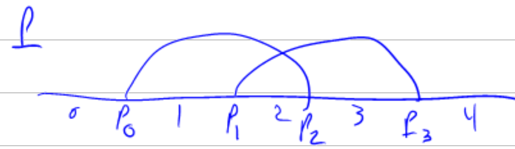
Rooting the BKT for FTI

Following joint work with Itai Bar-Natan, Iva Halacheva, and Nancy Scherich, I will show that the Best Known Time (BKT) to compute a typical Finite Type Invariant (FTI) of type d on a typical knot with n crossings is roughly equal to n^{d/2}, which is roughly the square root of what I believe was the standard belief before, namely about n^d.

$$t = \underset{\text{place}}{p} + \underset{\text{lookup}}{\ell} \qquad \varphi_t(G) = \sum_{T\in\binom{D}{t}} T$$



$$\varphi_t(G) = \binom{t}{p}^{-1} \sum_{P\in\binom{G}{p}} \sum_{\lambda:\, 2\ell \xrightarrow[\text{monotone}]{} \underset{\sim}{2p+1}} \sum_{\substack{L\in\binom{G}{\ell} \\ L_i\in(P_{\lambda(i)-1},\, P_{\lambda(i)})}} P\#_\lambda L$$

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$$= \binom{t}{p}^{-1} \sum_{P\in\binom{G}{p}} \sum_{\lambda:\, 2\ell \xrightarrow[\text{monotone}]{} \underset{\sim}{2p+1}} P\#_\lambda \left(\sum_{\prod_i (P_{\lambda(i)-1},\, P_{\lambda(i)})} \Theta \right)$$

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Thanks for inviting me to UTokyo! ωεβ:=http://drorbn.net/tok2309

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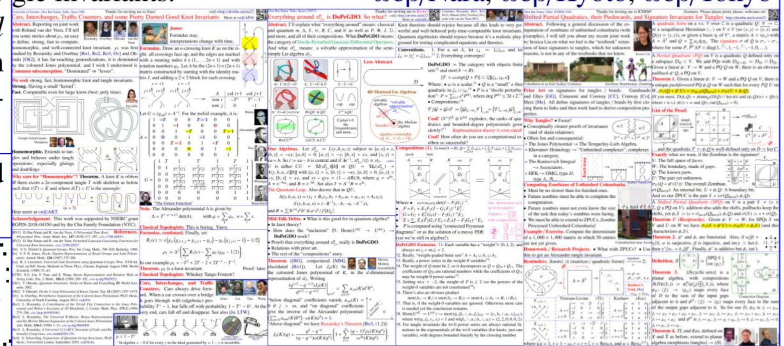
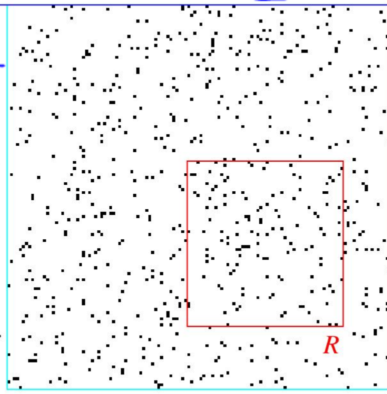


A: an enumerated subset

Convention. For complexity estimates, we ignore constant and logarithmic terms: $n^3 \sim 2023d!(\log n)^d n^3$. Define $\underline{n}$

A ~~Key~~ A Preliminary. Let $X \subset \underline{n}^d$ be a subset ~~of the $n \times n \times \cdots \times n$ grid in $\mathbb{N}^d$~~, with $1 \ll P = |X| \ll n^d$. In time $\sim P$ we can set up a ~~"summary database"~~ of size $\sim P$ so that we will be able to compute $|X \cap R|$ in time ~ 1, for any rectangle $R \subset \underline{n}^d$.

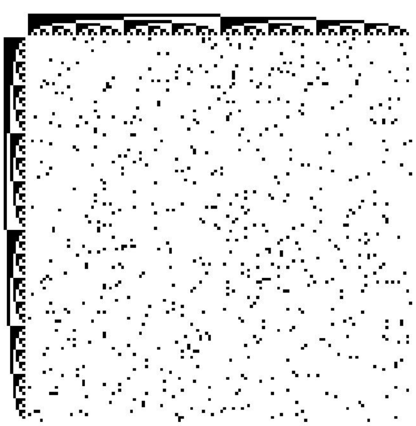


Fails. • Count after R is presented. • Make a ~~database~~ of $|X \cap R|$ counts for all R's.

lookup table

$q = |Q|$ instead of $P = |X|$?

make a $Q \rightsquigarrow O$ version.



