{(m)} = V|_{\mathcal{K}^m} = V(\times \cdots \times)$  of a type  $m$  invariant  $V$  is a constant in the sense that it does not see the difference between overcrossings and undercrossings and so it is blind to 3D topology. Indeed

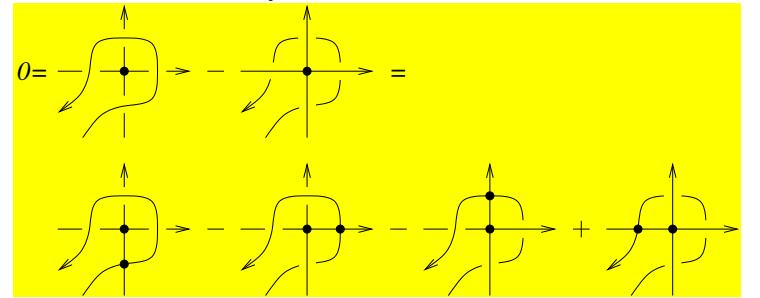
$$V(\times \cdots \times \nearrow\searrow) - V(\times \cdots \times \nwarrow\nearrow) = V(\times \cdots \times) = 0.$$

Also, clearly  $V^{(m)}$  determines  $V$  up to invariants of lower type. Hence a primary tool in the study of finite

type invariants is the study of the “top derivative”  $V^{(m)}$ , also known as “the weight system of  $V$ ”.

Blind to 3D topology,  $V^{(m)}$  only sees the combinatorics of the circle that parameterizes an  $m$ -singular knot. On this circle there are  $m$  pairs of points that are pairwise identified in the image; standardly one indicates those by drawing a circle with  $m$  chords marked (an “ $m$ -chord diagram”) as above. Let  $\mathcal{D}_m$  denote the space of all formal linear combinations with rational coefficients of  $m$ -chord diagrams. Thus  $V^{(m)}$  is a linear functional on  $\mathcal{D}_m$ .

I leave it for the reader to figure out or read in [Book, pp. 88] how the following figure easily implies the “4T” relations of the “easy side” of the theorem that follows:



**Theorem 3. (The Fundamental Theorem, details in [Book].)**

- (Easy side) If  $V$  is a rational valued type  $m$  invariant then  $V^{(m)}$  satisfies the “4T” relations shown above, and hence it descends to a linear functional on  $\mathcal{A}_m := \mathcal{D}_m / 4T$ . If in addition  $V^{(m)} \equiv 0$ , then  $V$  is of type  $m-1$ .
- (Hard side, slightly missstated by avoiding “framings”) For any linear functional  $W$  on  $\mathcal{A}_m$  there is a rational valued type  $m$  invariant  $V$  so that  $V^{(m)} = W$ .

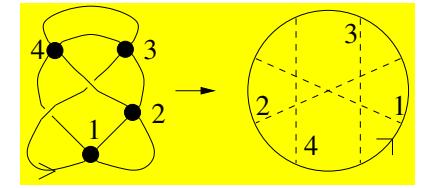
Thus to a large extent the study of finite type invariants is reduced to the finite (though super-exponential in  $m$ ) algebraic study of  $\mathcal{A}_m$ .

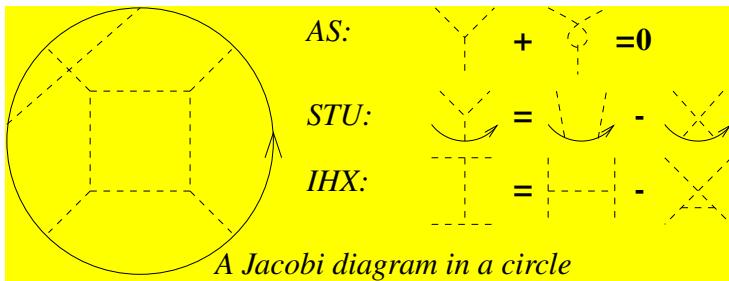
Much of the richness of finite type invariants stems from their relationship with Lie algebras. Theorem 4 below suggests this relationship on an abstract level and Theorem 5 makes that relationship concrete.

<sup>1</sup>Partially self-plagiarized from [BN2].

<sup>2</sup>Keep this apart from invariants of knots whose values are polynomials, such as the Alexander or the Jones polynomial. A posteriori related, these are a priori entirely different.

<sup>3</sup>As common in the knot theory literature, in the formulas that follow a picture such as  $\times \cdots \times \times$  indicates “some knot having  $m$  double points and a further (right-handed) crossing”. Furthermore, when two such pictures appear within the same formula, it is to be understood that the parts of the knots (or diagrams) involved outside of the displayed pictures are to be taken as the same.