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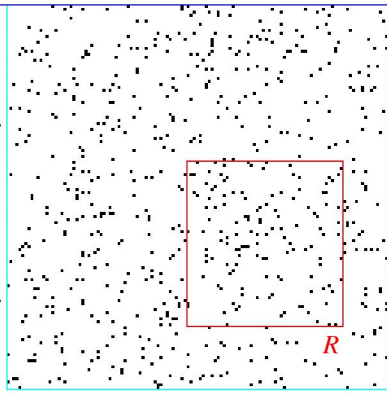

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$$\{R \to |Q \cap R|\},$$

but only for *dyadic* R's.

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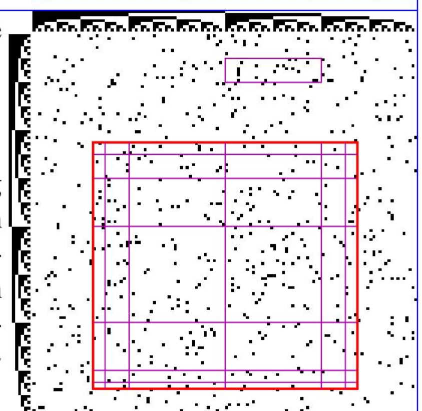

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Gauss Diagrams as in NCSU-1604

(Displaying P55 and its GD)?

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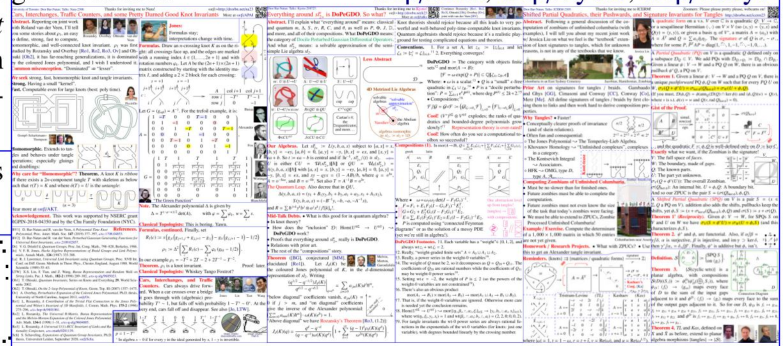
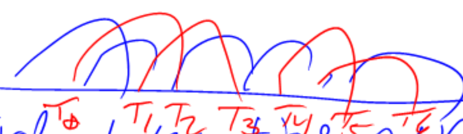


Def $\mathcal{G}$: all Gauss diagrams

$\varphi_t\colon \mathcal{G} \to \mathcal{G}_{\leq t}$ $\quad \varphi_t\colon \mathcal{G} \to \mathcal{G}_t$

Example φ_4



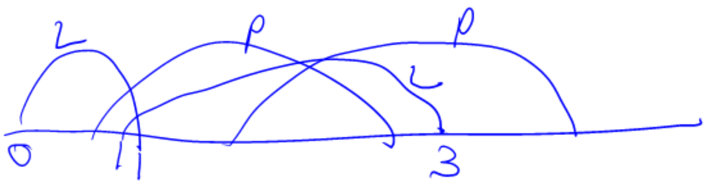
Naively, takes time $\sim n^t$

Thm FT iff $\exists w$ s.t. ... [GPV]

* Hard?
* Not every w gives an invariant.
* The theory of f.t. is rich & powerful.

Thm φ_t can be computed on an n-arrow Gauss diagram in time $\sim n^{\lceil t/2 \rceil}$

Idea



Then formulas...

Refs

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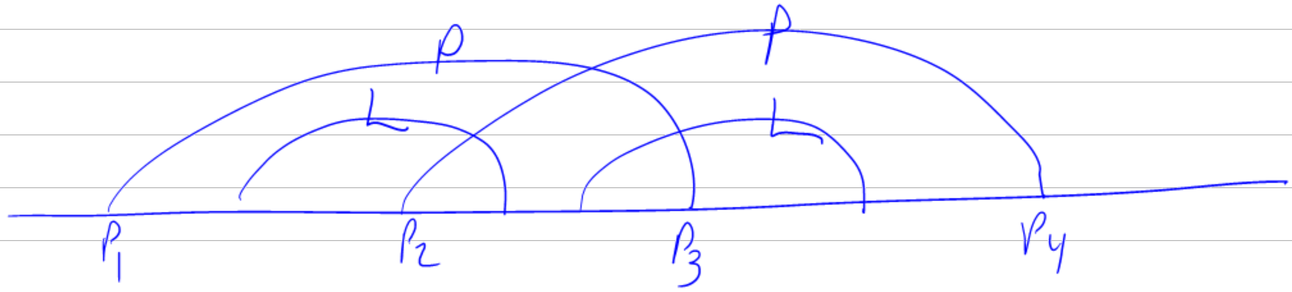

p: purple
L:

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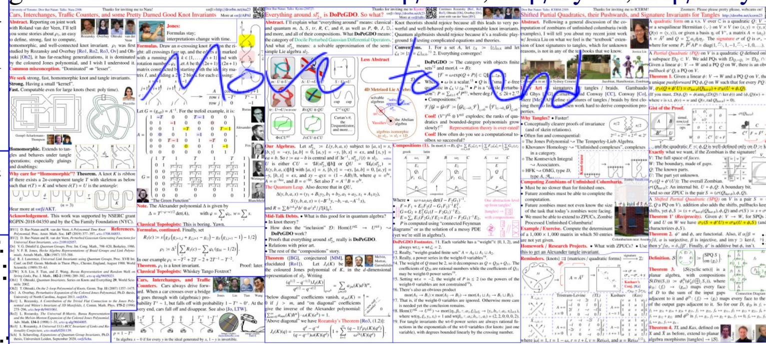

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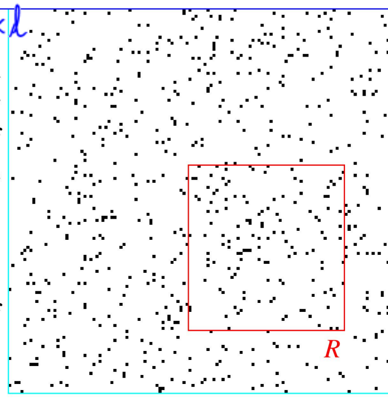

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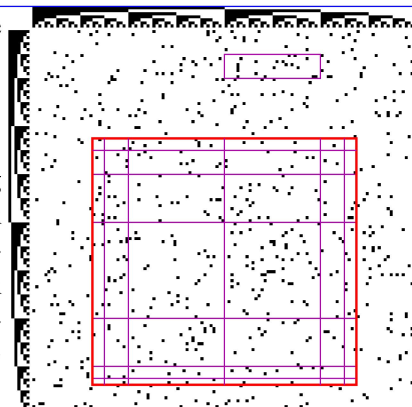

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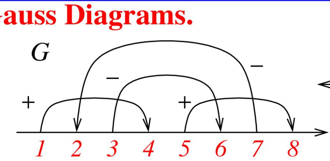
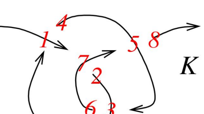
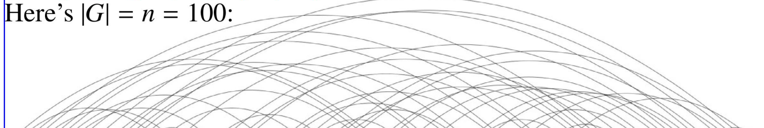


$\longleftrightarrow$



Gauss

Here's $|G| = n = 100$:



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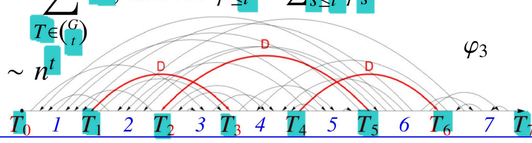

The GPV Theorem. A knot invariant is finite type of type t iff it is of the form $\omega \circ \varphi_{\leq t}$ for some $\omega \in \mathcal{G}^*_{\leq t}$.

Goussarov-Polyak-Viro

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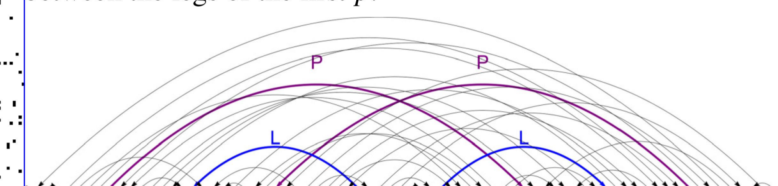

λ: 2 3 3 4

~~Details.~~ To reconstruct $D = P\#_\lambda L$ we need a non-decreasing "placement function" $\lambda\colon \underline{2l} \to \underline{2p+l}$.