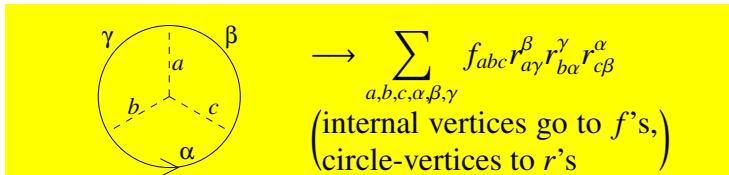
**Theorem 4.** [BN1] The space  $\mathcal{A}_m$  is isomorphic to the space  $\mathcal{A}'_m$  generated by “Jacobi diagrams in a circle” (chord diagrams that are also allowed to have oriented internal trivalent vertices) that have exactly  $2m$  vertices, modulo the AS, STU and IHX relations. See the figure above.

The key to the proof of Theorem 4 is the figure above, which shows that the  $4T$  relation is a consequence of two  $STU$  relations. The rest is more or less an exercise in induction.

Thinking of internal trivalent vertices as graphical analogs of the Lie bracket, the  $AS$  relation becomes the anti-commutativity of the bracket,  $STU$  becomes the equation  $[x, y] = xy - yx$  and  $IHX$  becomes the Jacobi identity. This analogy is made concrete within the following construction, originally due to Penrose [Pe] and to Cvitanović [Cv]. Given a finite dimensional metrized Lie algebra  $g$  (e.g., any semi-simple Lie algebra) and a finite-dimensional representation  $\rho : g \rightarrow \text{End}(V)$  of  $g$ , choose an orthonormal basis<sup>4</sup>  $\{X_a\}_{a=1}^{\dim g}$  of  $g$  and some basis  $\{v_\alpha\}_{\alpha=1}^{\dim V}$  of  $V$ , let  $f_{abc}$  and  $r_{ab}^\gamma$  be the “structure constants” defined by

$$f_{abc} := \langle [X_a, X_b], X_c \rangle \quad \text{and} \quad \rho(X_a)(v_\beta) = \sum_\gamma r_{ab}^\gamma v_\gamma.$$

Now given a Jacobi diagram  $D$  label its circle-arcs with Greek letters  $\alpha, \beta, \dots$ , and its chords with Latin letters  $a, b, \dots$ , and map it to a sum as suggested by the following example:



**Theorem 5.** This construction is well defined, and the basic properties of Lie algebras imply that it respects the AS, STU, and IHX relations. Therefore it defines a linear functional  $W_{g,\rho} : \mathcal{A}_m \rightarrow \mathbb{Q}$  for any  $m$ .

The last assertion along with Theorem 3 show that associated with any  $g, \rho$  and  $m$  there is a weight system and

hence a knot invariant. Thus knots are indeed linked with Lie algebras.

The above is of course merely a sketch of the beginning of a long story. You can read the details, and some of the rest, in [Book].

**What I like about [Book].** Detailed, well thought out, and carefully written. Lots of pictures! Many excellent exercises! A complete discussion of “the algebra of chord diagrams”. A nice discussion of the pairing of diagrams with Lie algebras, including examples aplenty. The discussion of the Kontsevich integral (meaning, the proof of the hard side of Theorem 3) is terrific — detailed and complete and full of pictures and examples, adding a great deal to the original sources. The subject of “associators” is huge and worthy of its own book(s); yet in as much as they are related to Vassiliev invariants, the discussion in [Book] is excellent. A great many further topics are touched — multiple  $\zeta$ -values, the relationship of the Hopf link with the Duflo isomorphism, intersection graphs and other combinatorial aspects of chord diagrams, Rozansky’s rationality conjecture, the Melvin-Morton conjecture, braids,  $n$ -equivalence, etc.

For all these, I’d certainly recommend [Book] to any newcomer to the subject of knot theory, starting with my own students.

However, some proofs other than that of Theorem 3 are repeated as they appear in original articles with only a superficial touch-up, or are omitted altogether, thus missing an opportunity to clarify some mysterious points. This includes Vogel’s construction of a non-Lie-algebra weight system and the Goussarov-Polyak-Viro proof of the existence of “Gauss diagram formulas”.

**What I wish there was in the book, but there isn’t.** The relationship with Chern-Simons theory, Feynman diagrams, and configuration space integrals, culminating in an alternative (and more “3D”) proof of the Fundamental Theorem. This is a major omission.

**Why I hope there will be a continuation book, one day.** There’s much more to the story! There are finite type invariants of 3-manifolds, and of certain classes of 2-dimensional knots in  $\mathbb{R}^4$ , and of “virtual knots”, and they each have their lovely yet non-obvious theories, and these theories link with each other and with other branches of Lie theory, algebra, topology, and quantum field theory. Volume 2 is sorely needed.

## REFERENCES

- [BN1] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology 34 (1995) 423–472.

<sup>4</sup>This requirement can easily be relaxed.

- [BN2] D. Bar-Natan, *Finite Type Invariants*, in *Encyclopedia of Mathematical Physics*, (J.-P. Francoise, G. L. Naber and Tsou S. T., eds.) Elsevier, Oxford, 2006 (vol. 2 p. 340).

[Book] The reviewed book.

[BL] J. S. Birman and X-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993) 225–270.

[Cv] P. Cvitanović, *Group Theory, Birdtracks, Lie's, and Exceptional Groups*, Princeton University Press, Princeton 2008 and <http://www.birdtracks.eu>.

[Go1] M. Goussarov, *A new form of the Conway-Jones polynomial of oriented links*, Zapiski nauch. sem. POMI **193** (1991) 4–9 (English translation in *Topology of manifolds and varieties* (O. Viro, editor), Amer. Math. Soc., Providence 1994, 167–172).

[Go2] M. Goussarov, *On  $n$ -equivalence of knots and invariants of finite degree*, Zapiski nauch. sem. POMI **208** (1993) 152–173 (English translation in *Topology of manifolds and varieties* (O. Viro, editor), Amer. Math. Soc., Providence 1994, 173–192).

[Jo] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985) 103–111.

[Ko1] M. Kontsevich, *Vassiliev's knot invariants*, Adv. in Sov. Math., **16(2)** (1993) 137–150.

[Ko2] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics **II** 97–121, Birkhäuser Basel 1994.

[Pe] R. Penrose, *Applications of negative dimensional tensors*, Combinatorial mathematics and its applications (D. J. A. Welsh, ed.), Academic Press, San-Diego 1971, 221–244.

[RT] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. **127** (1990) 1–26.

[Va1] V. A. Vassiliev, *Cohomology of knot spaces*, in *Theory of Singularities and its Applications (Providence)* (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[Va2] V. A. Vassiliev, *Complements of discriminants of smooth maps: topology and applications*, Trans. of Math. Mono. **98**, Amer. Math. Soc., Providence, 1992.

[Wi] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989) 351–399.

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## *My talk yesterday:*

**Dror Bar-Natan: Talks: Toronto-1912:** [osf:https://drorbn.net/19/](http://drorbn.net/19/)

## Geography vs. Identity

Thanks for inviting me to the *Topology session*!

**Abstract.** Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

**Geographers** care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".

**Identifiers** believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation  $m_{ab}^c$ , and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See [osf/reg](#), [osf/kbh](#).

**Braids.**

Geography: (captures quantum algebra!)

$$GB := \langle \gamma_i \rangle / \left( \begin{array}{l} \sigma_{ijkl} = \sigma_{iljk} \text{ when } |i-j| > 1 \\ \gamma_{ij} \gamma_{i+1,j} = \gamma_{i+1,j} \gamma_{ij} \end{array} \right) = B.$$

Identity: (captures quantum algebra!)

$$IB := \langle \sigma_{ijkl} \rangle / \left( \begin{array}{l} \sigma_{ijkl} = \sigma_{iljk} \text{ when } |i-j, k,l| = 4 \\ \sigma_{ijkl} \sigma_{ijk'l} = \sigma_{jik'l} \sigma_{ijkl} \text{ when } |i,j,k,l| = 3 \end{array} \right) = PB.$$

**Theorem.** Let  $S = \langle \tau \rangle$  be the symmetric group. Then  $vB$  is both  $PB \bowtie S \cong B \otimes S$  ( $\forall i, \tau = \tau_{ij}$  when  $i = j$ ,  $\tau(i+1) = (j+1)$ ) (and so  $PB$  is "bigger" than  $B$ , and hence quantum algebra doesn't see topology very well).