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Define $\theta_G\colon \cancel{\underline{n}}^{2l} \to \mathcal{G}_{2l}$ by $\underline{2n}$

$$(L_1, \ldots, L_{2l}) \mapsto \begin{cases} L & \text{if } (L_1, \ldots, L_{2l}) \text{ are the ends of some } L \subset G \\ 0 & \text{otherwise} \end{cases}$$

and now

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can be computed in time $\sim n^p + n^l$. Now choose $p = \lfloor t/2 \rfloor$ or $p = \lceil t/2 \rceil$. □

Question. For computations, planar projections are better than braids (as likely $l \sim n^{3/2}$). But are yarn balls better than planar projections (here likely $n \sim L^{4/3}$)? cut 1-2 lines

Length L

Knot: Piccirillo

n crossings

length l

2023/09/01@03:35

refs: [GPV], [BBHS1]

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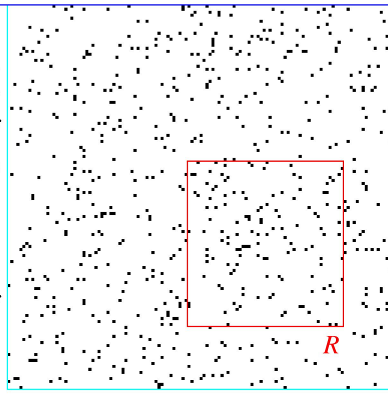

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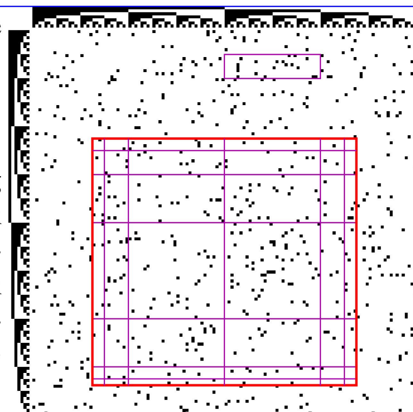

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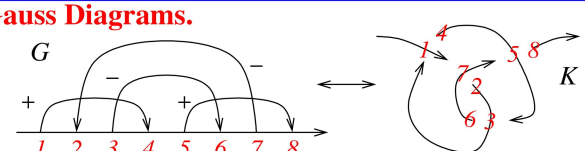

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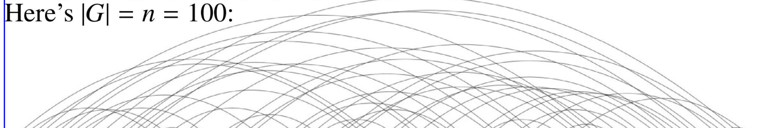

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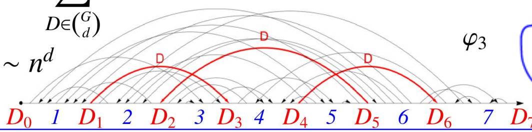

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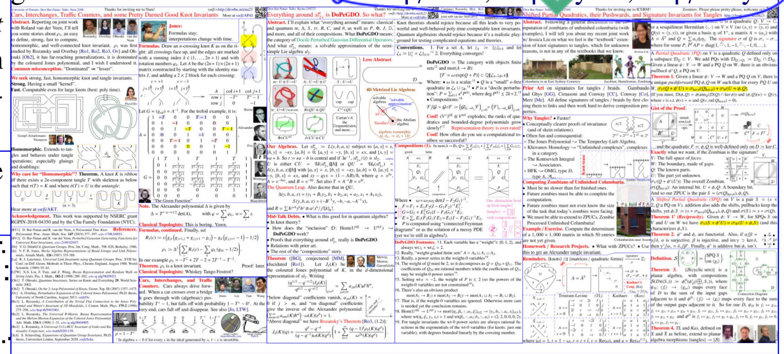

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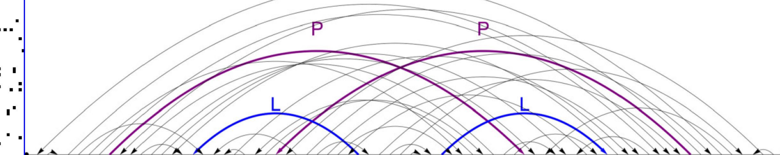

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Length L n crossings length l

References

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[BBHS]

A: [BBHS], wwp/Fields