**Proof.** Going left,  $\gamma_i \mapsto \sigma_{i,i+1}(i+1)$ . Going right, if  $i < j$  map  $\sigma_{ij} \mapsto \tau_{i,j-2} \dots \tau_{ij} \tau_{j,i+1} \dots \tau_{j,j}$  and if  $i > j$  use  $\sigma_{ij} \mapsto \tau_{j+1} \dots \tau_{ij} \tau_{i-1} \dots \tau_{j+1}$ .  
 $vB$  views of  $\sigma_{ijkl}$ :

The **Bureau Representation** of  $PB_n$  acts on  $R^n := \mathbb{Z}[t_1^{\pm 1}]^n = R(v_1, \dots, v_n)$  by

$$\sigma_{ij}v_k = v_k + \delta_{ij}(t-1)(v_j - v_i).$$

$\delta := \delta_{i,j,k,l} := \text{If } \{i = j, 1\} \text{ else } 0$  [osf/code](#)

$B_{n,k}[\mathcal{E}_i] := \mathcal{E}_i / \langle v_k - v_i \rangle \rightarrow v_k + \delta_{i,j,k,l}(\mathbf{t}-1)(v_j - v_i) // \text{Expand}$  Werner Bureau

$(\mathbf{bas3} = \{v_1, v_2, v_3\}) // B_{1,2}$   
 $\{v_1, v_1 - t v_1 + t v_2, v_3\}$   
 $\{v_1, v_1 - t v_1 + t v_2, v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3\}$

$\mathbf{bas3} // B_{2,3} // B_{1,3} // B_{1,2}$   
 $\{v_1, v_1 - t v_1 + t v_2, v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3\}$

$S_n$  acts on  $R^n$  by permuting the  $v_i$  so the Bureau representation extends to  $vB_n$  and restricts to  $B_n$ . With this,  $\gamma_i$  maps  $v_i \mapsto v_{i+1}$ ,  $v_{i+1} \mapsto t v_{i+1} - (t-1)v_{i+1}$ , and otherwise  $v_k \mapsto v_k$ .

**Geography view:**

$$\gamma_1 = \diagup \quad | \quad \gamma_2 = | \quad \diagdown \quad \gamma_3 = | \quad \times \quad \dots$$

so  $x$  is  $\gamma_2$ .

**Identity view:**

At  $x$  strand 1 crosses strand 3, so  $x$  is  $\sigma_{13}$ .

**The Gold Standard** is set by the "T-calculus" Alexander formulas ( $\omega/\phi/\text{mac}$ ). An  $S$ -component tangent  $T$  has

$$\Gamma(T) \in R_S \times M_{S \times S}(R_S) = \left\{ \frac{\omega}{S \mid A} \right\} \text{ with } R_S := \mathbb{Z}[\langle T_a : a \in S \rangle]$$

$$\begin{array}{c} 1 \quad a \quad b \\ a' \nearrow b' \nwarrow \\ a \quad | \quad 1 - T_a^{-1} \\ b \quad 0 \quad T_a^{-1} \end{array} \rightarrow \begin{array}{c} 1 \quad a \quad b \\ a \quad \alpha \quad B \quad \theta \\ m_{ab}^{ab} \\ b \quad \gamma \quad \epsilon \\ S \quad \phi \quad \Xi \end{array} \xrightarrow{T_1 \sqcup T_2} \begin{array}{c} \omega_{12} \omega_{34} \quad S_1 \quad S_2 \\ S_1 \quad A_1 \quad 0 \\ S_2 \quad 0 \quad A_2 \end{array}$$

$$\begin{array}{c} \omega \quad a \quad b \quad S \\ a \quad \alpha \quad B \quad \theta \\ m_{ab}^{ab} \\ b \quad \gamma \quad \epsilon \\ S \quad \phi \quad \Xi \end{array} \xrightarrow{T_a, T_b \rightarrow T_c} \begin{array}{c} (1-\beta)\omega \quad c \quad S \\ c \quad \gamma + \frac{\alpha\beta}{1-\beta} \quad \epsilon + \frac{\alpha\beta}{1-\beta} \\ S \quad \phi + \frac{\gamma\beta}{1-\beta} \quad \Xi + \frac{\gamma\beta}{1-\beta} \end{array}$$

The **Gassner Representation** of  $PB_n$  acts on  $V = R^n := \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = R(v_1, \dots, v_n)$  by

$$\sigma_{ij}v_k = v_k + \delta_{ij}(t_i - 1)(v_j - v_i).$$

$G_{n,k}[\mathcal{E}_i] := \mathcal{E}_i / \langle v_k - v_i \rangle \rightarrow v_k + \delta_{i,j,k,l}(\mathbf{t}-1)(v_j - v_i) // \text{Expand}$

$\mathbf{bas3} // G_{2,2} // G_{1,3} // G_{2,3} = (\mathbf{bas3} // G_{2,2} // G_{1,3}) // G_{2,3}$

True

$S_n$  acts on  $R^n$  by permuting the  $v_i$  and the  $t_i$ , so the Gassner representation extends to  $vB_n$  and then restricts to  $B_n$  as a  $\mathbb{Z}$ -linear co-dimensional representation. It then descends to  $PB_n$  as a finite rank  $R$ -linear representation, with lengthy non-local formulas.

**Geographers:** Gassner is an obscure partial extension of Burau.

**Identifiers:** Burau is a trivial silly reduction of Gassner.

**The Turbo-Gassner Representation.** With the same  $R$  and  $V$ ,  $TG$  acts on  $V \oplus (R^n \otimes V) \oplus (S^2 V \otimes V') = R(v_k, v_{k+1}, w_k, w_{k+1})$  by

$TG_{n,k}[\mathcal{E}_i] := \mathcal{E}_i - \{$

- $v_{k+1} + \delta_{k,j}(\mathbf{t}-1) \quad (v_k - v_j) + v_{i,j} - v_{i+1,j} + \delta_{k,j} (v_j - u_i) \quad u_i w_{j,j}$
- $v_{i,k+1} \rightarrow v_{i,k+1} - (\mathbf{t}-1) \times ( \delta_{k,j} (v_{i,j} - v_{i+1,j}) + (\delta_{k,i+1} - \delta_{i+1,j}) \mathbf{t}^{-1} \mathbf{t}_j ) \quad (u_k + \delta_{k,j}(\mathbf{t}-1) \quad (u_j - u_i)) \quad u_i w_j \},$
- $w_{k+1} \rightarrow w_k + (\delta_{k,j} - \delta_{i,j})(\mathbf{t}^{-1} - 1) \quad w_j // \text{Expand}$

$\mathbf{bas3} = \{v_1, v_2, v_3, v_{1,2}, v_{1,2,3}, v_{1,2,3,2}, v_{2,2,3}, v_{3,1}, v_{2,3}, v_{3,2}, v_{3,3}, v_{1,2} w_1, v_{1,2} w_2, w_1 w_2, v_{1,2} w_1, v_{1,2} w_2, v_{1,2} w_3, v_{1,2} w_4, v_{1,2} w_5, v_{1,2} w_6, v_{1,2} w_7, v_{1,2} w_8, v_{1,2} w_9, v_{1,2} w_10, v_{1,2} w_11, v_{1,2} w_12, v_{1,2} w_13, v_{1,2} w_14, v_{1,2} w_15, v_{1,2} w_16, v_{1,2} w_17, v_{1,2} w_18, v_{1,2} w_19, v_{1,2} w_20, v_{1,2} w_21, v_{1,2} w_22, v_{1,2} w_23, v_{1,2} w_24, v_{1,2} w_25, v_{1,2} w_26, v_{1,2} w_27, v_{1,2} w_28, v_{1,2} w_29, v_{1,2} w_30, v_{1,2} w_31, v_{1,2} w_32, v_{1,2} w_33, v_{1,2} w_34, v_{1,2} w_35, v_{1,2} w_36, v_{1,2} w_37, v_{1,2} w_38, v_{1,2} w_39, v_{1,2} w_40, v_{1,2} w_41, v_{1,2} w_42, v_{1,2} w_43, v_{1,2} w_44, v_{1,2} w_45, v_{1,2} w_46, v_{1,2} w_47, v_{1,2} w_48, v_{1,2} w_49, v_{1,2} w_50, v_{1,2} w_51, v_{1,2} w_52, v_{1,2} w_53, v_{1,2} w_54, v_{1,2} w_55, v_{1,2} w_56, v_{1,2} w_57, v_{1,2} w_58, v_{1,2} w_59, v_{1,2} w_60, v_{1,2} w_61, v_{1,2} w_62, v_{1,2} w_63, v_{1,2} w_64, v_{1,2} w_65, v_{1,2} w_66, v_{1,2} w_67, v_{1,2} w_68, v_{1,2} w_69, v_{1,2} w_70, v_{1,2} w_71, v_{1,2} w_72, v_{1,2} w_73, v_{1,2} w_74, v_{1,2} w_75, v_{1,2} w_76, v_{1,2} w_77, v_{1,2} w_78, v_{1,2} w_79, v_{1,2} w_80, v_{1,2} w_81, v_{1,2} w_82, v_{1,2} w_83, v_{1,2} w_84, v_{1,2} w_85, v_{1,2} w_86, v_{1,2} w_87, v_{1,2} w_88, v_{1,2} w_89, v_{1,2} w_90, v_{1,2} w_91, v_{1,2} w_92, v_{1,2} w_93, v_{1,2} w_94, v_{1,2} w_95, v_{1,2} w_96, v_{1,2} w_97, v_{1,2} w_98, v_{1,2} w_99, v_{1,2} w_100, v_{1,2} w_101, v_{1,2} w_102, v_{1,2} w_103, v_{1,2} w_104, v_{1,2} w_105, v_{1,2} w_106, v_{1,2} w_107, v_{1,2} w_108, v_{1,2} w_109, v_{1,2} w_110, v_{1,2} w_111, v_{1,2} w_112, v_{1,2} w_113, v_{1,2} w_114, v_{1,2} w_115, v_{1,2} w_116, v_{1,2} w_117, v_{1,2} w_118, v_{1,2} w_119, v_{1,2} w_120, v_{1,2} w_121, v_{1,2} w_122, v_{1,2} w_123, v_{1,2} w_124, v_{1,2} w_125, v_{1,2} w_126, v_{1,2} w_127, v_{1,2} w_128, v_{1,2} w_129, v_{1,2} w_130, v_{1,2} w_131, v_{1,2} w_132, v_{1,2} w_133, v_{1,2} w_134, v_{1,2} w_135, v_{1,2} w_136, v_{1,2} w_137, v_{1,2} w_138, v_{1,2} w_139, v_{1,2} w_140, v_{1,2} w_141, v_{1,2} w_142, v_{1,2} w_143, v_{1,2} w_144, v_{1,2} w_145, v_{1,2} w_146, v_{1,2} w_147, v_{1,2} w_148, v_{1,2} w_149, v_{1,2} w_150, v_{1,2} w_151, v_{1,2} w_152, v_{1,2} w_153, v_{1,2} w_154, v_{1,2} w_155, v_{1,2} w_156, v_{1,2} w_157, v_{1,2} w_158, v_{1,2} w_159, v_{1,2} w_160, v_{1,2} w_161, v_{1,2} w_162, v_{1,2} w_163, v_{1,2} w_164, v_{1,2} w_165, v_{1,2} w_166, v_{1,2} w_167, v_{1,2} w_168, v_{1,2} w_169, v_{1,2} w_170, v_{1,2} w_171, v_{1,2} w_172, v_{1,2} w_173, v_{1,2} w_174, v_{1,2} w_175, v_{1,2} w_176, v_{1,2} w_177, v_{1,2} w_178, v_{1,2} w_179, v_{1,2} w_180, v_{1,2} w_181, v_{1,2} w_182, v_{1,2} w_183, v_{1,2} w_184, v_{1,2} w_185, v_{1,2} w_186, v_{1,2} w_187, v_{1,2} w_188, v_{1,2} w_189, v_{1,2} w_190, v_{1,2} w_191, v_{1,2} w_192, v_{1,2} w_193, v_{1,2} w_194, v_{1,2} w_195, v_{1,2} w_196, v_{1,2} w_197, v_{1,2} w_198, v_{1,2} w_199, v_{1,2} w_200, v_{1,2} w_201, v_{1,2} w_202, v_{1,2} w_203, v_{1,2} w_204, v_{1,2} w_205, v_{1,2} w_206, v_{1,2} w_207, v_{1,2} w_208, v_{1,2} w_209, v_{1,2} w_210, v_{1,2} w_211, v_{1,2} w_212, v_{1,2} w_213, v_{1,2} w_214, v_{1,2} w_215, v_{1,2} w_216, v_{1,2} w_217, v_{1,2} w_218, v_{1,2} w_219, v_{1,2} w_220, v_{1,2} w_221, v_{1,2} w_222, v_{1,2} w_223, v_{1,2} w_224, v_{1,2} w_225, v_{1,2} w_226, v_{1,2} w_227, v_{1,2} w_228, v_{1,2} w_229, v_{1,2} w_230, v_{1,2} w_231, v_{1,2} w_232, v_{1,2} w_233, v_{1,2} w_234, v_{1,2} w_235, v_{1,2} w_236, v_{1,2} w_237, v_{1,2} w_238, v_{1,2} w_239, v_{1,2} w_240, v_{1,2} w_241, v_{1,2} w_242, v_{1,2} w_243, v_{1,2} w_244, v_{1,2} w_245, v_{1,2} w_246, v_{1,2} w_247, v_{1,2} w_248, v_{1,2} w_249, v_{1,2} w_250, v_{1,2} w_251, v_{1,2} w_252, v_{1,2} w_253, v_{1,2} w_254, v_{1,2} w_255, v_{1,2} w_256, v_{1,2} w_257, v_{1,2} w_258, v_{1,2} w_259, v_{1,2} w_260, v_{1,2} w_261, v_{1,2} w_262, v_{1,2} w_263, 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w_463, v_{1,2} w_464, v_{1,2} w_465, v_{1,2} w_466, v_{1,2} w_467, v_{1,2} w_468, v_{1,2} w_469, v_{1,2} w_470, v_{1,2} w_471, v_{1,2} w_472, v_{1,2} w_473, v_{1,2} w_474, v_{1,2} w_475, v_{1,2} w_476, v_{1,2} w_477, v_{1,2} w_478, v_{1,2} w_479, v_{1,2} w_480, v_{1,2} w_481, v_{1,2} w_482, v_{1,2} w_483, v_{1,2} w_484, v_{1,2} w_485, v_{1,2} w_486, v_{1,2} w_487, v_{1,2} w_488, v_{1,2} w_489, v_{1,2} w_490, v_{1,2} w_491, v_{1,2} w_492, v_{1,2} w_493, v_{1,2} w_494, v_{1,2} w_495, v_{1,2} w_496, v_{1,2} w_497, v_{1,2} w_498, v_{1,2} w_499, v_{1,2} w_500, v_{1,2} w_501, v_{1,2} w_502, v_{1,2} w_503, v_{1,2} w_504, v_{1,2} w_505, v_{1,2} w_506, v_{1,2} w_507, v_{1,2} w_508, v_{1,2} w_509, v_{1,2} w_510, v_{1,2} w_511, v_{1,2} w_512, v_{1,2} w_513, v_{1,2} w_514, v_{1,2} w_515, v_{1,2} w_516, v_{1,2} w_517, v_{1,2} w_518, v_{1,2} w_519, v_{1,2} w_520, v_{1,2} w_521, v_{1,2} w_522, v_{1,2} w_523, v_{1,2} w_524, v_{1,2} w_525, v_{1,2} w_526, v_{1,2} w_527, v_{1,2} w_528, v_{1,2} w_529, v_{1,2} w_530, v_{1,2} w_531, v_{1,2} w_532, v_{1,2} w_533, v_{1,2} w_534, v_{1,2} w_535, v_{1,2} w_536, v_{1,2} w_537, v_{1,2} w_538, v_{1,2} w_539, v_{1,2} w_540, v_{1,2} w_541, v_{1,2} w_542, v_{1,2} w_543, v_{1,2} w_544, v_{1,2} w_545, v_{1,2} w_546, v_{1,2} w_547, v_{1,2} w_548, v_{1,2} w_549, v_{1,2} w_550, v_{1,2} w_551, v_{1,2} w_552, v_{1,2} w_553, v_{1,2} w_554, v_{1,2} w_555, v_{1,2} w_556, v_{1,2} w_557, v_{1,2} w_558, v_{1,2} w_559, v_{1,2} w_560, v_{1,2} w_561, v_{1,2} w_562, v_{1,2} w_563, v_{1,2} w_564, v_{1,2} w_565, v_{1,2} w_566, v_{1,2} w_567, v_{1,2} w_568, v_{1,2} w_569, v_{1,2} w_570, v_{1,2} w_571, v_{1,2} w_572, v_{1,2} w_573, v_{1,2} w_574, v_{1,2} w_575, v_{1,2} w_576, v_{1,2} w_577, v_{1,2} w_578, v_{1,2} w_579, v_{1,2} w_580, v_{1,2} w_581, v_{1,2} w_582, v_{1,2} w_583, v_{1,2} w_584, v_{1,2} w_585, v_{1,2} w_586, v_{1,2} w_587, v_{1,2} w_588, v_{1,2} w_589, v_{1,2} w_590, v_{1,2} w_591, v_{1,2} w_592, v_{1,2} w_593, v_{1,2} w_594, v_{1,2} w_595, v_{1,2} w_596, v_{1,2} w_597, v_{1,2} w_598, v_{1,2} w_599, v_{1,2} w_600, v_{1,2} w_601, v_{1,2} w_602, v_{1,2} w_603, v_{1,2} w_604, v_{1,2} w_605, v_{1,2} w_606, v_{1,2} w_607, v_{1,2} w_608, v_{1,2} w_609, v_{1,2} w_610, v_{1,2} w_611, v_{1,2} w_612, v_{1,2} w_613, v_{1,2} w_614, v_{1,2} w_615, v_{1,2} w_616, v_{1,2} w_617, v_{1,2} w_618, v_{1,2} w_619, v_{1,2} w_620, v_{1,2} w_621, v_{1,2} w_622, v_{1,2} w_623, v_{1,2} w_624, v_{1,2} w_625, v_{1,2} w_626, v_{1,2} w_627, v_{1,2} w_628, v_{1,2} w_629, v_{1,2} w_630, v_{1,2} w_631, v_{1,2} w_632, v_{1,2} w_633, v_{1,2} w_634, v_{1,2} w_635, v_{1,2} w_636, v_{1,2} w_637, v_{1,2} w_638, v_{1,2} w_639, v_{1,2} w_640, v_{1,2} w_641, v_{1,2} w_642, v_{1,2} w_643, v_{1,2} w_644, v_{1,2} w_645, v_{1,2} w_646, v_{1,2} w_647, v_{1,2} w_648, v_{1,2} w_649, v_{1,2} w_650, v_{1,2} w_651, v_{1,2} w_652, v_{1,2} w_653, v_{1,2} w_654, v_{1,2} w_655, v_{1,2} w_656, v_{1,2} w_657, v_{1,2} w_658, v_{1,2} w_659, v_{1,2} w_660, v_{1,2} w_661, v_{1,2} w_662, v_{1,2} w_663, v_{1,2} w_664, v_{1,2} w_665, v_{1,2} w_666, v_{1,2} w_667, v_{1,2} w_668, v_{1,2} w_669, v_{1,2} w_670, v_{1,2} w_671, v_{1,2} w_672, v_{1,2} w_673, v_{1,2} w_674, v_{1,2} w_675, v_{1,2} w_676, v_{1,2} w_677, v_{1,2} w_678, v_{1,2} w_679, v_{1,2} w_680, v_{1,2} w_681, v_{1,2} w_682, v_{1,2} w_683, v_{1,2} w_684, v_{1,2} w_685, v_{1,2} w_686, v_{1,2} w_687, v_{1,2} w_688, v_{1,2} w_689, v_{1,2} w_690, v_{1,2} w_691, v_{1,2} w_692, v_{1,2} w_693, v_{1,2} w_694, v_{1,2} w_695, v_{1,2} w_696, v_{1,2} w_697, v_{1,2} w_698, v_{1,2} w_699, v_{1,2} w_700, v_{1,2} w_701, v_{1,2} w_702, v_{1,2} w_703, v_{1,2} w_704, v_{1,2} w_705, v_{1,2} w_706, v_{1,2} w_707, v_{1,2} w_708, v_{1,2} w_709, v_{1,2} w_710, v_{1,2} w_711, v_{1,2} w_712, v_{1,2} w_713, v_{1,2} w_714, v_{1,2} w_715, v_{1,2} w_716, v_{1,2} w_717, v_{1,2} w_718, v_{1,2} w_719, v_{1,2} w_720, v_{1,2} w_721, v_{1,2} w_722, v_{1,2} w_723, v_{1,2} w_724, v_{1,2} w_725, v_{1,2} w_726, v_{1,2} w_727, v_{1,2} w_728, v_{1,2} w_729, v_{1,2} w_730, v_{1,2} w_731, v_{1,2} w_732, v_{1,2} w_733, v_{1,2} w_734, v_{1,2} w_735, v_{1,2} w_736, v_{1,2} w_737, v_{1,2} w_738, v_{1,2} w_739, v_{1,2} w_740, v_{1,2} w_741, v_{1,2} w_742, v_{1,2} w_743, v_{1,2} w_744, v_{1,2} w_745, v_{1,2} w_746, v_{1,2} w_747, v_{1,2} w_748, v_{1,2} w_749, v_{1,2} w_750, v_{1,2} w_751, v_{1,2} w_752, v_{1,2} w_753, v_{1,2} w_754, v_{1,2} w_755, v_{1,2} w_756, v_{1,2} w_757, v_{1,2} w_758, v_{1,2} w_759, v_{1,2} w_760, v_{1,2} w_761, v_{1,2} w_762, v_{1,2} w_763, v_{1,2} w_764, v_{1,2} w_765, v_{1,2} w_766, v_{1,2} w_767, v_{1,2} w_768, v_{1,2} w_769, v_{1,2} w_770, v_{1,2} w_771, v_{1,2} w_772, v_{1,2} w_773, v_{1,2} w_774, v_{1,2} w_775, v_{1,2} w_776, v_{1,2} w_777, v_{1,2} w_778, v_{1,2} w_779, v_{1,2} w_780, v_{1,2} w_781, v_{1,2} w_782, v_{1,2} w_783, v_{$

More Dror:  $\omega\beta/talks$

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